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Retractable state-finite automata without outputs*

Attila Nagy†

Abstract

A homomorphism of an automaton **A** without outputs onto a subautomaton **B** of **A** is called a retract homomorphism if it leaves the elements of B fixed. An automaton **A** is called a retractable automaton if, for every subautomaton **B** of **A**, there is a retract homomorphism of **A** onto **B**. In [1] and [3], special retractable automata are examined. The purpose of this paper is to give a construction for state-finite retractable automata without outputs.

In this paper, by an automaton we mean an automaton without outputs, that is, a system $\mathbf{A} = (A, X, \delta)$ consisting of a non-empty *state set* A, a non-empty *input set* X and a *transition function* $\delta : A \times X \mapsto A$. If A has only one element then the automaton **A** will be called *trivial*. The function δ is extended to $A \times X^*$ (X^* denotes the free monoid over X) as follows. If a is an arbitrary state of **A** then $\delta(a, e) = a$ for the empty word e, and $\delta(a, qx) = \delta(\delta(a, q), x)$ for every $q \in X^*$, $x \in X$.

If B is a non-empty subset of the state-set of an automaton $\mathbf{A} = (A, X, \delta)$ such that $\delta(b, x) \in B$ for every $b \in B$ and $x \in X$, then $\mathbf{B} = (B, X, \delta_B)$ is an automaton, where δ_B denotes the restriction of δ to $B \times X$. This automaton is called a *subautomaton* (more precisely, an *A-subautomaton*) of **A**. A subautomaton **B** of an automaton **A** is called a *proper subautomaton* of **A** if B is a proper subset of A. A subautomaton **B** of an automaton **A** is said to be a *minimal subautomaton* of **A** if **B** has no proper subautomaton. If a subautomaton **B** of an automaton **A** has only one state then **B** is minimal; the state of **B** is called a *trap* of **A**. If an automaton $\mathbf{A} = (A, X, \delta)$ contains only one trap denoted by a_0 then **A** is called a *one-trap automaton* (or an OT-*automaton*). This fact will be denoted by $(A, X, \delta; a_0)$. If an automaton **A** has a subautomaton which is contained in every subautomaton of **A** then it is called the *kernel* of **A**. The kernel of **A** is denoted by $Ker\mathbf{A}$.

Let $\mathbf{A} = (A, X, \delta)$ be an automaton containing at most one trap. Let A^0 denote the following set. $A^0 = A$ if **A** does not contain a trap or **A** is trivial; $A^0 = A - \{a_0\}$ if **A** is a non-trivial OT-automaton and a_0 is the trap of **A**. Consider the mapping $\delta^0 : A^0 \times X \mapsto A^0$ which is defined for a couple $(a, x) \in A^0 \times X$ if and only if

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†Department of Algebra, Institute of Mathematics, Budapest University of Technology and Economics

$\delta(a,x) \in A^0$. In this case, let $\delta^0(a,x) = \delta(a,x)$. (A^0, X, δ^0) is a *partial automaton* which will be denoted by $\mathbf{A}^0$.

An equivalence relation α of the state set A of an automaton $\mathbf{A} = (A, X, \delta)$ is called a *congruence* of $\mathbf{A}$ if, for every $a, b \in A$ and $x \in X$, the assumption $(a,b) \in \alpha$ implies $(\delta(a,x), \delta(b,x)) \in \alpha$. It is easy to see that if $\mathbf{B}$ is a subautomaton of an automaton $\mathbf{A}$ then $\rho_B = \{(a,b) \in A \times A : a = b \text{ or } a, b \in B\}$ is a congruence of $\mathbf{A}$, which is called the *Rees congruence* of $\mathbf{A}$ induced by $\mathbf{B}$. The factor automaton $\mathbf{A}/\rho_B$ is called the *Rees factor automaton* of $\mathbf{A}$ modulo $\mathbf{B}$. If $\mathbf{B}$ is a subautomaton of an automaton $\mathbf{A}$ then we may describe the Rees factor $\mathbf{A}/\rho_B$ as the result of collapsing B into a trap a_0 of the Rees factor, while the elements of A outside of B retain their identity. Sometimes we can identify these elements a $(a \in A - B)$ with the one-element ρ_B-class $[a]$, that is, we can suppose that the state set of the Rees factor is $(A - B) \cup \{a_0\}$.

If a is a state of an automaton $\mathbf{A}$, then the smallest subautomaton $\mathbf{R}(a)$ of $\mathbf{A}$ containing the state a is called the *principal subautomaton* of $\mathbf{A}$ generated by a. It is easy to see that $R(a) = \delta(a, X^*) = \{\delta(a,p) : \ p \in X^*\}$. Clearly, every minimal subautomaton of an automaton is principal.

The relation $\mathcal{R}$ on an automaton $\mathbf{A}$ defined by $\mathcal{R} = \{(a,b) \in A \times A : \ R(a) = R(b)\}$ is an equivalence relation on A. The $\mathcal{R}$-class of A containing an element $a \in A$ is denoted by R_a. The subset $R(a) - R_a$ is denoted by $R[a]$. It is clear that $R[a]$ is either empty or $(R[a], X, \delta_{R[a]})$ is a subautomaton of $\mathbf{A}$. The factor automaton $\mathbf{R}\{a\} = \mathbf{R}(a)/\rho_{R[a]}$ is called a *principal factor* of $\mathbf{A}$. We note that if $\mathbf{R}[a] = \emptyset$ then $\mathbf{R}\{a\}$ is defined to be $\mathbf{R}(a)$. For example, if a is a trap then $R(a) = \{a\}$ and so $R[a] = \emptyset$.

