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Relationally defined clones of tree functions closed under selection or primitive recursion

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Abstract

We investigate classes of tree functions which are closed under composition and primitive recursion or selection (a restricted form of recursion). The main result is the characterization of those finitary relations ϱ (on the set of all trees of a fixed signature) for which the clone of tree functions preserving ϱ is closed under selection. Moreover, it turns out that such clones are closed also under primitive recursion.

Introduction

Classes of tree functions and primitive recursion for such functions were investigated, e.g., in [FülHVV93], [EngV91], [Hup78], [Kla84]. In this paper a tree function will be an operation $f : T^n \to T$ on the set T of all trees of a given finite signature (in general one allows trees of different signature).

If a class of operations is a clone (i.e. if it contains all projections and is closed with respect to composition), then it can be described by invariant relations (cf. e.g. [PösK79], [Pös80], [Pös01]).

In this paper we apply such results from clone theory and ask which finitary relations characterize clones of tree functions that in addition are closed under primitive recursion. The answer is given in Theorem 2.1 and shows that such relations are easy to describe: they are direct products of order ideals of trees (an order ideal contains with a tree also all its subtrees). Moreover it turns out that for such clones the closure under primitive recursion is equivalent to a much weaker closure (the so-called selection or S-closure, cf. 1.5) for which no real recursion is necessary.

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1 Notions and Notation

Let $\mathbb{N} := \{0, 1, 2, \ldots\}$ denote the set of natural numbers and let $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$.

1.1. Clone theoretic notions and notation. Let T be an arbitrary set (later we shall use only the set T of trees) and let $\mathrm{Op}(T)$ denote the set of all finitary operations on T, i.e. functions of the form $f : T^n \to T$ ($n \in \mathbb{N}_+$). A set $F \subseteq \mathrm{Op}(T)$ is called a *clone* if F contains all *projections* e_i^n ($n \in \mathbb{N}_+$, $i \in \{1, \ldots, n\}$) defined by

$$e_i^n(x_1, \ldots, x_n) = x_i \tag{1.1.1}$$

(for every $x_1, \ldots, x_n \in T$) and if F is closed with respect to composition, i.e. for every n-ary $f \in F$ and m-ary $g_1, \ldots, g_n \in F$ the m-ary *composition* $f(g_1, \ldots, g_n)$ defined by

$$f(g_1, \ldots, g_n)(x_1, \ldots, x_m) := f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m)) \tag{1.1.2}$$

also belongs to F, $n, m \in \mathbb{N}_+$. For technical reasons 0-ary operations will not be considered in clones; they are replaced by unary constant operations, i.e., a constant $t \in T$ is replaced by the unary constant operation $T \to T : x \mapsto t$.

The least clone containing a given set F of operations over T will be denoted by $\langle F \rangle$. The projection $e_1^1 : T \to T : x \mapsto x$ is the identity mapping and will be denoted simply by e. The clone $\langle e \rangle$ generated by e is the least clone: it consists exactly of all projections.

We say that an n-ary operation $f : T^n \to T$ *preserves* an m-ary relation $\varrho \subseteq T^m$ (or, equivalently, that ϱ is *invariant* for f) if $c_1, \ldots, c_n \in \varrho$ implies $f(c_1, \ldots, c_n) \in \varrho$, where

$$f(c_1, \ldots, c_n) := (f(c_{11}, \ldots, c_{1n}), \ldots, f(c_{m1}, \ldots, c_{mn})) \tag{1.1.3}$$

for m-tuples $c_1 = (c_{11}, \ldots, c_{m1}), \ldots, c_n = (c_{1n}, \ldots, c_{mn})$ (see Fig.1).

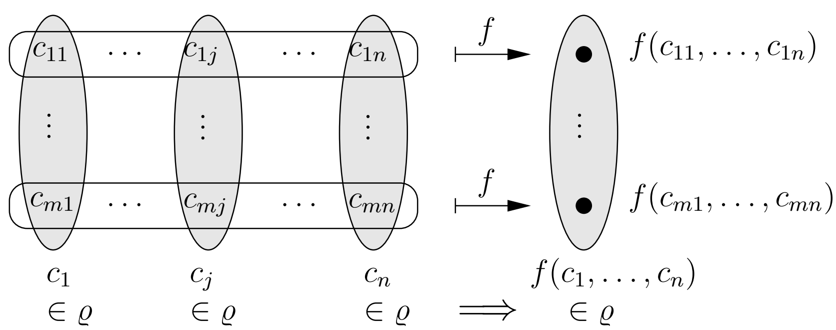

Figure 1: The operation f preserves the relation ϱ

The set of all operations preserving a given relation ϱ (so-called *polymorphisms*) will be denoted by

$$\operatorname{Pol}\varrho := \{f \mid f : T^n \to T \text{ preserves } \varrho,\, n \in \mathbb{N}_+\}\,. \tag{1.1.4}$$

It is well-known from clone theory (but also easy to see) that $\operatorname{Pol}\varrho$ is always a clone (e.g. [PösK79, 1.1.15]), moreover every clone on a finite set T can be characterized as $\operatorname{Pol} Q$ for some set Q of finitary relations ([PösK79, 1.2.1]), where

$$\operatorname{Pol} Q := \bigcap_{\varrho\in Q} \operatorname{Pol}\varrho\,.$$

For infinite T this remains true either if one considers so-called locally closed clones or if infinitary relations are allowed. We do not go into details here and refer to e.g. [Pös80], [Pös01].

