Infinite limits and R-recursive functions

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Abstract

In this paper we use infinite limits to define R-recursive functions. We prove that the class of R-recursive functions is closed under this operation.

Keywords: Theory of computation, Real recursive function.

1 Introduction

The theory of recursion had been originally formulated for enumerable domains [3, 9]. Later the extensions on the continuous domains were proposed (for example see [4]). During a few past years many authors have studied problems of recursion theory for reals [1, 2].

The new approach was given by Moore in [5]. He used not only continuous functions, but also continuous operators on real recursive functions. The set of R-recursive functions defined by Moore is the subclass of real functions constructed as the smallest set containing $0, 1$ and closed under operations of composition, differential recursion and μ-recursion.

Infinite limits are the natural operation on real functions. and can be viewed as a method to define new functions. It is mentioned in [5] that limits can be expressed in terms of μ-operation, but without giving the way of this 'translation'. In this paper we give a proper way to define limits by μ-recursion.

This result can be useful for a few reasons. First, infinite limits are natural operations in calculus in contrast to the μ-operation. Furthermore with infinite limits we can define a limit hierarchy and relate it to the μ-hierarchy. This would be a continuous analog of Shoenfield's theorem [8]. Infinite limits can also be useful to compare the μ-hierarchy with the levels of Rubel's [7] Extended Analog Computer.

2 Preliminaries

This section summarizes some notions and results taken from [5], which are useful in this paper. Let us start with the precise definition of an R-recursive function.

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Definition 2.1. *A function $h : R^m \to R^n$ is R-recursive if it can be generated from the constants 0 and 1 with the following operators:*

1. *composition:* $h(\bar{x}) = f(g(\bar{x}))$;
2. *differential recursion:* $h(\bar{x}, 0) = f(\bar{x})$, $\partial_y h(\bar{x}, y) = g(\bar{x}, y, h(\bar{x}, y))$ *(an equivalent formulation can be given by integrals:* $h(\bar{x}, y) = f(\bar{x}) + \int_0^y g(\bar{x}, y', h(\bar{x}, y'))dy'$*)*;
3. *μ-recursion* $h(\bar{x}) = \mu_y f(\bar{x}, y) = inf\{y : f(\bar{x}, y) = 0\}$, *where infimum chooses the number y with the smallest absolute value and for two y with the same absolute value the negative one.*

Several comments are needed to the above definition. A solution of a differential equation need not to be unique or can diverge. Hence, we assume that if h is defined by differential recursion then h is defined only where a finite and unique solution exists. This is why the set of R-recursive functions includes also partial functions. For coherence with Moore's paper[5] we use the name of R-recursive functions in the article, however we should remember that in reality we have partiality here (partial R-recursive functions).

The second problem arises with the operation of infimum. Let us observe that if an infinite number of zeros accumulate just above some positive y or just below some negative y then the infimum operation returns that y even if y itself is not a zero.

The above defintion creates the class of R-recursive functions with some interesting features. Let us cite a few results from [5].

Lemma 2.2. *The functions* $-x, x + y, xy, x/y, e^x, \ln x, x^y, \sin x, \cos x$ *and the projection functions* $I_n^i(x_1, \dots, x_n) = x_i$ *are R-recursive.*

The power of the system of R-recursive functions can be viewed from the following lemma, which is sufficient to solve the classical halting problem.

Lemma 2.3. *The function χ_S such that* $\chi_S(x) = \begin{cases} 1 & x \in S, \\ 0 & x \notin S, \end{cases}$ *is R-recursive for any partial N-recursive set S (S is partial N-recursive if $S = f_S(N)$ for $f_S : N \to N$, f_S is some N-recursive function).*

It is possible to define for every R-recursive function $f : R^n \times R \to R$ the characteristic function $\eta_y f$ for the set of $\bar{x}$ on which $\mu_y f$ is well-defined. Precisely this fact is stated by the below theorem:

Theorem 2.4. *If $f(\bar{x}, y)$ is R-recursive, then $\eta_y f(\bar{x}, y)$ is also R-recursive, where*

$$\eta_y f(\bar{x}, y) = \begin{cases} 1 & \exists y f(\bar{x}, y) = 0 \\ 0 & \forall y f(\bar{x}, y) \neq 0 \end{cases}$$

The operator μ is a key operator in generating the R-recursive functions. This fact suggests creating a μ-hierarchy, which is built with respect to the number of uses of μ in the definition of a given f.