A mapping ϕ (acting on the left) of the state set A of an automaton $\mathbf{A} = (A, X, \delta_A)$ into the state set B of an automaton $\mathbf{B} = (B, X, \delta_B)$ is called a *homomorphism* of $\mathbf{A}$ into $\mathbf{B}$ if $\phi(\delta_A(a,x)) = \delta_B(\phi(a), x)$ for every $a \in A$ and $x \in X$.

A mapping ϕ (acting on the left) of A^0 into B^0 is called a *partial homomorphism* of a partial automaton $\mathbf{A}^0 = (A^0, X, \delta_A^0)$ into a partial automaton $\mathbf{B}^0 = (B^0, X, \delta_B^0)$ if, for every $a \in A^0$, $x \in X$, the assumption $\delta_A(a,x) \in A^0$ implies $\delta_B(\phi(a), x) \in B^0$ and $\delta_B(\phi(a), x) = \phi(\delta_A(a,x))$.

Definition 1. *A subautomaton* $\mathbf{B}$ *of an automaton* $\mathbf{A}$ *is said to be a retract subautomaton if there is a homomorphism of* $\mathbf{A}$ *onto* $\mathbf{B}$ *which leaves the elements of* B *fixed. Such a homomorphism is called a retract homomorphism of* $\mathbf{A}$ *onto* $\mathbf{B}$.

Definition 2. *An automaton* $\mathbf{A}$ *is called a retractable automaton if every subautomaton of* $\mathbf{A}$ *is retract.*

Lemma 1. *Every subautomaton of a retractable automaton is retractable.*

Proof. As a subautomaton $\mathbf{C}$ of a subautomaton $\mathbf{B}$ of an automaton $\mathbf{A}$ is also a subautomaton of $\mathbf{A}$, and the retriction of a retract homomorphism of $\mathbf{A}$ onto $\mathbf{C}$ to $\mathbf{B}$ is a retract homomorphism of $\mathbf{B}$ onto $\mathbf{C}$, our assertion is obvious. □

Lemma 2. *If* $\mathbf{A}$ *is a retractable automaton and* $\{a_i : \ i \in I\}$ *are elements of* A *such that* $R(a_i) \subseteq R(b)$ *for an element* b *of* A *then there is an index* $j \in I$ *such that* $R(a_i) \subseteq R(a_j)$ *for every* $i \in I$.

Proof. Let $\mathbf{A} = (A, X, \delta)$ be a retractable automaton and $\{a_i : i \in I\}$ be arbitrary elements of A such that $R(a_i) \subseteq R(b)$ for an element b of A. Let $R = \cup_{i \in I} R(a_i)$. As $\mathbf{R} = (R, X, \delta_R)$ is a subautomaton of $\mathbf{A}$, there is a retract homomorphism λ_R of $\mathbf{A}$ onto $\mathbf{R}$. As $\lambda_R(b) \in R$, there is an index $j \in I$ such that $\lambda_R(b) \in R(a_j)$. Then $\lambda_R(\delta(b,p)) = \delta(\lambda_R(b), p) \in R(a_j)$ for every $p \in X^*$, and so $\lambda_R(R(b)) \subseteq R(a_j)$. As $R(a_i) \subseteq R \cap R(b)$ $(i \in I)$, we get $R(a_i) = \lambda_R(R(a_i)) \subseteq R(a_j)$ for every $i \in I$. □

Corollary 1. *Every subautomaton of a principal subautomaton of a retractable automaton is principal. In particular, for every state a of a retractable automaton $\mathbf{A}$, $R[a]$ is either empty or $\mathbf{R}[a]$ is a principal subautomaton of $\mathbf{A}$.*

Proof. Let $\mathbf{B}$ be a subautomaton of a principal subautomaton $\mathbf{R}(b)$ of a retractable automaton $\mathbf{A}$. Then $R(a) \subseteq R(b)$ for every $a \in B$. By Lemma 2, there is an element $c \in B$ such that $R(a) \subseteq R(c)$ for every $a \in B$. As $B = \cup_{a \in B} R(a)$, we get $B = R(c)$. □

Let T be a set with a partial ordering $\leq$ such that every two-element subset of T has a lower bound in T and every non-empty subset of T having an upper bound in T contains a greatest element. Then T is a semilattice under multiplication $*$ by letting $a * b$ $(a, b \in T)$ be the (necessarily unique) greatest lower bound of a and b in T. A semilattice which can be constructed as above is called a *tree* ([4]).

Corollary 2. *A state-finite retractable automaton $\mathbf{A}$ contains a kernel if and only if the principal subautomata of $\mathbf{A}$ form a tree with respect to inclusion.*

Proof. Let $\mathbf{A}$ be a state-finite retractable automaton. The inclusion (the inclusion of the state-sets) is a partial ordering on the set T of all principal subautomata of $\mathbf{A}$. By Lemma 2, every non-empty subset of T having an upper bound in T contains a greatest element. As every finite tree has a least element, T (which is finite) is a tree if and only if it has a least element. As the least element of T is the kernel of $\mathbf{A}$, our proof is complete. □

Lemma 3. *Every principal subautomaton of a state-finite retractable automaton contains exactly one minimal subautomaton.*

Proof. From the finiteness of the state set, it follows that every principal subautomaton contains a minimal subautomaton. As a minimal subautomaton is a principal subautomaton, our assertion follows from Lemma 2. □

Lemma 4. *If a_1, a_2 are states of a state-finite retractable automaton $\mathbf{A} = (A, X, \delta)$ such that $B_1 \subseteq R(a_1)$, $B_2 \subseteq R(a_2)$ for distinct minimal subautomata $\mathbf{B}_1$ and $\mathbf{B}_2$ of $\mathbf{A}$ then $R(a_1) \cap R(a_2) = \emptyset$.*

Proof. If $c \in R(a_1) \cap R(a_2)$ then, by Lemma 3, there is a minimal subautomaton $\mathbf{B}$ of $\mathbf{A}$ such that $B \subseteq R(c) \subseteq R(a_1) \cap R(a_2)$. Using again Lemma 3, we get $B_1 = B = B_2$ which is a contradiction. □

If $\mathbf{A}_i = (A_i, X, \delta_i)$, $i \in I$ are automata such that $A_i \cap A_j = \emptyset$ for every $i \neq j$, then $\mathbf{A} = (A, X, \delta)$ is an automaton, where $A = \cup_{i \in I} A_i$ and $\delta(a, x) = \delta_i(a, x)$ for every $a \in A_i$ and $x \in X$. The automaton $\mathbf{A}$ is called the *direct sum* of the automata $\mathbf{A}_i$, $i \in I$.