We say that the i-th and j-th component of an m-ary relation $\varrho \subseteq T^m$ coincide ($i, j \in \{1, \ldots, m\}$) if $a_i = a_j$ for *all* elements $(a_1, \ldots, a_m) \in \varrho$. A relation ϱ is called *reduced* if no two of its components coincide. A relation can always be reduced without changing the set of polymorphisms: if the i-th and j-th component of ϱ coincide then $\operatorname{Pol}\varrho = \operatorname{Pol}\varrho'$ where $\varrho' := \{(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_m) \mid (a_1, \ldots, a_m) \in \varrho\}$ is obtained from ϱ by deleting the j-th component. Thus we may consider reduced relations only.

1.2. Trees and subtrees. This is a well-known concept in mathematics and computer science; nevertheless we shall repeat it here in order to fix notions and notation. Let Σ be a finite *signature* (or *ranked alphabet*), i.e. a finite set of symbols such that to every symbol $\sigma \in \Sigma$ a *rank* (or *arity*) $r_\sigma \in \mathbb{N}$ is assigned. Let $\Sigma_n := \{\sigma \in \Sigma \mid r_\sigma = n\}$ denote the set of all symbols of rank n ($n \in \mathbb{N}$). We assume that $\Sigma_0 \neq \emptyset$. A *tree* (or *ground term*) over Σ is an expression which can be obtained inductively by the following rules:

(0) Every $\sigma \in \Sigma_0$ is a tree.

(1) If $\sigma \in \Sigma_r$ ($r \in \mathbb{N}_+$) and if $s_1, \ldots, s_r$ are trees, then the expression $\sigma(s_1, \ldots, s_r)$ is also a tree.

If t is a tree of the form $\sigma(s_1, \ldots, s_r)$ (according to (1)), then the trees $s_1, \ldots, s_r$ are called *maximal subtrees* of t, and we write $s \lessdot t$ if s is a maximal subtree of t. A tree s is called *subtree* of a tree t, notation $s \leq t$, if $s = t$ or if there is a finite sequence $s_0, \ldots, s_l$ of trees with

$$s = s_0 \lessdot s_1 \lessdot \ldots \lessdot s_l = t$$

(i.e., $\leq$ is the transitive and reflexive closure of $\lessdot$). Note that a tree $t \in \Sigma_0$ has no proper subtrees. As usual we write $s < t$ if $s \leq t$ and $s \neq t$.

From now on T will always denote the set of all trees over a fixed signature $\Sigma = (\Sigma_r)_{r\in\mathbb{N}}$.

A subset $I \subseteq T$ of trees is an *order ideal* (*down-set*) if it is closed with respect to subtrees, i.e. if $s \leq t$ and $t \in I$, then $s \in I$.

The *height* $h(t)$ of a tree t is defined inductively on the structure of trees (according to the above rules): $h(t) := 0$ for $t \in \Sigma_0$, and $h(\sigma(s_1, \ldots, s_r)) := 1 + \max\{h(s_1), \ldots, h(s_r)\}$ for $\sigma \in \Sigma_r$ ($r \in \mathbb{N}_+$) and $s_1, \ldots, s_r \in T$. For $k \in \mathbb{N}$ let T_k denote the set of all trees in T of height $\leq k$.

1.3. Primitive recursion for tree functions. An operation $f : T^n \to T$ will also be called *tree function*. Let $n \in \mathbb{N}_+$ and for every $\sigma \in \Sigma$ let $g_\sigma : T^{n+2r_\sigma} \to T$ be an $(n+2r_\sigma)$-ary tree function. The $(n+1)$-ary tree function $h : T^{n+1} \to T$ defined recursively (in its last argument) by

$$h(a_1, \ldots, a_n, \sigma) := g_\sigma(a_1, \ldots, a_n) \qquad \text{for } \sigma \in \Sigma_0\,, a_1, \ldots, a_n \in T\,, \tag{1.3.1}$$

$$\begin{aligned} &h(a_1, \ldots, a_n, \sigma(t_1, \ldots, t_r)) := \\ &\quad g_\sigma(a_1, \ldots, a_n, t_1, \ldots, t_r, h(a_1, \ldots, a_n, t_1), \ldots, h(a_1, \ldots, a_n, t_r)) \\ &\qquad\qquad \text{for } \sigma \in \Sigma_r,\ r \geq 1, \text{ and } a_1, \ldots, a_n, t_1, \ldots, t_r \in T\,, \end{aligned} \tag{1.3.2}$$

will be denoted by $\mathrm{PR}(g_\sigma)_{\sigma\in\Sigma}$; we say that h is obtained from $(g_\sigma)_{\sigma\in\Sigma}$ by *primitive recursion* (PR). By convention, equation 1.3.1 is considered as the special case of 1.3.2 for $r = 0$.