Definition 2.5. *For a given R-recursive expression $s(\bar{x})$, let $M_{x_i}(s)$ (the μ-number with respect to x_i) be defined as follows:*

$$M_x(0) = M_x(1) = M_x(-1) = 0,$$

$$M_x(f(g_1, g_2, \ldots)) = \max_j (M_{x_j}(f) + M_x(g_j)),$$

$$M_x(h = f + \int_0^y g(\bar{x}, y', h)dy') = \max(M_x(f), M_x(g), M_h(g)),$$

$$M_y(h = f + \int_0^y g(\bar{x}, y', h)dy') = \max(M_{y'}(g), M_h(g)),$$

$$M_x(\mu_y f(\bar{x}, y)) = \max(M_x(f), M_y(f)) + 1,$$

where x can by any $x_1, \ldots, x_n$ for $\bar{x} = (x_1, \ldots, x_n)$.

For an R-recursive function f, let $M(f) = \max_i M_{x_i}(s)$ minimized over all expressions s that define f. Now we are ready to define μ-hierarchy.

Definition 2.6. *The μ-hierarchy is a family of $M_j = \{f : M(f) \leq j\}$.*

Let us add that if f is in M_j then $\eta_y f$ is in M_{j+2}.

As it was mentioned we focus our interest on functions defined by the infinite limits.

Definition 2.7. *Let $g : R^n \times R \to R$, then we can say the function $f : R^n \to R$ is defined by an infinite limit from g if:*

$$f(\bar{x}) = \begin{cases} \lim_{y\to\infty} g(\bar{x}, y) & \lim_{y\to\infty} g(\bar{x}, y) \textit{ exists,} \\ \textit{undefined} & \textit{otherwise.} \end{cases}$$

3 Auxiliary results

In this section we give a few results which will be useful in the proof of the main theorem. We start with a slight modification of the definition of R-recursive functions.

Lemma 3.1. *Let us consider the set of functions generated from $0, 1, -1$ by the operations 1,2 from the Definition 2.1 and by absolute μ-recursion $\mu_y^A f(\bar{x}, y) = inf\{|y| : f(\bar{x}, y) = 0\}$. Then this set of functions is equal to the set of all R-recursive functions.*

Proof. Because -1 can be simply defined in the set of R-recursive functions, it is sufficient to prove that in definitions operation μ can be replaced by μ^A and vice versa.

As the first case we consider the method of replacing μ^A by μ. Let $h(\bar{x}) = \mu_y^A f(\bar{x}, y)$. Let us set $f'(\bar{x}, y) = f(\bar{x}, y)f(\bar{x}, -y)$. Clearly $h(x) = -\mu_y f'(\bar{x}, y)$.

Because $-x$ and multiplication are R-recursive, so h defined as above is R-recursive too.

Now we must show that function $h(\bar{x}) = \mu_y f(\bar{x}, y)$ can be defined by μ^A. Let us point out the fact that $|x| = \mu_y^A(x - y)$. Then we will use $f^+(\bar{x}, y) = f(\bar{x}, |y|), f^-(\bar{x}, y) = f(\bar{x}, -|y|)$ to define

$$h^+(\bar{x}) = \mu_y^A f^+(\bar{x}, y), \quad h^-(\bar{x}) = \mu_y^A f^-(\bar{x}, y).$$

It is simple to observe that h^+ gives as the result the smallest nonnegative zero of f and h^- the absolute value of the greatest nonpositive (the smallest with respect to absolute value) zero of f.

We will choose as $h(\bar{x})$ the proper value from $h^+(\bar{x}), -h^-(\bar{x})$. We can define $K_=(w, y) = \delta(w - y)$, where $\delta(x) = 1 - \mu_y^A(x^2 + y^2)(y - 1)$. From definition $K_=(w, y)$ is equal to 1 if $w = y$, 0 if $w \neq y$. Then the function $\Theta(w) = 1$ for $x \geq 0$, 0 otherwise, can be defined as $K_=(w, |w|)$.

We can conclude the proof by the following observation

$$h(\bar{x}) = \begin{cases} -h^-(\bar{x}) & h^+(\bar{x}) \geq h^-(\bar{x}), \\ h^+(\bar{x}) & h^+(\bar{x}) < h^-(\bar{x}). \end{cases}$$

Hence $h(\bar{x}) = -h^-(\bar{x})\Theta(h^+(\bar{x}) - h^-(\bar{x})) + h^+(\bar{x})(1 - \Theta(h^+(\bar{x}) - h^-(\bar{x})))$ and since $+$ is defined without μ and all remaining functions in the last equation are defined only by μ^A, so $h(\bar{x})$ can be defined by μ^A instead of μ. □

It is interesting to define the μ^A-hierarchy of R-recursive functions as the analog to μ-hierarchy.