Definition 3. *We say that an automaton* $\mathbf{A}$ *is a strong direct sum of a family of subautomata* $\mathbf{A}_i$*,* $i \in I$ *if* $\mathbf{A}$ *is a direct sum of* $\mathbf{A}_i$*,* $i \in I$ *and, for every couple* $(i, j) \in I \times I$*, there is a homomorphism of* $\mathbf{A}_i$ *into* $\mathbf{A}_j$*.*

Theorem 1. *A strong direct sum of retractable automata is retractable.*

Proof. Assume that an automaton $\mathbf{A} = (A, X, \delta)$ is a strong direct sum of automata $\mathbf{A}_i = (A_i, X, \delta_i)$, $i \in I$. Let $\phi_{i,j}$ be the corresponding homomorphism of $\mathbf{A}_i$ into $\mathbf{A}_j$ $(i, j \in I)$. Let $\mathbf{R}$ be an arbitrary subautomaton of $\mathbf{A}$. Let $R_i = R \cap A_i$. It is clear that R_i is either empty or $\mathbf{R}_i = (R_i, X, \delta_{R_i})$ is a subautomaton of $\mathbf{A}_i$. Let λ_{R_i} denote a retract homomorphism of $\mathbf{A}_i$ onto $\mathbf{R}_i$ if $R_i \neq \emptyset$, and let i_0 denote a fixed index, for which $R_{i_0} \neq \emptyset$. We define a mapping λ_R of A onto R as follows. If $a \in A_i$ and $R_i = \emptyset$, then let $\lambda_R(a) = \lambda_{R_{i_0}}(\phi_{i,i_0}(a))$; if $a \in A_i$ and $R_i \neq \emptyset$, then let $\lambda_R(a) = \lambda_{R_i}(a)$. It is clear that λ_R mapps A onto R and leaves the elements of R fixed. To prove that λ_R is a homomorphism of $\mathbf{A}$ onto $\mathbf{R}$, let $i \in I$, $a \in A_i$, $x \in X$ be arbitrary elements. In case $R_i = \emptyset$,

$$\lambda_R(\delta(a, x)) = \lambda_{R_{i_0}}(\phi_{i,i_0}(\delta_i(a, x))) = \lambda_{R_{i_0}}(\delta_{i_0}(\phi_{i,i_0}(a), x)) =$$

$$= \delta_{i_0}(\lambda_{R_{i_0}}(\phi_{i,i_0}(a)), x) = \delta(\lambda_R(a), x),$$

and, in case $R_i \neq \emptyset$,

$$\lambda_R(\delta(a, x)) = \lambda_{R_i}(\delta_i(a, x)) = \delta_i(\lambda_{R_i}(a), x) = \delta(\lambda_R(a), x),$$

because $a, \delta(a, x) \in A_i$. Hence λ_R is a retract homomorphism of $\mathbf{A}$ onto $\mathbf{R}$. Thus the theorem is proved. □

Theorem 2. *For a state-finite automaton* $\mathbf{A} = (A, X, \delta)$*, the following assertions are equivalent:*

(i) $\mathbf{A}$ *is retractable;*

(ii) $\mathbf{A}$ *is a direct sum of finite many state-finite retractable automata containing kernels being isomorphic to each other.*

(iii) $\mathbf{A}$ *is a strong direct sum of finite many state-finite retractable automata containing kernels.*

Proof. (i) implies (ii): Assume that $\mathbf{A}$ is retractable. As $\mathbf{A}$ is finite, it has a minimal subautomaton. Let $\{\mathbf{B}_i,\ i = 1, 2, \ldots r\}$ be the set of all distinct minimal subautomata of $\mathbf{A}$. Let $A_i = \cup_{a \in A}\{R(a) :\ B_i \subseteq R(a)\}$, $i = 1, 2, \ldots, r$. It is clear that $\mathbf{A}_i$ is a subautomaton of $\mathbf{A}$ and $\mathbf{B}_i$ is the kernel of $\mathbf{A}_i$ for every $i = 1, \ldots, r$. By Lemma 3, for every principal subautomaton $\mathbf{R}(a)$ of $\mathbf{A}$, there is a unique index i such that $B_i \subseteq R(a)$. Thus $A = \cup_{i=1}^{r} A_i$. By Lemma 4, $A_i \cap A_j = \emptyset$ for every

$i \neq j$. Hence $\mathbf{A}$ is a direct sum of the automata $\mathbf{A}_i$, $i = 1, \dots, r$. By Lemma 1, every automaton $\mathbf{A}_i$ is retractable. Let $i, j \in \{1, 2, \dots, r\}$ be arbitrary. As $\mathbf{B}_i$ is a minimal subautomaton of $\mathbf{A}$, the retract homomorphism λ_{B_i} of $\mathbf{A}$ onto $\mathbf{B}_i$ maps $\mathbf{B}_j$ onto $\mathbf{B}_i$. Thus $|B_j| \geq |B_i|$. Similarly, $|B_i| \geq |B_j|$. Thus $|B_i| = |B_j|$ and the restriction of λ_{B_j} to B_i is an isomorphism of $\mathbf{B}_i$ onto $\mathbf{B}_j$. Thus (ii) is satisfied.