A set $F \subseteq \mathrm{Op}(T)$ of tree functions is called PR-*closed* if $\mathrm{PR}(g_\sigma)_{\sigma\in\Sigma} \in F$ whenever $g_\sigma \in F$ for all $\sigma \in \Sigma$. For $F \subseteq \mathrm{Op}(T)$, by

$$\langle\!\langle F \rangle\!\rangle_{\mathsf{PR}}$$

we shall denote the least set of tree functions which contains F and which is both, a clone and PR-closed. The existence of $\langle\!\langle F \rangle\!\rangle_{\mathsf{PR}}$ is guaranteed because the intersection of PR-closed clones is again a PR-closed clone. Obviously, $\langle F \rangle \subseteq \langle\!\langle F \rangle\!\rangle_{\mathsf{PR}}$.

1.4 Examples. a) The least PR-closed clone is $\langle\!\langle \emptyset \rangle\!\rangle_{\mathsf{PR}}$; by definition it must contain all projections and we shall denote this clone also by $\langle\!\langle e \rangle\!\rangle_{\mathsf{PR}}$. Lemma 2.4 and Proposition 2.5 shall describe some further operations which also belong to $\langle\!\langle e \rangle\!\rangle_{\mathsf{PR}}$.

b) Let F_{base} consist of all the constant tree functions $\mathrm{const}_t : T^n \to T : (t_1, \ldots, t_n) \mapsto t$ ($t \in T$), and all the top concatenations $\mathrm{top}_\sigma : T^r \to T : (t_1, \ldots, t_r) \mapsto \sigma(t_1, \ldots, t_r)$ ($\sigma \in \Sigma_r$, $r \in \mathbb{N}_+$). Then $\langle\!\langle F_{\mathrm{base}} \rangle\!\rangle_{\mathsf{PR}}$ is the class PREC_Σ of all primitive recursive tree functions over Σ (cf. e.g. [EngV91, 4.6]).

We are particularly interested in the following very special form of primitive recursion.

1.5. S-closure for tree functions. For a family $(g'_\sigma)_{\sigma\in\Sigma}$ of $(n+r_\sigma)$-ary tree functions g'_σ ($\sigma \in \Sigma$) we define

$$\mathrm{S}(g'_\sigma)_{\sigma\in\Sigma} := \mathrm{PR}(g_\sigma)_{\sigma\in\Sigma} \tag{1.5.1}$$

where g_σ is the $(n+2r_\sigma)$-ary tree function defined by

$$g_\sigma := g'_\sigma(e_1^{n+2r_\sigma}, \dots, e_{n+r_\sigma}^{n+2r_\sigma}), \tag{1.5.2}$$

for every $\sigma \in \Sigma$, i.e. we have in particular (for the $(n+1)$-ary tree function $h = \mathrm{S}(g'_\sigma)_{\sigma\in\Sigma}$)

$$\begin{aligned} h(a_1, \dots, a_n, \sigma(t_1, \dots, t_{r_\sigma})) = \\ g_\sigma(a_1, \dots, a_n, t_1, \dots, t_{r_\sigma}, h(a_1, \dots, a_n, t_1), \dots, h(a_1, \dots, a_n, t_{r_\sigma})) \\ = g'_\sigma(a_1, \dots, a_n, t_1, \dots, t_{r_\sigma}). \end{aligned} \tag{1.5.3}$$

Analogously to the PR-closure and the notation $\langle\!\langle F \rangle\!\rangle_{\mathsf{PR}}$, we introduce the S-closure of F (using the special primitive recursion S instead of PR) and let

$$\langle\!\langle F \rangle\!\rangle_{\mathsf{S}}$$

denote the least S-closed clone containing F. We call this S-closure also *selection* closure because the functions g_σ just select and there is no real recursion (i.e. h is not allowed to call itself recursively, see 1.3.2 and 1.5.3). By definition we have

$$\langle\!\langle F \rangle\!\rangle_{\mathsf{S}} \subseteq \langle\!\langle F \rangle\!\rangle_{\mathsf{PR}} \tag{1.5.4}$$

and this inclusion is proper in general.

