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SW-type puzzles and their graphs

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Abstract

In this paper, we present the SW-type of truth-tellers and liars puzzles. We examine the SW-type puzzles where each person can utter a sentence about the person's type and in which he uses only the "and" connective. We make the graphs of these puzzles. The graph of a puzzle has all information about the puzzle if we have no other information to solve the puzzle than the statements given (clear puzzles). We analyze the graphs of the possible puzzles. We give some transformations of graphs based on local information, for instance arrow-adding steps. These local steps are very helpful to solve these puzzles. We show an example that we can solve using these local steps. After this, we examine into the global properties of the graphs. We show a special example when the local steps do not help, but the puzzle is solvable by using global information. Finally we show a graph-algorithm which is a combination of local and global information, and show that it can solve the SW-type puzzles.

Keywords: puzzles, truth-tellers and liars, graphs, graph-algorithm

1 Introduction

Games are as old as humanity. Nowadays most people connect them to computers. Game playing is also good time-spending activities. The problems needing more or less time to solve represent useful ways of spending one's spare time. A part of games are puzzles. Logical puzzles can be solved by a rational way of thinking. From children to very wise people everybody can find puzzles which develop their skills. It can be a good hobby as well. Therefore, logical puzzles are very useful to explore the ability of logical thought. There are many kind of puzzles. In this article, we consider a simple type called "truth-tellers and liars". In these puzzles, there are some people each of the following two types: either truth-teller, who can say only true statements; or liar, who can say only false statements. All participants have full information about the type of the others. Some of them claim about the type of the others. The puzzle is to figure out the types of each person. These problems are very popular. Smullyan examined such puzzles in scientifical and logical way ([7], [8], [9], [10]), where the participants was distinguished as

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knights and knaves. In [6], satellites send messages informing mechanics whether the neighbouring satellites work properly, or not. In [1], [2] and in [3], Aszalós solves many puzzles by using tableaux method and Prolog language, as well.

In this paper, we investigate a special type of truth-teller–liar puzzles. We show a characteristic example.

Example 1.1. Consider the following problem. There are four people: Alice, Bob, Charlie and David. Each of them is either a truth-teller or a liar. They say:

Alice says that Bob is a truth-teller and David is a liar. Bob says that Charlie is a liar. Charlie says that Alice is a truth-teller and David is a liar.

Using Smullyan's method [7] we write the example in the next logical form:

$$A \equiv B \wedge \neg D, \quad B \equiv \neg C, \quad C \equiv A \wedge \neg D$$

One can check easily that valuation $A = C = false$, $B = D = true$ gives valid formulae. Therefore a solution of the example that Alice and Charlie are truth-tellers; Bob and David are liars. Moreover it is easy to check that the variables A, B, C, D have not other values such that all given formulae are true. So our solution is unique.

In the next section we describe more precisely the SW-type truth-teller and liar puzzles. In [4] we presented a program in language C which can generate special SW-type puzzles. Now we will associate graphs to the puzzles, which are very useful to examine the structure of the puzzles, and we can solve a puzzle by using the associated graph.

In section 3 we analyse the SW-type puzzles and we show some useful steps to solve them by using local information, in this section we will solve Example 1.1.

In the next sections, we show an example such that the local information is not enough to solve it. We show how we can use the global information of the graph to get the solution. We present a general algorithm mixed the local and global information.

2 SW-type puzzles

We need a few concepts to the mathematical discussion and so; we give some basic definitions and notations.

The sentences which are not dividable to smaller sentences are called atomic (or simple) statements. In this paper, we use atomic statements only about a person's type as we seen in Example 1.1.

In Example 1.1 there are 5 atomic sentences. Alice and Charlie tell two-two atomic statements and Bob tells one.

In this puzzles, if a person is a truth-teller then the conjunction of his atomic statements must be true. If a person is a liar then the conjunction of his atomic statements is false. In [5] we used these definitions to define S(trong) Truth-tellers and W(eak) Liars.

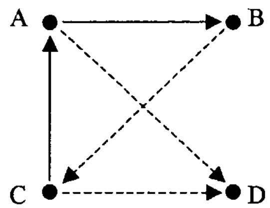

Figure 1: The graph of Example 1.1

If a person remains silent then he may be Truth-teller or he may be Liar. We may know it only from the atomic sentences about his type.