(ii) implies (iii): Assume that $\mathbf{A}$ is a direct sum of the state-finite retractable automata $\mathbf{A}_i$, $i = 1, 2, \dots, r$ such that each of $\mathbf{A}_i$ contains a kernel $\mathbf{B_i}$, and, for every $i, j \in \{1, 2, \dots, r\}$, there is an isomorphism $\phi_{i,j}$ of $\mathbf{B}_i$ onto $\mathbf{B}_j$. It is easy to see that $\Phi_{i,j}$ defined by

$$\Phi_{i,j}(a) = \phi_{i,j}(\lambda_{B_i}(a)), \ a \in A_i$$

is a homomorphism of $\mathbf{A}_i$ into $\mathbf{A}_j$, where λ_{B_i} denotes a retract homomorphism of $\mathbf{A}_i$ onto $\mathbf{B}_i$. Thus $\mathbf{A}$ satisfies (iii).

(iii) implies (i): By Theorem 1, it is obvious. □

By the previous theorem, we concentrate our attention to state-finite retractable automata containing a kernel. These automata will be described by Corollary 3 and Theorem 7. First consider some results and notions which will be needed for us.

Lemma 5. *Every principal factor of an automaton can contain at most one trap.*

Proof. If $R[a] = \emptyset$ for a state a then the principal factor $\mathbf{R}\{a\}$ has a trap only that case when a is a trap of $\mathbf{A}$, that is, the principal factor is trivial. If $R[a] \neq \emptyset$ then $R(b) = R(a)$ for every $b \in R_a = R(a) - R[a]$, and so $\mathbf{R}\{a\}$ contains only one trap, namely the $\rho_{R[a]}$-class $R[a]$ of $\mathbf{R}(a)$. □

Definition 4. *An automaton $\mathbf{A} = (A, X, \delta)$ is called strongly connected if, for every couple $(a, b) \in A \times A$, there is a word $p \in X^+$ (X^+ denotes the free semigroup over X) such that $b = \delta(a, p)$.*

We note that every strongly connected automaton can contain only one subautomaton, namely itself. We also note that if an automaton is trivial (has only one state which is a trap) then it is strongly connected. If an automaton has at least two state and has a trap then it is not strongly connected.

Definition 5. *A non-trivial OT-automaton $\mathbf{A} = (A, X, \delta; a_0)$ is called strongly trap-connected if, for every couple $(a, b) \in A \times A$, $a \neq a_0$, there is a word $p \in X^+$ such that $b = \delta(a, p)$.*

We note that every strongly trap-connected automaton $\mathbf{A} = (A, X, \delta; a_0)$ contains only two subautomaton, namely itself and $(\{a_0\}, X, \delta_{\{a_0\}})$. Moreover, for every state $a \neq a_0$ of $\mathbf{A}$ there is a word $p \in X^+$ such that $a = \delta(a, p)$.

Definition 6. *We say that a non-trivial OT-automaton $\mathbf{A} = (A, X, \delta; a_0)$ is strongly trapped if $\delta(a, x) = a_0$ for every $a \in A$ and $x \in X$.*

Theorem 3. *Every principal factor of an automaton is either strongly connected or strongly trap-connected or strongly trapped.*

Proof. If $R[a] = \emptyset$ then $\mathbf{R}\{a\} = \mathbf{R}(a)$ is strongly connected. If $R[a] \neq \emptyset$ then, by Lemma 5, $\mathbf{R}\{a\}$ is a non-trivial OT-automaton. Let a_0 denote the trap of $\mathbf{R}\{a\}$. If $|R_a| = 1$, that is, $R\{a\} = \{a, a_0\}$, then $\mathbf{R}\{a\}$ is either strongly trapped (if $\delta(a, x) \in R[a]$ in $\mathbf{A}$, that is, $\delta(a, x) = a_0$ in $\mathbf{R}\{a\}$ for every $x \in X$) or strongly trap-connected (if $a = \delta(a, x)$ for some $x \in X$). If $|R_a| > 1$ then, for every elements b, c of R_a, $c = \delta(b, p)$ for some $p \in X^+$. Moreover, for every $b \in R_a$, there is a word $p \in X^+$ such that $\delta(b, p) \in R[a]$ in $\mathbf{A}$, that is, $\delta(b, p) = a_0$ in $\mathbf{R}\{a\}$. Hence $\mathbf{R}\{a\}$ is strongly trap-connected. □

Definition 7. *An automaton* $\mathbf{A}$ *is called semiconnected if every principal factor of* $\mathbf{A}$ *is either strongly connected or strongly trap-connected.*

Theorem 4. *An automaton* $\mathbf{A} = (A, X, \delta)$ *is semiconnected if and only if every subautomaton* $\mathbf{B}$ *of* $\mathbf{A}$ *satisfies the following: for every* $a \in B$ *there are elements* $b \in B$ *and* $p \in X^+$ *such that* $a = \delta(b, p)$.

Proof. Let $\mathbf{A} = (A, X, \delta)$ be a semiconnected automaton and $\mathbf{B}$ be a subautomaton of $\mathbf{A}$. Let a be an arbitrary element of B. Then $R(a) \subseteq B$. If a is a trap then $a = \delta(a, x)$ for every $x \in X$. Consider the case when a is not a trap. Then $|R(a)| \geq 2$. If $R[a] = \emptyset$ then, by Theorem 3, $\mathbf{R}(a) = \mathbf{R}\{a\}$ is strongly connected which means that, for every $b \in R(a)$ there is a word $p \in X^+$ such that $a = \delta(b, p)$. If $R[a] \neq \emptyset$ then, by Theorem 3, $\mathbf{R}\{a\}$ is strongly trap-connected and so, for every element $b \in R_a$, there is a word $p \in X^+$ such that $a = \delta(b, p)$. Thus, in all cases, there is a state $b \in B$ and a word $p \in X^+$ such that $a = \delta(b, p)$.