1.6 Remarks. a) If we write $\mathrm{PR}(g_\sigma)_{\sigma\in\Sigma}$ (or $\mathrm{S}(g'_\sigma)_{\sigma\in\Sigma}$) we assume that the operations g_σ (or g'_σ) are of arity $n + 2r_\sigma$ (or $n + r_\sigma$) for some fixed $n \in \mathbb{N}_+$.

b) Usually the definition of primitive recursion also includes the case $n = 0$ in 1.3. Then however the operations g_σ are 0-ary constants for $\sigma \in \Sigma_0$ (cf. 1.3.1) which does not fit our convention not to consider 0-ary operations for clones (cf. 1.1). Nevertheless the restriction to $n \geq 1$ is no loss of generality: In fact, let $h : T \to T$ be the unary tree function obtained from (1.3.1) and (1.3.2) in case $n = 0$ with given constants $g_\sigma \in T$ for $r_\sigma = 0$ and operations $g_\sigma : T^{2r_\sigma} \to T$ for $r_\sigma \geq 1$. Further let $g'_\sigma : T^{1+2r_\sigma} \to T$ be the operations obtained from g_σ by adding a fictitious variable (at the first place), i.e. $g'_\sigma := g_\sigma(e_2^{1+2r_\sigma}, \dots, e_{1+2r_\sigma}^{1+2r_\sigma})$ for $r_\sigma \geq 1$, and $g'_\sigma(a) := g_\sigma$ for $r_\sigma = 0$, $a \in T$. Then, for $h' := \mathrm{PR}(g'_\sigma)$ as given with 1.3, we have $h = h'(e, e)$ and $h' = h(e_2^2)$. Thus h belongs to a clone F if and only if h' does.

2 Clones Pol ϱ of tree functions closed under primitive recursion

The following theorem is the main result of this paper. It characterizes finitary relations ϱ over trees with the property that the clone $\mathrm{Pol}\,\varrho$ of tree functions is closed with respect to primitive recursion.

2.1 Theorem. *Let $m \in \mathbb{N}_+$ and let $\varrho \subseteq T^m$ be a reduced relation. Then the following conditions are equivalent:*

(i) $\langle\!\langle \mathrm{Pol}\,\varrho \rangle\!\rangle_{\mathsf{PR}} = \mathrm{Pol}\,\varrho\,,$

(ii) $\langle\!\langle \mathrm{Pol}\,\varrho \rangle\!\rangle_{\mathsf{S}} = \mathrm{Pol}\,\varrho\,,$

(iii) *there exist order ideals* $I_1, \dots, I_m \subseteq T$ *such that* $\varrho = I_1 \times \cdots \times I_m\,.$

2.2 Remarks. The tree functions in $\mathrm{Pol}\,\varrho$ with ϱ as in 2.1(iii) can easily be described:

$$\mathrm{Pol}\,\varrho = \bigcap_{j=1}^{m} \mathrm{Pol}\,I_j$$

where $\mathrm{Pol}\,I_j$ is the set of tree functions preserving the order ideal I_j (considered as unary relation on T), i.e. each $f \in \mathrm{Pol}\,I_j$ maps trees from I_j to trees which are again in I_j. By Theorem 2.1, every clone $\mathrm{Pol}\,I_j$ is PR-closed. Thus any intersection – finite or infinite – of clones of the form $\mathrm{Pol}\,I$ for some order ideal $I \subseteq T$ gives a PR-closed clone. Consequently the implication (iii) $\Longrightarrow$ (i) of Theorem 2.1 can be generalized to infinitary relations $\varrho = \prod_{j \in J} I_j$ (J being an infinite index set) because $\mathrm{Pol}\,\varrho = \bigcap_{j \in J} \mathrm{Pol}\,I_j$. However, the converse (i) $\Longrightarrow$ (iii) does not remain true for infinitary relations (cf. 3.3).

As mentioned in 1.1 the restriction to reduced relations is not a loss of generality, however it is crucial for the formulation of Theorem 2.1. If ϱ is not reduced then it is no longer a direct product of order ideals (note $I_1 \times I_1 \neq \{(x,x) \mid x \in I_1\}$).

In clone theory usually relations are even further reduced to relations without "fictitious components". In the context of Theorem 2.1 we have: the j-th component of $\varrho = I_1 \times \cdots \times I_m$ is fictitious iff the order ideal I_j is trivial, i.e., $I_j = T$. Relations which differ only in fictitious components determine the same clone $\mathrm{Pol}\,\varrho$.

In the remainder of this section we shall prove Theorem 2.1. Note that 2.1(i) $\Longrightarrow$ (ii) is trivial because $\mathrm{Pol}\,\varrho \subseteq \langle\!\langle \mathrm{Pol}\,\varrho \rangle\!\rangle_{\mathsf{S}} \subseteq \langle\!\langle \mathrm{Pol}\,\varrho \rangle\!\rangle_{\mathsf{PR}}$ (cf. 1.5.4). We start with the following more or less straightforward part:

Proof of 2.1 (iii) $\Longrightarrow$ (i).

