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Properties of Composite of Closure Operations and Choice Functions

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Abstract

The equivalence of the family of FDs is among many hottest topics that get a lot of attention and consideration currently. There are many equivalent descriptions of the family of FDs. The closure operation and choice function are two of them. Major results of this paper are the properties of the composite function of the choice functions and closure operations. The first parts of this paper address the theories of the composite function of two choice functions and the sufficient and necessary condition of a composite function of two choice functions to be a choice function. Rest of the paper addresses the sufficient and necessary condition of a composite function of more than two choice functions to be a choice function and a composite function of more than two closure operations to be a closure operation.

Keywords: composite function, choice function, closure operation.

1 Introduction

Equivalent descriptions of the family of functional dependencies (FDs) have been widely studied. Based on the equivalent descriptions, we can obtain many important properties of the family of FDs. Choice function and closure operation are two of many equivalent descriptions of the family of FDs. In this paper, we mostly investigate the choice functions. We show some properties of choice functions, and focus on the comparison between and composite function of two, and more than two choice functions. At the end of this paper, we show a theory of the composite function of two and more than two closure operations.

The results of this paper are divided into four parts. First, some properties of the composite function of two choice functions appear in Section 2. Section 3 presents the results about the composite function of more than two choice functions, and that of more than two closure operations. In the conclusion section, we introduce our plans for future research.

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Let us give some necessary definitions that are used in the next section. Those well-known concepts in relational database given in this section can be found in [1, 2, 3, 4, 5, 6, 8].

Definition 1 *Let $U = \{a_1, ..., a_n\}$ be a nonempty finite set of attributes. A functional dependency is a statement of the form $A \to B$, where $A, B \subseteq U$. The FD $A \to B$ holds in a relation $R = \{h_1, ..., h_m\}$ over U if $\forall\ h_i, h_j \in R$ we have $h_i(a) = h_j(a)$ for all $a \in A$ implies $h_i(b) = h_j(b)$ for all $b \in B$. We also say that R satisfies the FD $A \to B$.*

A family of FDs satisfying Armstrong's Axioms is called an f-family over U. Given a family F of FDs over U, there exits a unique minimal f-family F^+ that contains F. It can be seen that F^+ contains all FDs which can be derived from F by Armstrong Axioms.

A relation scheme s is a pair $< U, F >$, where U is a set of attributes, and F is a set of FDs over U.

Let U be a nonempty finite set of attributes and $P(U)$ its power set. A map $L : P(U) \to P(U)$ is called a closure over U if it satisfies the following conditions:

(1) $A \subseteq L(A)$,
(2) $A \subseteq B$ implies $L(A) \subseteq L(B)$
(3) $L(L(A)) = L(A)$.

Set $L(A) = \{a : A \to \{a\} \in F^+\}$, we can see that L is a closure over U. There is a 1-1 correspondence between closures and f-families on U.

A map $C : P(U) \to P(U)$ is called a choice function, if every $A \in P(U)$, then $C(A) \subseteq A$.
If we assume that $C(A) = U - L(U - A)$ (*), we can easily see that C is a choice function.

The relationship like (*) is considered as a 1-1 correspondence between closures and choice functions, which satisfies the following two conditions:
For every $A, B \subseteq U$,

(1) If $C(A) \subseteq B \subseteq A$, then $C(A) = C(B)$
(2) If $A \subseteq B$, then $C(A) \subseteq C(B)$

We call all of choice functions satisfying those two above conditions special choice functions.

There is a 1-1 correspondence between special choice functions and f-families on U.

We define Γ as a set of all of special choice (SC) functions on U. Now we investigate some properties of those functions.

2 Properties of the SC functions

In this section, we give some results related to the composite function of two choice functions.