Definition 3.2. *For a given R-recursive expression $s(\bar{x})$, let $M_{x_i}^A(s)$ (the μ^A-number with respect to x_i) be defined as follows:*

$$M_x^A(0) = M_x^A(1) = M_x^A(-1) = 0,$$

$$M_x^A(f(g_1, g_2, \ldots)) = \max_j(M_{x_j}^A(f) + M_x^A(g_j)),$$

$$M_x^A(h = f + \int_0^y g(\bar{x}, y', h)dy') = \max(M_x^A(f), M_x^A(g), M_h^A(g)),$$

$$M_y^A(h = f + \int_0^y g(\bar{x}, y', h)dy') = \max(M_{y'}^A(g), M_h^A(g)),$$

$$M_x^A(\mu_y^A f(\bar{x}, y)) = \max(M_x^A(f), M_y^A(f)) + 1,$$

where x can by any $x_1, \ldots, x_n$ for $\bar{x} = (x_1, \ldots, x_n)$.

For an R-recursive function f, let $M^A(f) = \max_i M_{x_i}^A(s)$ minimized over all expressions s that define f and $M_j^A = \{f : M^A(f) \leq j\}$.

Now we can add a corollary from the previous lemma.

Corollary 3.3. *In the above lemma we use only one μ instead of μ^A, when we change the definition of function with μ^A by the definition with μ. Hence if some function f is from M_k^A then $f \in M_k$.*

The reverse relation is more complicated. In the definition of h given by $h(\bar{x}) = -h^-(\bar{x})\Theta(h^+(\bar{x}) - h^-(\bar{x})) + h^+(\bar{x})(1 - \Theta(h^+(\bar{x}) - h^-(\bar{x})))$ the operation μ^A is used 3 times. So each function from M_j belongs to M_{3j}^A.

Lemma 3.4. *Let $g : R^{n+1} \to R$ be an R-recursive function. Then there are R-recursive functions $G : R^{n+1} \to R$, $S : R \to R$ such that*

$$\inf_y g(\bar{x}, y) = S(\mu_w^A G(\bar{x}, w)).$$

Proof. We can distinguish three cases in the proof:

1. $(\forall \bar{x}, y) g(\bar{x}, y) \geq 0$. Then we can write

   $$\inf_y g(\bar{x}, y) = \inf_w \{|w| : G'(\bar{x}, w) = 0\},$$

   where

   $$G'(\bar{x}, w) = \begin{cases} 0 & (\exists y) g(\bar{x}, y) - w = 0, \\ 1 & \text{otherwise.} \end{cases}$$

   The condition $(\exists y) g(\bar{x}, y) - w = 0$ is equivalent to the fact that $\mu_y(g(\bar{x}, y) - w)$ is defined. But the last statement can easily be checked by the function η. Finally we have

   $$G'(\bar{x}, y) = 1 - \eta_y(g(\bar{x}, y) - w) \text{ and } \inf_y g(\bar{x}, y) = \mu_w^A G'(\bar{x}, w),$$

   so in this case $S = I_1^1, G = G'$.

2. $(\forall \bar{x}, y) g(\bar{x}, y) < 0$. In this case we can observe that $\inf_y g(\bar{x}, y)$ is a negative number and the infimum 'searches' in the direction of the smallest negative numbers, whereas μ^A gives us a positive result and its 'search' is oriented to zero. That is why we must use some transformation in a construction of the proper result. The simplest way to change 'the orientation' of the infimum is the expression $\frac{1}{g(\bar{x}, y)}$. Let our expression be equal to

   $$G''(\bar{x}, w) = \begin{cases} 0 & (\exists y) \frac{1}{g(\bar{x}, y)} - w = 0, \\ 1 & \text{otherwise.} \end{cases}$$

   then $G''(\bar{x}, w)$ is zero iff $\frac{1}{w}$ is a value of $g(\bar{x}, y)$ for some y. Hence an infimum on g is equal to:

   $$\frac{-1}{\inf_w \{|w| : G''(\bar{x}, w) = 0\}}$$

   and (like in the previous step) we eliminate the quantifier:

$$G''(\bar{x}, w) = 1 - \eta_y(\frac{1}{g(\bar{x}, y)} - w).$$

We can write now that

$$\inf_y g(\bar{x}, y) = S(\mu_w^A G(\bar{x}, w)), \quad S(z) = -1/z, G = G''.$$

3. Now we are prepared to consider the general case, where $g : R^{n+1} \to R$ is an arbitrary function. Let us observe, that if there is one point y_0^- such that $g(\bar{x}, y_0^-) < 0$ then $\inf_y g(\bar{x}, y)$ must be negative and if such point y_0^- does not exist then the first case of our proof solves the problem.
To check an existance of the point y_0^- it is sufficient to use the condition $\eta_y K_<(g(\bar{x}, y), 0) = 1$ where $K_<(z, y) = 0 \iff z < y$. Then we can find y_0^-. We will use the following method (we must remember that μ_A gives us the absolute value of the proper solution)

$$y_0^- = \begin{cases} \mu_y^A K_<(g(\bar{x}, y), 0) & g(\bar{x}, \mu_y^A K_<(g(\bar{x}, y), 0)) \leq 0, \\ -\mu_y^A K_<(g(\bar{x}, y), 0) & \text{otherwise.} \end{cases}$$

Then we can define the function:

$$g^-(\bar{x}, y) = \begin{cases} g(\bar{x}, y) & g(\bar{x}, y) < 0, \\ g(\bar{x}, y_0^-) & \text{otherwise.} \end{cases}$$

This function for a given $\bar{x}$ has the same infimum as g, but its values are always negative.
As a summary of the previous considerations we give the conditional definition of $q(\bar{x})$, where $q(\bar{x})$ denotes the expression $\inf_y g(\bar{x}, y)$:

$$q(\bar{x}) = \begin{cases} \inf_w\{|w| : 1 - \eta_y[g(\bar{x}, y) - w] = 0\} & \text{if } (\forall y) g(\bar{x}, y) > 0, \\ \frac{-1}{\inf_w\{|w|:1-\eta_y[\frac{1}{g(\bar{x},y)} - w]=0\}} & \text{if } (\forall y) g(\bar{x}, y) < 0, \\ \frac{-1}{\inf_w\{|w|:1-\eta_y[\frac{1}{g^-(\bar{x},y)} - w]=0\}} & \text{otherwise.} \end{cases}$$

Let us add that the condition $(\forall y) g(\bar{x}, y) > 0$ is equivalent to the statement y_0^- *does not exist*, but this last phrase can be expressed by $\eta_y K_<(g(\bar{x}, y), 0) = 0$. The similar translation of $(\forall y) g(\bar{x}, y) < 0$ is: $\eta_y K_<(0, g(\bar{x}, y)) = 0$.

It is obvious that such $q(x)$ is R-recursive ($\inf_w\{|w| : \ldots\}$ can be replaced by μ_A). The final forms of functions S and G can be obtained from the above definition of q. □

Remark 3.5. *Let us observe that in the general case (the third point of the above proof) we used for the construction of the definition of q the function η, infimum and y_0^-, which gives the number of used μ^A operations equal to 5. This is the maximal number of μ^A operations for all cases. Hence for $g \in M_j$ we have $q \in M_{j+5}$.*

We also need a similiar result, but with restricted infimum.

Lemma 3.6. *Let $g : R^{n+1} \to R$ be an R-recursive function. Then there are R-recursive functions $G : R^{n+2} \to R$, $S : R \to R$ such that for all $z \in R$*

$$\inf_{y \in (z,\infty)} g(\bar{x}, y) = S(\mu_w^A G(\bar{x}, w, z)).$$

Proof. Let us consider the set $S_z^{g,\bar{x}}$ such that

$$S_z^{g,\bar{x}} = \{w : (\exists y \geq z) g(\bar{x}, y) = w\}.$$

We will use the characteristic function of this set: $\chi_z^{g,\bar{x}}(w) = 0 \iff w \in S_z^{g,\bar{x}}$ $\chi_z^{g,\bar{x}}(w) = 1 \iff w \notin S_z^{g,\bar{x}}$. From the Lemma 3.4 we have S_g, G_g for a given g and the problem of unrestricted infimum. It is clear that

$$\inf_{y \in (z,\infty)} g(\bar{x}, y) = S_g(\mu_w^A(|G_g(\bar{x}, w)| + |\chi_z^{g,\bar{x}}(w)|)).$$