Let $\varrho = I_1 \times \cdots \times I_m$ where $I_1, \dots, I_m \subseteq T$ are order ideals. Then $\mathrm{Pol}\,\varrho = \bigcap_{j=1}^{m} \mathrm{Pol}\,I_j$ (cf. 2.2) and (iii) $\Longrightarrow$ (i) of Theorem 2.1 follows from the following lemma.

2.3 Lemma. *Let* $I \subseteq T$ *be an order ideal. Then* $\langle\!\langle \mathrm{Pol}\,I \rangle\!\rangle_{\mathsf{PR}} = \mathrm{Pol}\,I$.

Proof. We have to show that $\mathrm{Pol}\,I$ is PR-closed. Let $n \in \mathbb{N}_+$ and, for every $\sigma \in \Sigma$, let g_σ be an $(n + 2r_\sigma)$-ary operation in $\mathrm{Pol}\,I$. We must show $h \in \mathrm{Pol}\,I$ for the $(n+1)$-ary operation $h := \mathrm{PR}(g_\sigma)_{\sigma \in \Sigma}$, i.e., $s_1, \dots, s_n, s \in I$ implies

$$h(s_1, \dots, s_n, s) \in I\,. \tag{2.3.1}$$

Thus let $s_1, \dots, s_n, s \in I$. We show 2.3.1 for $s \in T_k$ by induction on k:

For $s = \sigma \in \Sigma_0$ (i.e. $k = 0$) we have

$$h(s_1, \dots, s_n, s) = g_\sigma(s_1, \dots, s_n) \in I$$

because $g_\sigma \in \operatorname{Pol} I$ by assumption and $s_1, \ldots, s_n \in I$.

Now assume that 2.3.1 holds for every $s \in T_{k-1}$ $(k \geq 1)$. Let $s \in T_k$ be of the form $s = \sigma(t_1, \ldots, t_{r_\sigma})$, $r_\sigma \geq 1$. Then $t_1, \ldots, t_{r_\sigma} \in T_{k-1}$ and we get $h(s_1, \ldots, s_n, t_j) \in I$ for $j \in \{1, \ldots, r_\sigma\}$ by induction hypothesis. Note that $t_1, \ldots, t_{r_\sigma} \in I$ because I is an order ideal and $\sigma(t_1, \ldots, t_{r_\sigma}) = s \in I$ by assumption. Consequently

$$\begin{aligned} &h(s_1, \ldots, s_n, s) \\ &\qquad = g_\sigma(s_1, \ldots, s_n, t_1, \ldots, t_{r_\sigma}, h(s_1, \ldots, s_n, t_1), \ldots, h(s_1, \ldots, s_n, t_{r_\sigma})) \in I \end{aligned}$$

because $g_\sigma \in \operatorname{Pol} I$ and every argument of g_σ belongs to I. $\square$

The part (ii) $\Longrightarrow$ (iii) is crucial for the proof of Theorem 2.1. It is based on Proposition 2.5 which might be of independent interest (as well as Lemma 2.4): here we describe properties of the least S-closed clone $\langle\!\langle e \rangle\!\rangle_{\mathsf{S}}$.

2.4 Lemma. *Let $s, t \in T$ and $s \leq t$. Then there exists an operation $f_{t,s} \in \langle\!\langle e \rangle\!\rangle_{\mathsf{S}}$ such that $f_{t,s}(t) = s$.*

Proof. The case $t = s$ is trivial (take $f_{t,s} := e$). Thus assume $s < t$.

At first we consider the case $s \lessdot t$, i.e. t has the form $t = \delta(s_1, \ldots, s_k)$ with $s_i = s$ and $\delta \in \Sigma_k$ for some $k \geq 1$ and $i \in \{1, \ldots, k\}$. We construct a binary operation $h \in \langle\!\langle e \rangle\!\rangle_{\mathsf{S}}$ with $h(t,t) = s$. Define $(g'_\sigma)_{\sigma \in \Sigma}$ as follows:

$$g'_\sigma := \begin{cases} e & \text{for } \sigma \in \Sigma_0 \\ e^{1+r}_{1+\min\{i,r\}} & \text{for } \sigma \in \Sigma_r,\ r \in \mathbb{N}_+ \end{cases}.$$

In particular we have $g'_\delta := e^{1+k}_{1+i}$. We note that for $\sigma \in \Sigma \setminus \{\delta\}$ the operations g'_σ will play no essential role in the following and could be chosen arbitrarily in $\langle\!\langle e \rangle\!\rangle_{\mathsf{S}}$. Let $h := \mathrm{S}(g'_\sigma)_{\sigma \in \Sigma}$ (cf. 1.5.1 and 1.3 with $n = 1$). Since all g'_σ are projections (and therefore belong to $\langle e \rangle$) we have $h \in \langle\!\langle e \rangle\!\rangle_{\mathsf{S}}$. Further we have

$$\begin{aligned} h(t,t) &= h(t, \delta(s_1, \ldots, s_i, \ldots, s_k)) \\ &= g'_\delta(t, s_1, \ldots, s_i, \ldots, s_k) \\ &= s_i = s\,. \end{aligned}$$

Choosing $f_{t,s} := h(e,e) \in \langle\!\langle e \rangle\!\rangle_{\mathsf{S}}$ we have $f_{t,s}(t) = h(t,t) = s$ and consequently $f_{t,s} \in \langle\!\langle e \rangle\!\rangle_{\mathsf{S}}$ exists for $s \lessdot t$.