Let $f, g \in \Gamma$, and we determine a map k as a composite function of f and g as the following:

$$k(X) = f(g(X)) = f.g(X) = fg(X) \text{ for every } X \subseteq U.$$

Let U be a nonempty finite set of attributes, and $f, g \in \Gamma$. We say that f is smaller than g, denoted as $f \leq g$ or $g \geq f$, if for every $X \subseteq U$ we always have $f(X) \subseteq g(X)$.
The "smaller" relation, $\leq$, satisfies these following properties. For every $f, g, h \in \Gamma$:

1) $f = f$ (Reflexive)
2) If $f \leq g$, and $g \leq f$, then $g = f$. (Symmetric)
3) If $f \leq g$, and $g \leq h$, then $f \leq h$. (Transitive)

Proposition 1 *If $f, g \in \Gamma$, then*

$$1)\ fg \leq f, \quad 2)\ fg \leq g,$$
$$3)\ gf \leq f, \quad 4)\ gf \leq g.$$

Proof. Since $f, g \in \Gamma$, f and g must be SC functions on U. Therefore, we have $g(X) \subseteq X$ for every $X \subseteq U$, then $f(g(X)) \subseteq f(X)$. And f is a SC function on U; so $f(g(X)) \subseteq g(X)$. So we can conclude that $fg \leq f$ and $fg \leq g$. Similarly, we can easily prove $gf \leq f$ and $gf \leq g$. □

Proposition 2 *If f, h and $g \in \Gamma$ and $f \leq g$, then*

$$1)\ fh \leq gh,$$
$$2)\ hf \leq hg.$$

Proof. Because f, g and h are three SC functions and $f \leq g$, we always have $f(h(X)) \subseteq g(h(X))$, for every $X \subseteq U$. Since $f \leq g$, we have $f(X) \subseteq g(X)$. h is a SC function, so we have $h(f(X)) \subseteq h(g(X))$. We can conclude that $fh \leq gh$ and $hf \leq hg$. □

Proposition 3 *If f, g, h and $k \in \Gamma$, and $f \leq g$, and $k \leq h$, then $fk \leq gh$.*

Proof. Assume $f, g, h, k \in \Gamma$ and $f \leq g$, and $k \leq h$. According to Proposition 2, we have $fk \leq gk$ and $gk \leq gh$. Therefore, according to the transitive property, we have $fk \leq gh$. □

Theorem 1 *If $f, g \in \Gamma$, then these following two conditions are equivalence:*

$$1)\ f \leq g,$$
$$2)\ fg = f.$$

Proof. ($1 \to 2$) Assume $f, g \in \Gamma$ and $f \leq g$. Since f is a SC function, f must satisfies this property: if $f(X) \subseteq Y \subseteq X$, then $f(X) = f(Y)$. Therefore, we have $f \leq g$ or $f(X) \subseteq g(X) \subseteq X$ for every $X \subseteq U$, so $f(g(X)) = f(X)$ or we conclude that $fg = f$.
($2 \to 1$) Assume $f, g \in \Gamma$ and $fg = f$. Since f and g are SC functions, according to Proposition 1, we have $fg \leq g$, but $fg = f$, so we have $f \leq g$. The proof is completed. □

From the Theorem 1, we can easily see that if $f \leq g$, then fg is a SC function (since $fg = f$, and f is a SC function).

Lemma 1 *If $f \in \Gamma$, then $ff = f$.*

Proof. It can be seen easily that Lemma 1 holds directly from the Theorem 1. □

Theorem 2 *Let $f, g \in \Gamma$. A composite function of f and g, denoted as fg, is a SC function if and only if $fgf = fg$:*

$$(fg \text{ is a SC function} \Leftrightarrow fgf = fg).$$

Proof. First, we need to prove that fg is a choice function.
For every $X \subseteq U$, we have $g(X) \subseteq X$ because g is a SC function. And f also is a SC function, so if $g(X) \subseteq X$, then $f(g(X)) \subseteq f(X) \subseteq X$. Therefore, we can conclude that $fg(X) \subseteq X$, in other word, we can say that fg is a choice function. Similarly, we can prove that gf is also a choice function.

Now, we prove that fg is a SC function $\Leftrightarrow fgf = fg$. First, we need to prove the statement: if fg is a SC function, then $fgf = fg$. According to Proposition 1, we have $fg \leq f$. And fg is a SC function, so $fgf = fg$ due to Theorem 1.