We call our puzzles SW-type because each person is an S truth-teller, or a W liar.

In [4] we investigated clear and non-clear puzzles. A puzzle is clear, if we have no other information to solve the puzzle than the statements given.

An example of a non-clear puzzle is someone's type or the number of truth-tellers being known independently of the statements. In this paper, we examine only clear puzzles. Our example is a clear SW-puzzle.

We use the value 1 to represent the truth-tellers, and the value 0 for the liars.

Definition 2.1. *The* solution *of a puzzle is a function which assigns either a 1 or a 0 to each person, who is in the puzzle depending upon the truth-values of the statements he or she makes. Two solutions are different, if there is a person, whose type is not the same in these two functions.*

We say that a puzzle is good *if it has a unique solution.*

In this paper we use three type of participants in dynamic way. The initial type is the unknown. The other two types are the known 1 and known 0. We will sign the known values at the participants who have it. Usually we will use the sets T and L as sets of truth-tellers and sets of liars, respectively.

This paper will investigate clear puzzles only with solutions and the most of our results are about good puzzles.

Definition 2.2. *Puzzles are represented by directed* graphs *with a node of the graph for person in the puzzle. There are two types of arrows: if A said that B is a truth-teller then we use a solid arrow from A to B; if A said that B is a liar then we use a dashed arrow from A to B. We will use $\mathbf{N}$ as the set of nodes and $\mathbf{t}, \mathbf{l}$ as the sets of solid and dashed edges, respectively.*

We will use the names of persons as names of nodes, and sometimes as logical statements which can have either 0 or 1 values. We use the following notation: $P(\mathbf{N},\mathbf{t},\mathbf{l})$ (where the nodes are $B_i \in \mathbf{N}$, the solid and dashed edges $t_j \in \mathbf{t}$ and $l_k \in \mathbf{l}$ respectively, and they are sorted pairs of nodes) as the associated graph of a puzzle.

In Fig. 1 we show the graph of Example 1.1.

We say that the puzzles P and Q are equivalent if the solution(s) of P are the same as the solution(s) of Q.

Now we continue the describing the SW-type puzzles, which are widely used. One can find them in almost every book on puzzles.

According to our concepts, in these puzzles we have only S truth-tellers, and W liars, hence we can use conjunction in logical form of the sentences. In an SW-type puzzle, every person can make at most one complex assertion which has a truth-value corresponding to the type of the person. The possible atomic sentences are: the man names a person, and he states that this person is a truth-teller (or he states, that this person is a liar).

So in an SW-type puzzle with n persons, each person can claim at most one sentence, and it must be the following type: he names m persons ($0 \leq m \leq n$), and he states that all of them are truth-tellers, and he names k persons ($0 \leq k \leq n$), and he states that all of them are liars. So, there are two – maybe one of them or both empty, and not necessarily disjoint – sets of the persons for each member in the puzzle, about whom that member claims something.

Definition 2.3. (Formal definition of SW-puzzles) *Suppose that there are n people* $B_1, B_2, \ldots B_n$.
From [7] and [4] we write the following logical form from the statements: if $\exists j(B_iB_j \in t$ *or* $B_iB_j \in l)$ *then*

$$(*) \qquad B_i \equiv (\bigwedge_{B_iB_j \in t} B_j) \wedge (\bigwedge_{B_iB_k \in l} \neg B_k).$$

If all the conditions of a puzzle can be written by this way then it is a clear SW-puzzle.

Now we consider that P is a graph of an SW-type puzzle. Let us examine the meaning of the arrows.

If A states the atomic statement, that B is a truth-teller then $A \supset B$ is valid (this is what the solid arrow means), or if A states, that B is a liar then the formula $A \supset \neg B$ is true (it is a dashed arrow). In $(*)$ we have equivalence, so we need one more concept: the relevant edges.

Definition 2.4. *In the graph P, we call an edge* relevant edge *if it is possible for it to stand for a liar's actual, false atomic statement.*

We will use this concept in dynamic way. First we assume, that all arrows are relevant. (We do not know yet if it was not possible for an edge to stand for a liar's false statement.) And while it turns out that an edge might not be relevant we assume that it is possible for it to stand for a liar's actual, false atomic statement. It is evident that there is at least 1 relevant arrow from the nodes which are type 0, assuming that person said something.