Conversely, assume that every subautomaton of an automaton $\mathbf{A}$ satisfies the condition of the theorem. We show that $\mathbf{A}$ is semiconnected. Let a be an arbitrary element of A. If a is a trap of $\mathbf{A}$ then the principal factor $\mathbf{R}\{a\}$ is trivial (and so it is strongly connected). Consider the case when a is not a trap of $\mathbf{A}$. Then a is an element of $\mathbf{R}\{a\}$ (and is not the trap of $\mathbf{R}\{a\}$). By Theorem 3, it is sufficient to show that the principal factor $\mathbf{R}\{a\}$ is not strongly trapped. As $\mathbf{R}(a)$ is a subautomaton of $\mathbf{A}$, by the condition of the theorem, there are elements $b \in R(a)$ $p \in X^*$ and $x \in X$ such that $a = \delta(b, px) = \delta(\delta(b, p), x)$ in $\mathbf{A}$. It is clear that $b' = \delta(b, p) \notin R[a]$ and so $a = \delta(b', x)$ in $\mathbf{R}\{a\}$. Thus $\mathbf{R}\{a\}$ is not strongly trapped. □

Definition 8. *Let* $\mathbf{B} = (B, X, \delta_B)$ *be a subautomaton of an automaton* $\mathbf{A} = (A, X, \delta)$. *We say that* $\mathbf{A}$ *is a dilation of* $\mathbf{B}$ *if there is a mapping* ϕ *of* A *onto* B *which leaves the elements of* B *fixed and* $\delta(a, x) = \delta_B(\phi(a), x)$ *for all* $a \in A$ *and* $x \in X$.

Theorem 5. *Every dilation of a retractable automaton is retractable.*

Proof. Let $\mathbf{A} = (A, X, \delta)$ be a dilation of a retractable subautomaton $\mathbf{B} = (B, X, \delta_B)$. Then there is a mapping ϕ of A onto B which leaves the elements of B fixed and $\delta(a, x) = \delta_B(\phi(a), x)$ for every $a \in A$ and $x \in X$. Let $\mathbf{R}$ be a subautomaton of $\mathbf{A}$. Then, for every $c \in R$ and $x \in X$, $\delta(c, x) \in R \cap B$. Let $\lambda_{R \cap B}$ denote a retract homomorphism of $\mathbf{B}$ onto the subautomaton $\mathbf{R}\cap\mathbf{B}$. Define a mapping λ_R of A onto R as follows. Let $\lambda_R(a)=a$ if $a\in R$, and let $\lambda_R(a)=\lambda_{R\cap B}(\phi(a))$ if $a\notin R$. We show that λ_R is a homomorpism of $\mathbf{A}$ onto $\mathbf{R}$. Let $a\in A$ and $x\in X$ be arbitrary elements. If $a\in R$ then

$$\delta(\lambda_R(a),x)=\delta(a,x)=\lambda_R(\delta(a,x)).$$

Assume $a\notin R$. Then

$$\delta(\lambda_R(a),x)=\delta_B(\lambda_{R\cap B}(\phi(a)),x)=$$

$$=\lambda_{R\cap B}(\delta_B(\phi(a),x))=\lambda_R(\delta(a,x)),$$

because $\lambda_R(a),\delta(a,x)\in B$ and the restriction of λ_R to B equals $\lambda_{R\cap B}$. Hence λ_R is a homomorphism of $\mathbf{A}$ onto $\mathbf{R}$. As λ_R leaves the elements of R fixed, it is a retract homomorphism of $\mathbf{A}$ onto $\mathbf{R}$. Consequently, $\mathbf{A}$ is a retractable automaton. □

Theorem 6. *Every retractable automaton is a dilation of a semiconnected retractable automaton.*

Proof. Let $\mathbf{A}=(A,X,\delta)$ be a retractable automaton and let $B=\delta(A,X)$. Then $\mathbf{B}=(B,X,\delta_B)$ is a subautomaton of $\mathbf{A}$ and so there is a retract homomorphism ϕ of $\mathbf{A}$ onto $\mathbf{B}$. Let $a\in A$, $x\in X$ be arbitrary elements. Then $\delta(a,x)=\phi(\delta(a,x))=\delta_B(\phi(a),x)$. Hence $\mathbf{A}$ is a dilation of $\mathbf{B}$. By Lemma 1, $\mathbf{B}$ is retractable. Let $\mathbf{R}$ be an arbitrary subautomaton of $\mathbf{B}$. If $c\in R$ is an arbitrary element, then $c=\delta(a,x)$ for some $a\in A$ and $x\in X$. Let λ_R denote the retract homomorphism of $\mathbf{A}$ onto $\mathbf{R}$. Then $\lambda_R(a)\in R$ and

$$c=\lambda_R(c)=\lambda_R(\delta(a,x))=\delta(\lambda_R(a),x).$$

Thus, by Theorem 4, $\mathbf{B}$ is semiconnected. □

Corollary 3. *An automaton is retractable if and only if it is a dilation of a semiconnected retractable automaton.*

Proof. By the previous two theorems, it is evident. □

Theorem 2 shows that the state-finite retractable automata are exactly the direct sums of finite many state-finite retractable automata such that each component in a mentioned direct sum contains a kernel, and these kernels are isomorphic with each other. Corollary 3 and the remark after Theorem 2 show that every component in a direct sum is a dilation of a state-finite semiconnected retractable automaton containing a kernel. Theorem 7 will show how we can construct the state-finite semiconnected retractable automata containing a kernel. These results togethet give a complete description of state-finite retractable automata.