Then, we just need to prove that if $fgf = fg$, then fg is a SC function. In other words, we need to prove that if $fgf = fg$, then fg satisfies these two conditions (1) and (2):
If $X \subseteq Y$, then $fg(X) \subseteq fg(Y)$, and if $fg(X) \subseteq Y \subseteq X$, then $fg(X) = fg(Y)$.
When $X \subseteq Y$, we have $g(X) \subseteq g(Y)$ since g is a SC function. And when $g(X) \subseteq g(Y)$, we have $f(g(X)) \subseteq f(g(Y))$ or $fg(X) \subseteq fg(Y)$ since f is also a SC function.

We have $fg(X) \subseteq Y \subseteq X$, so $g(fg(X)) \subseteq g(Y) \subseteq g(X)$ or $gfg(X) \subseteq g(Y) \subseteq g(X)$ since g is a SC function. And since f is also a SC function, we also have $f(gfg(X)) \subseteq f(g(Y)) \subseteq f(g(X))$ or $fgfg(X) \subseteq fg(Y) \subseteq fg(X)$. However, fgf =fg, so that leads to that $fgg(X) = fgfg(X) \subseteq fg(Y) \subseteq fg(X)$. We can rewrite that expression as $fgg(X) \subseteq fg(Y) \subseteq fg(X)$. According to Lemma 1, we have $gg(X) = g(X)$, so $fgg(X) = fg(X) \subseteq fg(Y) \subseteq fg(X)$. Therefore, $fg(X) = fg(Y)$.
Consequently, we can conclude that fg is a SC function iff $fgf = fg$. The proof is completed. □

Theorem 3 *Let $f, g \in \Gamma$. Then fg and gf are simultaneously SC functions if and only if $fg = gf$.*

Proof. In the proof of Theorem 2, already we have proved that fg and gf are always choice functions when f and g are SC functions.
We need to prove this statement: if fg and gf are simultaneously SC functions, then $fg = gf$, for $f, g \in \Gamma$.
According to Proposition 1, we have $fg \leq g$ and $fg \leq f$. So due to Proposition 3, we have $(fg)(fg) \leq gf$. But we also have fg is a SC function, so $(fg)(fg) = fg$ due to Lemma 1. Thus, $(fg)(fg) = fg \leq gf$. Similarly, we also have $gf \leq fg$. Hence, we have $fg \leq gf \leq fg$, so we can conclude that $fg = gf$.
We just need to prove that: if $fg = gf$, then fg and gf are simultaneously SC functions for f,g $\in \Gamma$. In other words, we need to prove that if $fg = gf$, then fg and gf satisfies these two conditions (1) and (2):
If $X \subseteq Y$, then $fg(X) \subseteq fg(Y)$ and $gf(X) \subseteq gf(Y)$.
If $fg(X) \subseteq Y \subseteq X$, then $fg(X) = fg(Y)$, and if $gf(X) \subseteq Y \subseteq X$, then $gf(X) = gf(Y)$.

In the proof of Theorem 2, we have already proved: if $X \subseteq Y$, then $fg(X) \subseteq fg(Y)$. Similarly, we also can prove that $gf(X) \subseteq gf(Y)$.
We have $fg(X) \subseteq Y \subseteq X$, so $g(fg(X)) \subseteq g(Y) \subseteq g(X)$ or $gfg(X) \subseteq g(Y) \subseteq g(X)$ since g is a SC function. And since f is also a SC function, we also have $f(gfg(X)) \subseteq f(g(Y)) \subseteq f(g(X))$ or $fgfg(X) \subseteq fg(Y) \subseteq fg(X)$. However, $fg = gf$, so that leads to that $ffgg(X) = fgfg(X) \subseteq fg(Y) \subseteq fg(X)$. We can rewrite that expression as $ffgg(X) \subseteq fg(Y) \subseteq fg(X)$. According to Lemma 1, we have $gg = g$ and $ff = f$, so $ffgg(X) = fg(X) \subseteq fg(Y) \subseteq fg(X)$. Therefore, $fg(X) = fg(Y)$.
Similarly, we also prove that if $gf(X) \subseteq Y \subseteq X$, then $gf(X) = gf(Y)$.
Consequently, we can say that fg and gf are simultaneously SC functions if and only if $fg = gf$ for $f, g \in \Gamma$. The proof is completed. □

So far, we have covered some properties of the composition of two SC functions and found out some interesting results. However, we would like to raise the following two questions:
Can we generalize the Theorem 2 for the composition of more than two SC functions? Will we get the same answer? More generally, what is a necessary and sufficient condition such that a composite function of more than two SC functions is a SC function?