Notation 2.5. In the graph we cross the non-relevant edges. An edge is annotated with the sign '!' if there is only this relevant edge starting from that node. We will use these notations in figures. In form $P(\mathbf{N},\mathbf{t},\mathbf{l})$ we will use for a non-relevant edge AB ($A, B \in \mathbf{N}$) the sign $\overline{AB}$ (overlying). And for the unique relevant edge CD (where $C, D \in \mathbf{N}$) we use the $\underline{CD}$ (underlying) form in text instead of the sign '!'. From here $AB \in \mathbf{t}$ means that one of the AB, $\overline{AB}$ and $\underline{AB}$ is in the set $\mathbf{t}$, and similarly for the set $\mathbf{l}$.

The following assertion shows how we can use the relevant edges to make the ($*$) form from the implications.

If there is only one relevant edge from a node B, then we use equivalence in the formula instead of implication: $\underline{BC} \in \mathbf{t}$ means that $B \equiv C$; $\underline{BC} \in \mathbf{l}$ means that $B \equiv \neg C$.

Hence we can see that the graph of a puzzle represents really the sets of logical formulas of the puzzle. The nodes (B_i) like atomic statements ("B_i is a truth-teller"), the arrows like logical connectives between them, and hence we can use them as logical statements also.

3 Local steps in the graph

Solving the puzzles requires some techniques which modify the graph of the puzzle. However our new graph is equivalent to the original. These local techniques are the following (we will give detailed description of them in this section):

Definition 3.1.

a) An arrow-adding step *is the following: add a new (non-relevant) arrow to the graph such that all solutions remain such that our new graph is equivalent to the original.*

b) A node-union *is when it turns out that two nodes must be same types, and we need use only one of them as common node.*

c) A subgraph is a basic scheme *if the type of one of the nodes of this part can be only one of* $\{0, 1\}$ *according to the arrows in this part.*

d) An arrow is a valuable arrow, *if we know the type of starting or the ending node, and we can infer the type of the other node (using only the information about the type of the first node, and the type of connection.)*

e) In arrow deletion *and* arrow change to irrelevant *steps we will delete the arrows, which are non useful (we cannot use them to get new information, for example we know the type of both end-nodes), or we cross the edges, which we cannot use as relevant edges, but we may will use as valuable arrows.*

The basic schemes and the valuable arrows change the types of the nodes to known value. The node-union step decrease the number of nodes and use the new node as endpoint of the edges, which had endpoint one of the joined nodes. In an arrow-adding step we increase the number of edges, while in an arrow deletion we decrease it.

Now we have some lemmas about these local steps: when and how we can use them.

First we show the arrow-adding steps. We add a new non-relevant edge to the graph in the following cases. (The new edge is non-relevant, because it is not a real statement in the original puzzle.)

Lemma 3.2. *The following steps are arrow-adding steps. The graphs before and after a step are equivalent.*

	edge(s) in the graph before the step	*the new irrelevant edge(s)*
a)	$AC \in \boldsymbol{l}$	$\overline{AC} \in \boldsymbol{l}$
b)	$AB, BC \in \boldsymbol{t}$	$\overline{AC} \in \boldsymbol{t}$
c)	$AB \in \boldsymbol{t}$ *and* $(BC \in \boldsymbol{l}$ *or* $CB \in \boldsymbol{l})^{\clubsuit}$	$\overline{AC}, \overline{CA} \in \boldsymbol{l}^{\diamondsuit}$
d)	$AB \in \boldsymbol{l}$ *and* $(\underline{BC}$ *or* $\underline{CB} \in \boldsymbol{l})^{\heartsuit}$	$\overline{AC} \in \boldsymbol{t}$.

If the new type of arrow with this direction has directly connected the nodes A and C, then we do not need the new edge (we already have edge which means this type of connection).