Construction. Let T be a finite tree (under partial ordering $\leq$) with the least element i_0. Let $i\succ j$ $(i,j\in T)$ denote the fact that $i>j$ and, for every $k\in T$, $i\geq k\geq j$ implies $i=k$ or $j=k$.

Let $\mathbf{A}_i=(A_i,X,\delta_i)$, $i\in T$ be a family of disjunct automata such that

(i) $\mathbf{A}_{i_0}$ is strongly connected and $\mathbf{A}_i$ is a strongly trap-connected OT-automaton for every $i\in T$ with $i\neq i_0$.

(ii) Let $\phi_{i,i}$ denote the identity mapping of $\mathbf{A}_i$, and assume that, for every $i,j\in T$ with $i\succ j$, there is a partial homomorphism $\phi_{i,j}$ of $\mathbf{A}_i^0$ into $\mathbf{A}_j^0$ such that

(iii) for every $i\succ j$ there are elements $a\in A_i^0$ and $x\in X$ such that $\delta_i(a,x)\notin A_i^0$ and $\delta_j(\phi_{i,j}(a),x)\in A_j^0$.

For arbitrary elements $i,j\in T$ with $i\geq j$, define a partial homomorphism $\Phi_{i,j}$ of $\mathbf{A}_i^0$ into $\mathbf{A}_j^0$ as follows. $\Phi_{i,i}=\phi_{i,i}$ and, if $i>j$ such that $i\succ k_1\succ\dots k_n\succ j$ then let

$$\Phi_{i,j}=\phi_{k_n,j}\circ\phi_{k_{n-1},k_n}\circ\dots\circ\phi_{k_1,k_2}\circ\phi_{i,k_1}.$$

(We note that if $i\geq j\geq k$ are arbitrary elements of T then $\Phi_{i,k}=\Phi_{j,k}\circ\Phi_{i,j}$.)

Let $A=\cup_{i\in T}A_i^0$. Define a transition function $\delta':A\times X\mapsto A$ as follows. If $a\in A_i^0$ and $x\in X$ then let $\delta'(a,x)=\delta_{i'[a,x]}(\Phi_{i,i'[a,x]}(a),x)$, where $i'[a,x]$ denotes the greatest element of the set $\{j\in T:\ \delta_j(\Phi_{i,j}(a),x)\in A_j^0\}$.

It is easy to see that $\mathbf{A}=(A,X,\delta')$ is an automaton which will be denoted by $(A_i,X,\delta_i;\phi_{i,j},T)$.

Theorem 7. *A finite automaton is a semiconnected retractable automaton containing a kernel if and only if it is isomorphic to an automaton* $(A_i,X,\delta_i;\phi_{i,j},T)$ *constructed as above.*

Proof. Let $\mathbf{R}$ be a subautomaton of an automaton $(A_i,X,\delta_i;\phi_{i,j},T)$. As every automaton $\mathbf{A}_i$ $(i\in T-\{i_0\})$ is strongly trap-connected and $\mathbf{A}_{i_0}$ is strongly connected, it follows that $R=\cup_{j\in\Gamma}A_j^0$ for some non-empty subset Γ of T. We show that Γ is an ideal of T, that is, $i\in\Gamma$ and $j\leq i$ together imply $j\in\Gamma$ for all $i,j\in T$. Let i be an arbitrary element of T such that $i\in\Gamma$, $i\neq i_0$. If $j\in T$ with $i\succ j$ then, by (iii), there are elements $a\in A_i^0$ and $x\in X$ such that $\delta_i(a,x)\notin A_i^0$ and $\delta_j(\phi_{i,j}(a),x)\in A_j^0$. Then $\delta'(a,x)\in A_j^0$. Hence $A_j^0\cap R\neq\emptyset$ which implies that $A_j^0\subseteq R$ and so $j\in\Gamma$. This implies that Γ is an ideal of T. As T is a tree,

$$\pi:\ i\mapsto\ \max\{\gamma\in\Gamma:\ \gamma\leq i\}$$

is a well-defined mapping of T onto Γ which leaves the elements of Γ fixed (in fact, π is a retract homomorphism of the semigroup T onto the ideal Γ of T (see [4])). We define a retract homomorphism λ_R of $\mathbf{A}$ onto $\mathbf{R}$. For an arbitrary element $a\in A$, let

$$\lambda_R(a)=\Phi_{i,\pi(i)}(a)$$

if $a\in A_i^0$. It is easy to see that λ_R leaves the elements of R fixed. We prove that λ_R is a homomorphism of $\mathbf{A}$ onto $\mathbf{R}$. Let $x\in X$, $a\in A_i^0$ be arbitrary elements. Using $\delta'(a,x)=\delta_{i'[a,x]}(\Phi_{i,i'[a,x]}(a),x)\in A_{i'[a,x]}^0$ and the fact that $\Phi_{i'[a,x],\pi(i'[a,x])}$ is a partial homomorphism, we get

$$\lambda_R(\delta'(a,x))=\lambda_R(\delta_{i'[a,x]}(\Phi_{i,i'[a,x]}(a),x))=$$

$$=\Phi_{i'[a,x],\pi(i'[a,x])}(\delta_{i'[a,x]}(\Phi_{i,i'[a,x]}(a),x))=$$

$$= \delta_{\pi(i'[a,x])}(\Phi_{i,\pi(i'[a,x])}(a), x) \in A^0_{\pi(i'[a,x])}.$$

Using $\Phi_{i,\pi(i)}(a) \in A^0_{\pi(i)}$, we have

$$\delta'(\lambda_R(a), x) = \delta'(\Phi_{i,\pi(i)}(a), x) =$$

$$= \delta_{(\pi(i))'[\Phi_{i,\pi(i)}(a),x]}(\Phi_{\pi(i),(\pi(i))'[\Phi_{i,\pi(i)}(a),x]}(\Phi_{i,\pi(i)}(a)), x) =$$

$$= \delta_{(\pi(i))'[\Phi_{i,\pi(i)}(a),x]}(\Phi_{i,(\pi(i))'[\Phi_{i,\pi(i)}(a),x]}(a), x) \in A^0_{(\pi(i))'[\Phi_{i,\pi(i)}(a),x]}.$$