Finally, if s is not a maximal subtree of t, then there exists a chain

$$s = s_l \lessdot \ldots \lessdot s_1 \lessdot s_0 = t$$

and as shown above for every $j \in \{0, 1, \ldots, l-1\}$, there exists $f_{s_j, s_{j+1}} \in \langle\!\langle e \rangle\!\rangle_{\mathsf{S}}$ with $f_{s_j, s_{j+1}}(s_j) = s_{j+1}$; thus the composition

$$f := f_{s_{l-1}, s_l}(\ldots(f_{s_1, s_2}(f_{s_0, s_1}))\ldots)$$

satisfies $f(t) = s$ and belongs to $\langle\!\langle e \rangle\!\rangle_{\mathsf{S}}$. $\square$

2.5 Proposition. *For every $k \in \mathbb{N}$, $n, r \in \mathbb{N}_+$ and every family $(f_{b_1,\dots,b_r} \mid (b_1,\dots,b_r) \in T_k^r)$ of n-ary operations in $\langle\!\langle e \rangle\!\rangle_{\mathsf{S}}$, there exists an $(n+r)$-ary operation $h \in \langle\!\langle e \rangle\!\rangle_{\mathsf{S}}$ such that for all $a_1,\dots,a_n \in T$ and all $b_1,\dots,b_r \in T_k$ we have*

$$h(a_1,\dots,a_n,b_1,\dots,b_r) = f_{b_1,\dots,b_r}(a_1,\dots,a_n)\,. \tag{2.5.1}$$

Proof. In order to get h we construct (by double induction and using S-closure) auxiliary operations $h_{b_{i+1},\dots,b_r} \in \langle\!\langle e \rangle\!\rangle_{\mathsf{S}}$ satisfying the following statement $R(k,n,r,i)$ for $i \in \{0,1,\dots,r\}$:

$R(k,n,r,i) :\Longleftrightarrow$ [for every $b_{i+1},\dots,b_r \in T_k$, there is an $(n+i)$-ary operation $h_{b_{i+1},\dots,b_r} \in \langle\!\langle e \rangle\!\rangle_{\mathsf{S}}$ such that for every $\tilde{a} \in T^n$ and $b_1,\dots,b_i \in T_k$:

$$h_{b_{i+1},\dots,b_r}(\tilde{a},b_1,\dots,b_i) = f_{b_1,\dots,b_r}(\tilde{a})\,, \tag{2.5.2}$$

where

$$(f_{b_1,\dots,b_r} \mid (b_1,\dots,b_r) \in T_k^r) \text{ is an arbitrary family of } n\text{-ary operations in } \langle\!\langle e \rangle\!\rangle_{\mathsf{S}}. \tag{*}$$

We shall use the convention that $b_{i+1},\dots,b_r$ is the empty sequence for $i = r$ (i.e., $h_{b_{i+1},\dots,b_r}$ means h and $g_\sigma^{b_{i+1},\dots,b_r}$ below will mean g_σ in case $i = r$). Therefore, $R(k,n,r,r)$ is nothing else than the statement of Proposition 2.5, and we are done if we can prove

$$\forall k \in \mathbb{N}\,\forall n \in \mathbb{N}_+\,\forall r \in \mathbb{N}_+\,\forall (f_{b_1,\dots,b_r})_{\text{satisfying }(*)}\,\forall i \in \{0,1,\dots,r\} : R(k,n,r,i)\,. \tag{2.5.3}$$

We prove 2.5.3 by induction on k.

$\boxed{k=0}$ Let $n, r \in \mathbb{N}_+$ and $(f_{b_1,\dots,b_r} \mid (b_1,\dots,b_r) \in T_0^r)$ be a family of n-ary operations in $\langle\!\langle e \rangle\!\rangle_{\mathsf{S}}$. Note that $T_0 = \Sigma_0$. We prove $R(0,n,r,i)$ for every $i \in \{0,1,\dots,r\}$ by induction on i.