3 Composite of more than two SC functions and more than two closure operations

In order to generalize the Theorem 2, we first need to observe the composition of three SC functions before we can go any further.

Theorem 4 *Let f, g; and $h \in \Gamma$. A composite function of f, g, and h, denoted as fgh, is a SC function if and only if $fghfg = fgh$:*

$$(fgh \text{ is a SC function} \Leftrightarrow fghfg = fgh)$$

Proof. We can easily prove that fgh is a choice function.
For every $X \subseteq U$, we have $h(X) \subseteq X$ because g is a SC function. And f and g also are SC functions, so if $h(X) \subseteq X$, then $g(h(X)) \subseteq h(X) \subseteq X$, then $f(g(h(X))) \subseteq g(h(X)) \subseteq h(X) \subseteq X$. Therefore, we can conclude that $fgh(X) \subseteq X$, in other word, we can say that fgh is a choice function. Now, we must prove that fgh is a SC function $\Leftrightarrow fghfg = fgh$.

First, we need to prove the statement: if fgh is a SC function, then $fghfg = fgh$.
According to Proposition 1, we have $gh \leq g$ or $g(h(X)) \subseteq g(X)$, for every $X \subseteq U$. And f is a SC function, so $f(g(h(X)) \subseteq f(g(X))$, and $f(g(X)) \subseteq g(X) \subseteq X$. Thus, we have that $f(g(h(X))) \subseteq f(g(X)) \subseteq X$, so we have $f(g(h(f(g(X))))) = f(g(h(X)))$ or $fghfg = fgh$ since fgh is a SC function.

Then, we just need to prove that if $fghfg = fgh$, then fgh is a SC function. In other words, we need to prove that if $fghfg = fgh$, then fgh satisfies these two conditions (1) and (2):
If $X \subseteq Y$, then $fgh(X) \subseteq fgh(Y)$, and if $fgh(X) \subseteq Y \subseteq X$, then $fgh(X) = fgh(Y)$.

When $X \subseteq Y$, we have $h(X) \subseteq h(Y)$ since h is a SC function. And when $h(X) \subseteq h(Y)$, we have $g(h(X)) \subseteq g(h(Y))$ or $gh(X) \subseteq gh(Y)$ since g is a SC function. And since f is also a SC function, we have $f(gh(X)) \subseteq f(gh(Y))$ or $fgh(X) \subseteq fgh(Y)$.

We have $fgh(X) \subseteq Y \subseteq X$, so $h(fgh(X)) \subseteq h(Y) \subseteq h(X)$ or $hfgh(X) \subseteq h(Y) \subseteq h(X)$ since h is a SC function. And since g is also a SC function, we also have $g(hfgh(X)) \subseteq g(h(Y)) \subseteq g(h(X))$ or $ghfgh(X) \subseteq gh(Y) \subseteq gh(X)$. Similarly, we have $fghfgh(X) \subseteq fgh(Y) \subseteq fgh(X)$ since f is a SC function. However, $fghfg = fgh$, so that leads to that $fghfgh(X) = fghh(X) \subseteq fgh(Y) \subseteq fgh(X)$. We can rewrite that expression as $fghh(X) \subseteq fgh(Y) \subseteq fgh(X)$. According to Lemma 1, we have $hh(X) = h(X)$, so $fghh(X) = fgh(X) \subseteq fgh(Y) \subseteq fgh(X)$. Therefore, $fgh(X) = fgh(Y)$.
Consequently, we can conclude that fgh is a SC function iff $fghfg = fgh$. The proof is completed. □

It can be seen easily that we can generalize the Theorem 4 for the composite of more than three SC functions with the result and proof analogous to Theorem 4.