Proof. It is from the logical meanings of the edges. The case a) is from $A \supset \neg B$ is equivalent to $B \supset \neg A$, which is the meaning of the dashed arrow for opposite direction, case b) is from: $A \supset B$, $B \supset C$ are the original arrows, and their logical consequence is $A \supset C$, the new arrow. In case c) $\clubsuit$ means that we can use this step independently the direction of the dashed edge between B and C, because of point a), and $\diamondsuit$ signs that we can add arrows with both directions because we have $A \supset B$, $B \supset \neg C$, therefore $A \supset \neg C$ is valid, and than using point a) $C \supset \neg A$ also valid. In case d) the sign $\heartsuit$ notes that the dashed edge between B and C must be only one relevant at least from one direction and we have no restriction about the relevance of other edges in these steps. Then $A \supset \neg B$, $B \equiv \neg C$, which is implies that $C \equiv \neg B$, hence $A \supset C$. $\square$

The meanings of these new edges are about "that person could say these things also". In [3] Aszalós examines this modal operator in puzzles.

According to the point a) of the previous lemma, if the relevance is not important then we can use only dashed line instead arrows. (But sometime we need the directions of these arrows for using relevance.)

Let us see how we modify the graph of the example using these steps.

We note by sign ! the edge from B because it is unique. We can use arrow-adding steps b) and c) among A, B and C (Fig. 2).

Now, we can use arrow-adding step c) for $\overline{CB} \in \mathbf{t}, BC \in \mathbf{l}$ or for $CA \in \mathbf{t}, \overline{AC} \in \mathbf{l}$.

Now we go back to the theory. We have a basic scheme:

Lemma 3.3. *If the graph of a puzzle $P(\boldsymbol{N},\boldsymbol{t},\boldsymbol{l})$ contains dashed loop-edge $AA \in \boldsymbol{l}$, then the node A is type* 0.

Proof. If there is a dashed loop edge at node A, then $A \supset \neg A$, and it means that A must be a liar. $\square$

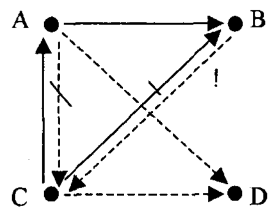

Figure 2: The graph of the Example 1.1 after some arrow-adding steps

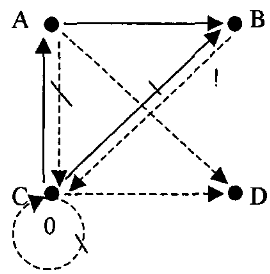

Figure 3: The graph after evaluating basic scheme at C

Let us see our example. We get basic scheme at C (Fig. 3). Using the scheme first we write the sign 0 to the node (it is a known type), and after this we examine the arrows, which remain relevant. It is very important part of this local method. In the next part we examine the valuable arrows, and the arrows, which become irrelevant. So if we know the type of the node of an end of an edge, what do we know about the other end?

Lemma 3.4. *Let $A, C \in N$ such a way that A is known type (let T and L be the known nodes with type 1 and type 0, respectively) as in the first column of the table and the noted edge between them is in P. The noted edge is a valuable arrow in the SW-puzzle if it is one of the following:*

	the case of valuable arrow	*after valuation we have these information*
a)	$A \in T$, $AC \in \mathbf{t}$	*C is type 1 also*
b)	$A \in T$, (AC *or* $CA \in \mathbf{l}$)	*C is type 0*
c)	$A \in L$, $\underline{AC} \in \mathbf{t}$	*C is type 0*
d)	$A \in T$, $\underline{CA} \in \mathbf{t}$	*C is type 1*
e)	$A \in L$, $CA \in \mathbf{t}$	*C is type 0*
f)	$A \in L$, ($\underline{AC}$ *or* $\underline{CA} \in \mathbf{l}$)$^{\spadesuit}$	*C is type 1*

$\spadesuit$ *this arrow must be unique relevant at least from one direction to use this step*

Proof. It is evident from logical meaning of the arrows. □

At this point we detail the arrow-deletions and arrows changing to irrelevant steps. First of all, we note that the point d) and some special case of point c) in the previous lemma are this kind of steps also, as we will show in the next lemma.

Lemma 3.5.

a) Let A, B and C be three nodes. If $AB \in \boldsymbol{l}$ and $\underline{BC}$ or $\underline{CB} \in \boldsymbol{l}$ and the edge $AC \in \boldsymbol{t}$ is already in the graph such that both arrows from A (the dashed edge to B and the solid one to C) are relevant then we cross out one of them.

b) Our graph will be equivalent to the previous one in the following case also: let A, B and C be nodes such a way that $AB \in \boldsymbol{t}$, ($\underline{BC}$ or $\underline{CB} \in \boldsymbol{l}$) and $AC \in \boldsymbol{l}$ where both arrows from A are relevant. In this case we cross out one of them from A.