$\boxed{i=0}$ $R(0,n,r,0)$ trivially holds by defining

$$h_{b_1,\dots,b_r} := f_{b_1,\dots,b_r} \in \langle\!\langle e \rangle\!\rangle_{\mathsf{S}}\,. \tag{2.5.4}$$

$\boxed{i-1 \to i}$ Let $i \in \{1,\dots,r\}$ and assume that $R(0,n,r,i-1)$ holds, i.e., for every $b_i, b_{i+1},\dots,b_r \in T_0$ there exists $h_{b_i,b_{i+1},\dots,b_r} \in \langle\!\langle e \rangle\!\rangle_{\mathsf{S}}$ fulfilling 2.5.2 for every $\tilde{a} \in T^n$ and $b_1,\dots,b_{i-1} \in T_0$.

Now, let $b_{i+1},\dots,b_r$ be arbitrary elements in T_0. For every $\sigma \in \Sigma$ we define the $(n+i-1+r_\sigma)$-ary operation $g_\sigma^{b_{i+1},\dots,b_r}$ as follows:

$$g_\sigma^{b_{i+1},\dots,b_r} := \begin{cases} h_{\sigma,b_{i+1},\dots,b_r} & \text{if } \sigma \in \Sigma_0 \\ e_1^{n+i-1+r_\sigma} & \text{otherwise} \end{cases}\,. \tag{2.5.5}$$

All these operations are in $\langle\!\langle e\rangle\!\rangle_{\mathrm{S}}$ (for $\sigma \in \Sigma_0$ by induction hypothesis and otherwise by definition). Then the $(n+i)$-ary operation $h_{b_{i+1},\dots,b_r}$ is defined by selection closure as follows:

$$h_{b_{i+1},\dots,b_r} := \mathrm{S}(g_\sigma^{b_{i+1},\dots,b_r})_{\sigma\in\Sigma}\,. \tag{2.5.6}$$

Thus $h_{b_{i+1},\dots,b_r} \in \langle\!\langle e\rangle\!\rangle_{\mathrm{S}}$, too. Moreover, condition 2.5.2 is satisfied because for every $b_1,\dots,b_i \in T_0$ and $\tilde{a} \in T^n$ we have

$$\begin{aligned} h_{b_{i+1},\dots,b_r}(\tilde{a},b_1,\dots,b_i) &= g_{b_i}^{b_{i+1},\dots,b_r}(\tilde{a},b_1,\dots,b_{i-1}) && \text{by 2.5.6 (cf. 1.5.3)}\\ &= h_{b_i,b_{i+1},\dots,b_r}(\tilde{a},b_1,\dots,b_{i-1}) && \text{by 2.5.5}\\ &= f_{b_1,\dots,b_r}(\tilde{a}) && \text{by } R(0,n,r,i-1). \end{aligned}$$

Thus $R(0,n,r,i)$ holds for all $n,r \in \mathbb{N}_+$ and $i \in \{0,1,\dots,r\}$.

We can continue the induction on k.

$\boxed{k-1 \to k}$ Let $k \geq 1$. By induction we assume that $R(k-1,n',r',i')$ holds for every $n',r' \in \mathbb{N}_+$, every family $(f_{b_1,\dots,b_{r'}} \mid (b_1,\dots,b_{r'}) \in T_{k-1}^{r'})$ of n'-ary operations in $\langle\!\langle e\rangle\!\rangle_{\mathrm{S}}$ and every $i' \in \{0,1,\dots,r'\}$. Now, let $n,r \in \mathbb{N}_+$ and let $(f_{b_1,\dots,b_r} \mid (b_1,\dots,b_r) \in T_k^r)$ be a family of n-ary operations in $\langle\!\langle e\rangle\!\rangle_{\mathrm{S}}$. We prove $R(k,n,r,i)$ for every $i \in \{0,1,\dots,r\}$ by induction on i.

$\boxed{i=0}$ $R(k,n,r,0)$ trivially holds as in case $k=0$ and $i=0$ (cf. 2.5.4).

$\boxed{i-1 \to i}$ Let $i \in \{1,\dots,r\}$ and assume that $R(k,n,r,i-1)$ holds. We prove $R(k,n,r,i)$. Thus let $b_{i+1},\dots,b_r$ be arbitrary elements in T_k. Let $\sigma \in \Sigma$ and consider the family $(f^\sigma_{t_1,\dots,t_{r_\sigma}} \mid t_1,\dots,t_{r_\sigma} \in T_{k-1})$ of $(n+i-1)$-ary operations given by

$$f^\sigma_{t_1,\dots,t_{r_\sigma}} := h_{\sigma(t_1,\dots,t_{r_\sigma}),b_{i+1},\dots,b_r}\,. \tag{2.5.7}$$