To prove that $\lambda_R(\delta'(a,x)) = \delta'(\lambda_R(a), x)$, it is sufficient to show that

$$(\pi(i))'[\Phi_{i,\pi(i)}(a), x] = \pi(i'[a,x]).$$

First, assume $i'[a,x] \geq \pi(i)$ (and so $\pi(i'[a,x]) = \pi(i)$). As $\phi_{i'[a,x],\pi(i)}$ is a partial homomorphism of $A^0_{i'[a,x]}$ into $A^0_{\pi(i)}$ and $\delta_{i'[a,x]}(\Phi_{i,i'[a,x]}(a), x) \in A^0_{i'[a,x]}$, we get

$$\delta_{\pi(i)}(\Phi_{i,\pi(i)}(a), x) = \delta_{\pi(i)}(\Phi_{i'[a,x],\pi(i)}(\Phi_{i,i'[a,x]}(a)), x) =$$

$$= \Phi_{i'[a,x],\pi(i)}(\delta_{i'[a,x]}(\Phi_{i,i'[a,x]}(a), x)) \in A^0_{\pi(i)}$$

and so

$$(\pi(i))'[\Phi_{i,\pi(i)}(a), x] = \pi(i) = \pi(i'[a,x]).$$

Next, consider the case when $i'[a,x] < \pi(i)$ (and so $\pi(i'[a,x]) = i'[a,x]$). If $j \in T$ with $\pi(i) \geq j > i'[a,x]$ then we have

$$\delta_j(\Phi_{\pi(i),j}(\Phi_{i,\pi(i)}(a)), x) = \delta_j(\Phi_{i,j}(a), x) \notin A^0_j.$$

Then

$$(\pi(i))'[\Phi_{i,\pi(i)}(a), x] \leq i'[a,x].$$

As

$$\delta_{i'[a,x]}(\Phi_{\pi(i),i'[a,x]}(\Phi_{i,\pi(i)}(a)), x) = \delta_{i'[a,x]}(\Phi_{i,i'[a,x]}(a), x) \in A^0_{i'[a,x]},$$

we get

$$(\pi(i))'[\Phi_{i,\pi(i)}(a), x] \geq i'[a,x].$$

Hence

$$(\pi(i))'[\Phi_{i,\pi(i)}(a), x] = i'[a,x] = \pi(i'[a,x]).$$

Consequently, $(\pi(i))'[\Phi_{i,\pi(i)}(a), x] = \pi(i'[a,x])$ in both cases. Hence λ_R is a (retract) homomorphism of $\mathbf{A}$ onto $\mathbf{R}$. Thus $\mathbf{A} = (A_i, X, \delta_i; \phi_{i,j}, T)$ is a retractable automaton.

We show that $\mathbf{A}$ is semiconnected. If $\mathbf{R}$ is an arbitrary subautomaton of $\mathbf{A}$, then there is an ideal Γ of T such that $R = \cup_{j\in\Gamma} A^0_j$ (see above). Let $a \in R$ be an arbitrary element. Then $a \in A^0_k$ for some $k \in \Gamma$. As A_k is strongly connected or strongly trap-connected, there are elements $b \in A^0_k$ and $p \in X^+$ such that $a = \delta_k(b, p) = \delta'(b, p)$. By Theorem 4, it means that $\mathbf{A}$ is semiconnected. As i_0 is contained in every ideal of T, $\mathbf{A}_{i_0}$ is the kernel of $(A_i, X, \delta_i; \phi_{i,j}, T)$.

Conversely, let $\mathbf{A}$ be a finite semiconnected retractable automaton containing a kernel. Let $Prf(A)$ denote the set of all principal factors of $\mathbf{A}$. By Corollary 2, $Prf(A)$ is a (finite) tree under partial ordering $\leq$ defined by $\mathbf{R}\{a\} \leq \mathbf{R}\{b\}$ if and only if $R(a) \subseteq R(b)$. As $\mathbf{A}$ is semiconnected, the least element of $Prf(A)$ is strongly connected, the other ones are strongly trap-connected.

Let T be a set with $|T| = |Prf(A)|$. Denote a bijection of T onto $Prf(A)$ by f. Define a partial ordering $\leq$ on T by $i \leq j$ $(i, j \in T)$ if and only if $f(i) \leq f(j)$. Let i_0 denote the least element of T. Clearly, T is a finite tree with the least element i_0. For every element $i \in T$, fix an element a_i in A such that $f(i) = \mathbf{R}\{a_i\}$. (We note that $\mathbf{R}\{a_i\} = \mathbf{R}\{a_j\}$ iff $a_i = a_j$ iff $i = j$). As $\mathbf{R}\{a_{i_0}\}$ is strongly connected and $\mathbf{R}\{a_i\}$ is strongly trap-connected if $i \neq i_0$, condition (i) of the Construction is satisfied.