As we used to mention in the Introduction part, there is a relation (*) between the choice function and closure. For every $A \in P(U)$, if we assume that $C(A) = U - L(U - A)(*)$, we can prove that C is a choice function. After investigating some properties of the composite of choice functions, we are willing to show that the closure operation has similar property. First, we need to give a definition of the composite function of closure operations.

Let $f, g \in L$, a set of all of closure operation on U. We determine a map k as a composite function of f and g as the following:

$$k(X) = f(g(X)) = f.g(X) = fg(X) \text{ for every } X \subseteq U.$$

We have similar definition of the composite function of more than two closure operations.
Here is the result about the composite of closure operations.

Theorem 5 *Let f, g and $h \in L$, a set of all of closure operation on U. A composite function of f, g and h, denoted as fgh, is a closure (or closure operation) if and only if $fghfg = fgh$.*

$$(\textit{That is, } fgh \textit{ is a closure } \Leftrightarrow fghfg = fgh)$$

Proof. First we prove this statement: if f, g, h and fgh are closures, then $fghfg = fgh$.
For every $X \subseteq U$, we have $X \subseteq h(X)$ since h is a closure. From $X \subseteq h(X)$, we have $g(X) \subseteq g(h(X))$ since g is a closure. Similarly, we have $f(g(X)) \subseteq f(g(h(X)))$. Since f is a closure, we have $g(X) \subseteq f(g(X))$. And since g is a closure, we have $X \subseteq g(X)$. Thus, $X \subseteq f(g(X))$. So we can lead to $X \subseteq f(g(X)) \subseteq f(g(h(X)))$. We can rewrite in the other form $X \subseteq fg(X) \subseteq fgh(X)$. Since fgh is a closure, we have $fgh(X) \subseteq fgh(fg(X)) \subseteq fgh(fgh(X))$. Because fgh is a closure, we have $fgh(fgh(X)) = fgh(X)$. Hence $fgh(X) \subseteq fgh(fg(X)) \subseteq fgh(fgh(X)) = fgh(X)$. So we can conclude that $fgh(fg(X)) = fgh(X)$ or $fghfg(X) = fgh(X)$.

Now, we move to prove the reversed statement: if $fghfg = fgh$, then fgh is a closure.
In order to prove fgh is a closure, we need to prove that fgh satisfies those three conditions:

1) $X \subseteq fgh(X)$,
2) $X \subseteq Y$ implies $fgh(X) \subseteq fgh(Y)$, for X and $Y \subseteq U$, and
3) $fgh(fgh(X)) = fgh(X)$.

We have already proved 1) above.
Since h is a closure, from $X \subseteq Y$, we have $h(X) \subseteq h(Y)$. Similarly, we have $g(h(X)) \subseteq g(h(Y))$, then $f(g(h(X))) \subseteq f(g(h(Y)))$ or $fgh(X) \subseteq fgh(Y)$. Thus, fgh satisfies 2).
Since $fghfg = fgh$, we have $fgh(fgh(X)) = fghfgh(X) = fghfg(h(X)) = fgh(h(X)) = fghh(X) = fgh(X)$ since h is a closure, which satisfies the third condition $hh(X) = h(X)$. Therefore, fgh also satisfies three conditions. So fgh is a closure if $fghfg = fgh$. The proof is completed. □

Similarly to the SC function, we can generalize Theorem 5 for the composite of more than three closure operations with analogous result and proof.

4 Open problems

Our further research will be devoted to following open problems:

Open Problem 1. Is the union, intersection, or subtraction of two SC functions a SC function?

Open Problem 2. We would like to apply above results and Theorems into design of algorithm. We have two relation schemes $s =< U, F >$ and $t =< U, V >$, where U is a set of attributes and F and V are two different sets of FDs over U. We define F^+ and V^+ be a set of all FDs that can be derived from F and V respectively. Is it possible build a closure f and a closure g from F^+ and V^+ respectively such that $fg = fgf$? If so, how can we design fg? In other word, how can we design a relation scheme $w =< U, H >$ from which we can build H^+, from which we can design the closure $fg = fgf$? If so, is it possible to generalize this design for more than two closure operations?

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