Now we should prove only that $\chi_z^{g,\bar{x}}(w)$ is an R-recursive function. But $\chi_z^{g,\bar{x}}(w)$ can be written in the form

$$\chi_z^{g,\bar{x}}(w) = 0 \iff (\exists y)[(g(\bar{x}, y) = w) \wedge (y \geq z)],$$

$$\chi_z^{g,\bar{x}}(w) = 1 \iff (\forall y)[(g(\bar{x}, y) \neq w) \vee (y < z)].$$

These last equations define $\chi_z^{g,\bar{x}}(w)$ as $1 - \eta_y(|g(\bar{x}, y)| + K_{\geq}(y, z))$, which ends this proof. □

Remark 3.7. *Because $\chi_z^{g,\bar{x}}(w)$ is defined by means of η and $K_{\geq}$, so it uses 3 μ^A operations but unrestricted infimum uses μ_A five times. Hence if $g \in M_j$ then the* $\inf_{y \in (z,\infty)} g(\bar{x}, y)$ *belongs to* M_{j+5}.

4 Main theorem

In this section we prove that the class of R-recursive functions is closed under the operation of defininig functions by infinite limits.

Theorem 4.1. *Let $F : R^{n+1} \to R$ be an R-recursive function. Let us define $f : R^n \to R$ in the following way $f(\bar{x}) = \lim_{y \to \infty} F(\bar{x}, y)$. Then there exist such R-recursive functions $G : R^{n+1} \to R$, $S : R \to R$ that*

$$f(\bar{x}) = S(\mu_w^A G(\bar{x}, w)).$$

Proof. Let us consider the function $f(\bar{x})$ defined as above. The function $f(\bar{x})$ is defined in the point $\bar{x}$ if there exist limits

$$\liminf_{y \to \infty} F(\bar{x}, y), \ \limsup_{y \to \infty} F(\bar{x}, y)$$

and they are equal to each other in this point.

Let us recall the definitions:

$$\liminf_{y\to\infty} F(\bar{x},y) = \sup_z \inf_{y>z} F(\bar{x},y), \quad \limsup_{y\to\infty} F(\bar{x},y) = \inf_z \sup_{y>z} F(\bar{x},y).$$

To check that the function f is defined in $\bar{x}$ first we must check the conditions: $\sup_z \inf_{y>z} F(\bar{x},y)$, $\inf_z \sup_{y>z} F(\bar{x},y)$ are defined for $\bar{x}$. So it will be helpful if we prove that there exist functions K^i, K^s such that

$$K^i(\bar{x}) = \begin{cases} 1 & \sup_z \inf_{y>z} F(\bar{x},y) \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

and K^s is analogously defined for $\inf_z \sup_{y>z} F(\bar{x},y)$.

It is easy to see that we can replace the expression 'sup F' by the expression '$-\inf(-F)$' in the above equations. So we can apply Lemmas 3.4 and 3.6, which means, that there are R-recursive functions S^s, G^s, S^i, G^i such that

$$\sup_z \inf_{y>z} F(\bar{x},y) = S^i(\mu_w^A G^i(\bar{x},w)),$$

$$\inf_z \sup_{y>z} F(\bar{x},y) = S^s(\mu_w^A G^s(\bar{x},w)).$$

The left sides in the last two lines are defined iff there exist such w_i, w_s, that $G^i(\bar{x},w_i) = G^s(\bar{x},w_s) = 0$. This condition can be checked by the R-recursive functions $\eta_w G^i(\bar{x},w), \eta_w G^s(\bar{x},w)$. The above considerations imply that K^i, K^s exist and they are R-recursive.

Now to end the proof it is sufficient to define:

$$f(\bar{x}) =$$

$$= \begin{cases} S^i(\mu_w^A G^i(\bar{x},w)) & K^i(\bar{x})K^s(\bar{x})K_=(S^i(\mu_w^A G^i(\bar{x},w)), S^s(\mu_w^A G^s(\bar{x},w))) = 1 \\ \text{undefined} & \text{otherwise.} \end{cases}$$

This definition of f by R-recursive functions is in the obvious way equivalent to the definition by the operation of infinite limit. □

Let us point out that in the above proof we use two operations of infimum for lim inf: the outer (unrestricted) infimum, which is obtained from the transformed supremum and the second - inner (restricted) infimum. We have the analogous construction for lim sup. Hence and from remarks below the Lemmas 3.4, 3.6 we can give the following result:

Theorem 4.2. *If F is a function from M_j then f defined as in the Theorem 4.1 is in M_{j+10}.*

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