Let $\lambda_{R(a_j)}$ $(j \in T)$ denote a fix retract homomorphism of $\mathbf{A}$ onto $\mathbf{R}(a_j)$. For every $i, j \in T$ with $i \succeq j$, let $\lambda_{i,j}$ denote the restriction of $\lambda_{R(a_j)}$ to $R(a_i)$. It is obvious that $\lambda_{i,j}$ is a retract homomorphism of $\mathbf{R}(a_i)$ onto $\mathbf{R}(a_j)$ for every $i \succeq j$, $(i, j \in T)$. Moreover, $\lambda_{i,i}$ is the identity mapping of $\mathbf{R}(a_i)$, for every $i \in T$. We show that $\lambda_{i,j}$ maps R_{a_i} into R_{a_j}. Let $a \in R_{a_i}$ be an arbitrary element (so $R(a) = R(a_i)$). Then, for every $p \in X^*$, $\lambda_{i,j}(\delta(a, p)) = \delta(\lambda_{i,j}(a), p)$. If $\lambda_{i,j}(a)$ was in $R[a_j]$ then we would have $\lambda_{i,j}(\delta(a, p)) \in R[a_j]$ for every $p \in X^*$, because $\mathbf{R}[a_j]$ is a subautomaton of $\mathbf{A}$. This would imply that $\lambda_{i,j}(R(a_i)) \subseteq R[a_j]$ which is impossible, because $\lambda_{i,j}$ maps $R(a_i)$ onto $R(a_j) = R_{a_j} \cup R[a_j] \supset R[a_j]$. Hence $\lambda_{i,j}$ maps R_{a_i} into R_{a_j} and so $\lambda_{i,j}$ can be considered as a mapping of $R^0\{a_i\}$ into $R^0\{a_j\}$. If $\delta(a, x) \in R_{a_i}$ for some $a \in R_{a_i}$ and $x \in X$ then $\delta(\lambda_{i,j}(a), x) = \lambda_{i,j}(\delta(a, x)) \in R_{a_j}$. Hence $\lambda_{i,j}$ is a partial homomorphism of the partial automaton $\mathbf{R}^0\{a_i\}$ into the partial automaton $\mathbf{R}^0\{a_j\}$. Thus condition (ii) of the Construction is satisfied (for $\mathbf{A}_i = \mathbf{R}\{a_i\}$, $\phi_{i,j} = \lambda_{i,j}$).

Assume $i \succ j$. Let $b \in R_{a_j}$ be an arbitrary element. Then $a_i \neq b \in R(a_i)$ and so there is a word $p = x_1x_2 \dots x_n \in X^+$ $(x_1, x_2, \dots x_n \in X)$ such that $b = \delta(a_i, p)$. Let m be the least index such that $\delta(a_i, x_1 \dots x_m) \in R_{a_j}$. Consider an element a of R_{a_i} (or of $\mathbf{R}^0\{a_i\}$) as follows. Let $a = a_i$ if $m = 1$. Let $a = \delta(a_i, x_1 \dots x_{m-1})$ if $m > 1$. Then $\delta(a, x_m) \notin R_{a_i}$ (or $\delta(a, x_m) \notin \mathbf{R}^0\{a_i\}$). On the other hand,

$$\delta(\lambda_{i,j}(a), x_m) = \lambda_{i,j}(\delta(a, x_m)) = \delta(a, x_m) \in R_{a_j} = R^0\{a_j\},$$

because $\lambda_{i,j}$ leaves the elements of $R(a_j)$ fixed. Thus (iii) of the Construction is satisfied (for $\phi_{i,j} = \lambda_{i,j}$, $x = x_m$).

For arbitrary elements $i, j \in T$ with $i \geq j$, define the mapping $\Phi_{i,j}$ as follows. Let $\Phi_{i,i} = \lambda_{i,i}$ and, if $i > j$ with $i \succ k_1 \succ k_2 \succ \dots k_n \succ j$ then let

$$\Phi_{i,j} = \lambda_{k_n,j} \circ \dots \circ \lambda_{i,k_1}.$$

It is clear that $\Phi_{i,j}$ is a retract homomorphism of $\mathbf{R}(a_i)$ onto $\mathbf{R}(a_j)$ such that it maps R_{a_i} into R_{a_j}. Thus $\Phi_{i,j}$ can be considered as a partial homomorphism of $\mathbf{R}^0\{a_i\}$ into $\mathbf{R}^0\{a_j\}$. Moreover, $\Phi_{i,k} = \Phi_{j,k} \circ \Phi_{i,j}$ for every $i, j, k \in T$ with $i \geq j \geq k$.

Construct the automaton $\mathbf{R} = (R\{a_i\}, X, \delta_i; \lambda_{i,j}, T)$, where δ_i is the transitive function of the factor automaton $\mathbf{R}\{a_i\}$ induced by δ. It is clear that the state sets of the automata **R** and **A** are the same. We show that the transitive functions δ of **A** equals the transitive function δ' of **R**. Let $i \in T$, $a \in R_{a_i} = R^0\{a_i\}$, $x \in X$ be arbitrary elements. Assume $\delta(a, x) \in R_{a_j}$ $(i \geq j)$. Let $k \in T$ with $i \geq k > j$. Then $\delta(a, x) \in R[a_k] \subset R(a_k)$ and so

$$\delta(\Phi_{i,k}(a), x) = \Phi_{i,k}(\delta(a, x)) = \delta(a, x) \notin R_{a_k} = R^0\{a_k\},$$

because $\Phi_{i,k}$ leaves the elements of $R(a_k)$ fixed. If $j \geq k$ then

$$\delta(\Phi_{i,k}(a), x) = \Phi_{i,k}(\delta(a, x)) =$$

$$= \Phi_{j,k} \circ \Phi_{i,j}(\delta(a, x)) = \Phi_{j,k}(\delta(a, x)) \in R_{a_k} = R^0\{a_k\},$$

because $\Phi_{i,j}$ leaves the element $\delta(a, x) \in R_{a_j} = R^0\{a_j\}$ fixed, and $\Phi_{j,k}$ maps R_{a_j} into R_{a_k}. Consequently $i'[a, x] = j$. Hence

$$\delta(a, x) = \Phi_{i,j}(\delta(a, x)) = \delta(\Phi_{i,j}(a), x) = \delta_j(\Phi_{i,j}(a), x) =$$

$$= \delta_{i'[a,x]}(\Phi_{i,i'[a,x]}(a), x) = \delta'(a, x).$$

Thus the theorem is proved. □

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