Proof. The connection between B and C means that B and C are different types. Hence if an arrow above is relevant in the solution then the other is relevant also. □

Now we will show the other cases when we cross out or delete an arrow.

Lemma 3.6. *Each arrow which starts from a type* 1 *node will be irrelevant.*

Proof. Trivially, it is from the definition of relevant edge. □

The following two lemmas are about that when we delete edges. We delete only edges, which have a known type end. If we cannot use an edge to get more information then we delete it.

Lemma 3.7. *Let T and L be the set of known type nodes (T is the nodes type 1, and L is the nodes type 0). Let A be a node, whose type is unknown at this time. If not only one relevant arrow started from the node A, but there is an arrow which goes to a known type node C, like*

a) $AC \in \boldsymbol{t}$, $C \in T$, and/or

b) $AC \in \boldsymbol{l}$, $C \in L$,

then this arrow will be deleted.

Proof. From the logical meaning of the arrows from A, we can use the $(*)$ formula. We have a conjunction in left hand side, these edges means values 1 in this conjunction. We can delete these values if it is not alone in this side. But there is other relevant edge from A, therefore we delete these arrows. □

Lemma 3.8. *If we know about the node A that it is type* 0 *($A \in L$), and there is a relevant arrow from A to a node C, like*

a) $AC \in \boldsymbol{t}$, $A, C \in L$, or

b) $AC \in \boldsymbol{l}$, $A \in L$ and $C \in T$

then we delete all other arrows starting from A.

Proof. We can use $(*)$. This formula must be valid independently the values of the other atomic sentences in the left hand side. □

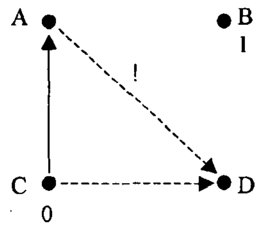

Figure 4: The graph after deleting edges

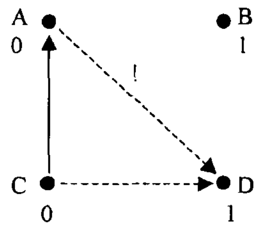

Figure 5: The solution of Example 1.1 on the graph

After using basic schemes we can use the point b) of lemma 3.7. We delete and cross the edges, which were relevant until this step.

Now we continue the solution of our example.

We can evaluate the value of B by step f) of lemma 3.4, hence B is type 1. We delete the edges AB, CB, BC, AC and CC (lemma 3.7). And we can sign the arrow AD by ! (Fig. 4).

We can use the step lemma 3.5. to cross out an edge from C. But after this we can ! sign the other arrow from C hence we can use valuable-arrow steps and get the value 0 for A and 1 for D. Therefore the result is: Alice and Charlie are liars, Bob and David are truth-tellers (Fig. 5).

In general case it is possible that we need the node-union step. We can use node union step in the following situation:

Lemma 3.9. *If there are two nodes A and B, such that $\underline{AB} \in \mathbf{t}$ (the unique relevant arrow from A goes to B, and it is solid), then we unite these nodes. The united node has label "A, B" and we have all edges at this node which were into/from A and B but the $\underline{AB} \in \mathbf{t}$ edge. And if there was relevant BA also, then after the node-union we have a relevant loop arrow at this node. The new graph is equivalent to the original one in the following sense. In the solution of the previous graph the nodes A and B have the same value as in the solution of the new graph the united node with label "A, B"; and all other nodes have the same value, respectively.*

Proof. From the logical meaning of the edge $\underline{AB} \in \mathrm{t}$ we know, that $A \equiv B$. So all edge, which had endpoint A or B must be valid in the new graph. There were not more relevant edges from A, and we have all relevant edges from B. ☐

Sometimes we have information that two nodes are the same type but we cannot use node-union. Therefore we need the concept of parity of nodes, what we can use usually after the arrow-adding steps.

Definition 3.10. *Two nodes are in* parity, *if they are connected by solid arrows by both directions. We will use the notation $A \Leftrightarrow B$ to show that there are solid arrows between them in both way.*

(And we may use, that each node in parity with itself, because we can add solid loop arrow for each node, $A \supset A$ must be valid.)

Lemma 3.11. *The nodes in parity have same value.*

Proof. Assume, that A and B are in parity. Then from logical meanings of the solid arrows: $A \supset B$, $B \supset A$: $A \equiv B$. □

Lemma 3.12. *If from a node there are more same-type (solid or dashed) relevant edges going to the nodes which are in parity, then we can keep only one of them relevant, and we cross the others.*

Proof. The nodes in parity are same type, so all these relevant edges mean true atomic statements, or all of them mean false atomic statements. So it is equivalent to only one independent statement. Easy to show, that the new graph is equivalent to the previous one. □

Remark 3.13. *If from a node A there are more than 1 same-type edge going to the node B, then we leave only one of them. If there was relevant one among them, then we keep a relevant one, and delete the others.*

Remark 3.14. *If an edge is relevant in the solution, then it must be relevant also in the original graph of the puzzle.*

4 The global properties of the possible puzzle-graphs

Now, before we examine how we can use the global information of a puzzle-graph, we make some statements about the possible graphs.

Lemma 4.1. *There is no good and clear SW-type puzzle only with solid arrows.*

Proof. It has at least two different solutions: everybody is truth-teller; or each person is a liar. □

Lemma 4.2. *There is no good and clear SW-type puzzle, whose solution is that each person is a truth-teller.*

Proof. In the graph of this puzzle there are only solid arrows. So according to the previous lemma, our statement is true. □

Lemma 4.3. *If the graph of a clear and good puzzle has two or more components, then this puzzle falls apart: we have two or more less clear and good puzzle.*

Proof. In a clear puzzle in a component there is no information about the nodes in other components. □

According to the previous lemma, we assume that our graph has only one component, or we can solve the less one-component's puzzles.

The following lemma plays important rule when we use global information of a graph.

Lemma 4.4. *There is no dashed edge between two type* 1 *nodes in the solution.*

Proof. If a node is type 1, then all dashed arrows from it must go to type 0 nodes. □

Lemma 4.5. *We know from Lemma 4.1 that there is a dashed edge in the puzzle, but from the previous lemma we know that this edge cannot be between type* 1 *nodes. So there must be a type* 0 *node in an end of each dashed edge.*

Lemma 4.6. *Let T be the set of the truth-tellers in the solution. Then there is no solid arrow from this set which goes outside T.*

Proof. If a solid arrow starting from a type 1 node goes to a node A, then A must be type 1 also. □

Lemma 4.7. *If there is a directed circle built by solid arrows, then all nodes in this circle are in parity.*

Proof. Easy by using arrow adding steps b). □

Lemma 4.8. *Parity is an equivalence relation among nodes.*

Proof. Each node is in parity with itself, according to the note after the Definition 3.10. The symmetry come from the definition. It is transitive (if $A \Leftrightarrow B$, $B \Leftrightarrow C$ then $A \Leftrightarrow C$) because we can use step b) of Lemma 3.2. □

Lemma 4.9. *Let P be the graph of a good, clear SW-puzzle, and T be the set of type* 1 *nodes in the solution. If it is a node B, who is liar in the solution of P and he remained silent, then we can use arrow-adding steps for a new dashed edge from B to a truth-teller, or we can use node-union step to join B to an another liar.*

Proof. We assume that the graph is connected. If there is dashed edge from T to B, then we can use arrow-adding step a), and we get a new dashed edge from B to a truth-teller. If there is no edges between T and B originally, then must be an arrow from a liar to B. Let L is the set of nodes, which are not in T and differ from B. In this case originally there is not edge between T and B. (From B does not start any, and from T to B there is no solid arrow (Lemma 4.6.), and we assumed that there is nor dashed edge.) But the P was connected, so there must be edges from L to B. If there was a solid arrow from L to B, which is uniquely relevant from where it starts then we must use node-union step. And in final case there is no uniquely relevant arrow to B, which means that all non-silent liars have other arrows meaning his lie. But in this case it is also a solution, when $T \cup \{B\}$ is the sets of truth-tellers, and L is the set of liars. So in this case we get contradiction. □

Lemma 4.10. *Let P be the graph of a good, clear SW-puzzle, and T be the set of type 1 nodes in the solution. After all usual arrow-adding and node-union steps for P there is not possible only one node, which is not in T, and not connected with an element of T by dashed edge.*

Proof. We can assume, that P is connected, and we have no usual arrow-adding or node-union steps. From Lemma 4.2. we know, that there must be a node outside of T. Now we have two possibilities: B said something, or he remained silent. If he said something, and in the solution he is a liar, then must start a relevant arrow from B. If he said about a truth-teller C, that C is a liar, then this edge is dashed between B and T. If he said about a liar C that, C is truth-teller, then – because of all liar, but B are connected with T by dashed edge – the arrow-adding step c) (Lemma 3.2) is useful, and we get a dashed edge between B and T. In the case when B was silent, we can use the previous Lemma 4.9 for using node-union step. □

Lemma 4.11. *If in the good and clear puzzle's solution everybody is a liar, and the graph of the puzzle is connected, then after the possible arrow-adding and node-union steps we get a puzzle with only one node with two kind of loop edges.*

Proof. Easy to show, that for one node it is the unique puzzle. We will show that if we have more nodes then we can use node-union steps (and we get smaller and smaller puzzle with same solution).

If there was a node without starting relevant edges, then we can use node-union step by using Lemma 4.9. Now, we assume that we already used all possible arrow-adding steps. It is evident, that all relevant edges in the solution are solid, because each person is a liar. If from a node there is only one relevant arrow, then we must use node-union step. In other case from each node must start at least two relevant arrows. Let A be a node. Let T_A be the set of nodes, which we can reach from A by directed solid arrows. (By using arrow-adding steps it is evident, that we have a direct arrow from A to each element of T_A.) The set T_A is finite, let $Y_1, Y_2, \ldots Y_k$ the subsets of T_A, such that all nodes in a Y_i are in parity. Then we can use Lemma 3.12, so in each Y_i there at most only one relevant edge from each node is inside of Y_i. If there is a node, for which only one relevant edge remains, then we can use node-union. If such a node does not exist, then an other relevant arrow starting from all node in Y_i to outside of Y_i. So there is at least one set Y_j, which differs from Y_i, and there is relevant arrow from Y_i to Y_j. But there is the same situation with Y_j. So if we cannot use a node-union step inside of Y_j then a relevant edge must go to another Y_k. But we have only finite number of set Y_n. So we must have a circle by using directed solid arrows among the sets Y inside in T_A. But it means, that two or more sets are in parity. It is a contradiction.

So it is not possible that we cannot use node-union step, if we have at least two nodes. □

The following theorem is a summary of the previous lemmas. It shows the global information of the graph, what we can use in the next section.

Theorem 4.12. *Let P be the graph of a good, clear SW-puzzle, and T be the set of type 1 nodes in the solution. After all usual arrow-adding and node-union steps put L be the set of the nodes which are connected to T by a dashed edge. If P is connected then there is no node in P which is not in $T \cup L$.*

Proof. Let S be the set of the nodes, which are nor in T, neither in L. We will show that S will be empty set. From Lemma 4.10. we know, that it is impossible that S has only one element. Let us see how the set S connected to the other sets. According to Lemma 4.6. and the definition of set S from T there is no arrow to S. And there is no relevant edge from S to T (the solid arrows are not relevant, and there is no dashed edge between S and T). So from the nodes in S all relevant edges go to liars. If there is a relevant arrow from S to L, it must be solid, therefore we can use arrow adding step c), and we have a dashed edge between T and S, which contradicts the definition of S. So all relevant edges from S are in inside of S. But if there is a node A in set S from which there are not at least two relevant edges, then we can use node-union step (which is contradict to our assumption, that we already used these steps). So we are in the same situation as the proof of Lemma 4.11. As we state there, because these sets are finite, we have contradiction. So S must be the empty set. □

5 The general solving method

We know everything which we need to solve puzzles with the graph method.

Now we describe our method:

Algorithm 5.1.

0. Let B_i be the nodes of the graph. Draw the initial graph of the puzzle using only relevant edges.

Part I. (Graph-changing, by using local information) We try to use the following steps.

1. Use all possible node-union steps. (Lemma 3.9.)
2. Use all possible arrow-adding steps. (Lemma 3.2.)
3. Cross as many arrow possible. (Lemmas 3.5, 3.12, and 3.13)

If these steps cannot be repeated any more, then we continue by Part II.

Part II. (Choosing the set of truth-tellers, T, by using global information)

4. Make the matrix of the subgraph of dashed edges. (Use only the edges of l.)
5. Choose a maximal set of nodes T, which are not connected with dashed edges.

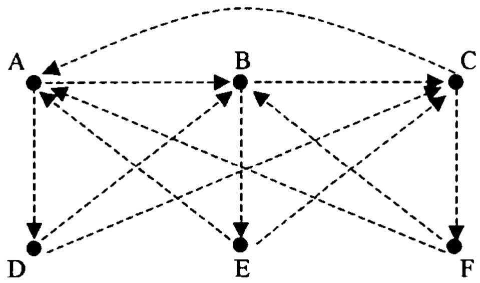

Figure 6: Graph of the Example 5.5

6. Check the following property: if there is a relevant arrow from each node which is not in T and is not silent originally, like dashed arrow to inside T, or solid arrow to outside T, then we have the solution.

If the property in step 6 is not true, then we choose another set T in step 5.
The solution is: all persons in T are truth-teller, and the others are liars.

Theorem 5.1. (Completeness and soundness of the algorithm) *Let P be the graph of a connected, good and clear SW-type puzzle. We can solve P by using Algorithm 5.1.*

Proof. It is clear, that the Part I. of the algorithm stops, because P is finite. It is evident, that we have only finite possibility to choose the set T. Let us assume, that we finished Part I. Let T' be the set of truth-tellers in the solution. T' is maximal because of Theorem 4.12, so we can choose T' as set T. We show that the property in point 6 must be true for this unique solution. Indirectly, assume that there is an – originally not silent – node B not in T for which we have nor solid relevant arrow to outside T neither dashed relevant edge to inside T. Then B did not lie originally, but he said something, hence he must be a truth-teller. But we have a dashed edge between the truth-tellers and B. It contradicts to Lemma 4.4. □

Remark 5.2. *In the case when each person is a liar our T set is empty. In this case according to Lemma 4.11 we have a puzzle with only one node after the steps of Part I.*

Remark 5.3. *Our algorithm detect if a puzzle has not any solution.*

And now we show an interesting example. In Lemma 4.1 we can see, that there is no good and clear SW-type puzzle only with solid arrows, now we show an example of a good and clear SW-type puzzle with only dashed edges. It is very nice symmetric example.

Example 5.4. A: B and D are liars. B: C and E are liars. C: A and F are liars. D: B and C are liars. E: A and C are liars. F: A and B are liars.

	A	B	C	D	E	F
A		x	x	x	x	x
B	x		x	x	x	x
C	x	x		x	x	x
D	x	x	x			
E	x	x	x			
F	x	x	x			

Table 1: The matrix of the graph

We can see that we have no local graph-step to use, so we cannot solve this puzzle without global information. There are two arrows starting from each nodes.

Now we solve this puzzle: As we can see, that we cannot use any steps of Part I of our algorithm. So we use Part II.

Let us make the matrix of the graph, which shows if two nodes are directly connected by a dashed edge. We use step 4.

The matrix of the graph in the table.

Our maximal T sets are the following: $\{A\}$, $\{B\}$, $\{C\}$, $\{D, E, F\}$. Easy to show in the original graph, that the condition of point 6 is not true for the first 3 sets. Our solution is D, E and F are truth-tellers, A, B and C are liars.

6 Summary

In this paper we defined and examined the SW-type of truth-tellers and liars puzzles. We represented these puzzles with graphs, which are very useful to examine and solve these puzzles. We examine what the edges of the graphs mean logically. The graph of a puzzle has all information about the puzzle in case of clear puzzle. We took some interesting statement about the possible structure of the puzzles. We used some local information steps in a graph as valuable arrows, arrow-adding, node-union steps and basic schemes. We showed that there is no clear and good SW-type puzzle with only solid arrows. Later on we presented a special example, when we have only dashed arrows. Finally we showed a graph-algorithm, which based on both local and global information of the graph, and it can solve the clear and good SW-type puzzles. The advantage of this method is to avoid case separations, which occurs for instance in tableaux method and requires great care for programmers. Using our method we need memory only size of n^2 for a puzzle with n persons to store our graph.

Using this approach from graph theory we can solve the puzzles in a new thinking way. Our theory connects the special type of satisfiability problems to graph theoretical problems.

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