(Let us agree to include the case $r_\sigma = 0$ just by deleting all $t_1,\dots,t_{r_\sigma}$ whenever they appear, i.e. $f^\sigma := h_{\sigma,b_{i+1},\dots,b_r}$.) According to 2.5.7, all operations in this family exist and belong to $\langle\!\langle e\rangle\!\rangle_{\mathrm{S}}$ by induction hypothesis $R(k,n,r,i-1)$ (note that $\sigma(t_1,\dots,t_{r_\sigma}) \in T_k$ for $t_1,\dots,t_{r_\sigma} \in T_{k-1}$). Now, by induction hypothesis $R(k-1,n+i-1,r_\sigma,r_\sigma)$, there exists an $(n+i-1+r_\sigma)$-ary operation $g_\sigma^{b_{i+1},\dots,b_r} \in \langle\!\langle e\rangle\!\rangle_{\mathrm{S}}$ such that

$$g_\sigma^{b_{i+1},\dots,b_r}(a_1,\dots,a_{n+i-1},t_1,\dots,t_{r_\sigma}) = f^\sigma_{t_1,\dots,t_{r_\sigma}}(a_1,\dots,a_{n+i-1}) \tag{2.5.8}$$

for every $a_1,\dots,a_{n+i-1} \in T$ and $t_1,\dots,t_{r_\sigma} \in T_{k-1}$ (by our convention, it follows from 2.5.7 and 2.5.8 that $g_\sigma^{b_{i+1},\dots,b_r}$ is defined for $r_\sigma = 0$ as in 2.5.5). Now, by S-closure, we define the $(n+i)$-ary operation $h_{b_{i+1},\dots,b_r} = \mathrm{S}(g_\sigma^{b_{i+1},\dots,b_r})_{\sigma\in\Sigma}$ as in 2.5.6. Thus $h_{b_{i+1},\dots,b_r} \in \langle\!\langle e\rangle\!\rangle_{\mathrm{S}}$. Moreover, let $b_1,\dots,b_{i-1},b_i \in T_k$ and let b_i be of the form $b_i = \sigma(t_1,\dots,t_{r_\sigma})$ for some $\sigma \in \Sigma$, then $t_1,\dots,t_{r_\sigma} \in T_{k-1}$ and we have for every

$\tilde{a} \in T^n$

$$\begin{aligned} h_{b_{i+1},\dots,b_r}(\tilde{a}, b_1, \dots, b_i) &= g_\sigma^{b_{i+1},\dots,b_r}(\tilde{a}, b_1, \dots, b_{i-1}, t_1, \dots, t_{r_\sigma}) && \text{by 2.5.6} \\ &= f^\sigma_{t_1,\dots,t_{r_\sigma}}(\tilde{a}, b_1, \dots, b_{i-1}) && \text{by 2.5.8} \\ &= h_{\sigma(t_1,\dots,t_{r_\sigma}),b_{i+1},\dots,b_r}(\tilde{a}, b_1, \dots, b_{i-1}) && \text{by 2.5.7} \\ &= f_{b_1,\dots,b_r}(\tilde{a}) && \text{by } R(k,n,r,i-1). \end{aligned}$$

Thus $R(k, n, r, i)$ also holds. Both inductions (on i and on k) are done and 2.5.3 is proved. As mentioned before 2.5.3, this finishes the proof of Proposition 2.5. □

2.6 Remark. From the proofs it follows that 2.4 and 2.5 remain true if $\langle\!\langle e \rangle\!\rangle_{\mathsf{S}}$ is substituted by $\langle\!\langle e \rangle\!\rangle_{\mathsf{PR}}$.

Now we can proceed with the

Proof of 2.1 (ii) $\Longrightarrow$ (iii).

Let ϱ be an m-ary reduced relation ($m \in \mathbb{N}_+$) and $\langle\!\langle \operatorname{Pol} \varrho \rangle\!\rangle_{\mathsf{S}} = \operatorname{Pol} \varrho$. We have to show that ϱ is of the form as indicated in 2.1(iii). For $j \in \{1, \dots, m\}$ let

$$I_j := \{t_j \in T \mid \exists\, t_1, \dots, t_m : (t_1, \dots, t_j, \dots, t_m) \in \varrho\} \tag{2.6.1}$$

be the set of all j-th components of ϱ.

At first we show that every I_j is an order ideal. In fact, let $s < t$ and $t \in I_j$, then by Lemma 2.4 there exists an operation $f_{t,s} \in \langle\!\langle e \rangle\!\rangle_{\mathsf{S}} \subseteq \langle\!\langle \operatorname{Pol} \varrho \rangle\!\rangle_{\mathsf{S}}$ with $f_{t,s}(t) = s$. Applying $f_{t,s}$ to an m-tuple in ϱ with t in its j-th component we get an m-tuple in ϱ with s in its j-th component, i.e. $s \in I_j$. Thus I_j is an order ideal.

Now we are going to show that $\varrho = I_1 \times \cdots \times I_m$ which will finish the proof of 2.1(ii) $\Longrightarrow$ (iii) (and thus finish the proof of Theorem 2.1).

Let $t_{11} \in I_1, \dots, t_{mm} \in I_m$. We have to show $(t_{11}, \dots, t_{mm}) \in \varrho$. By 2.6.1 there exist elements $(t_{1j}, \dots, t_{jj}, \dots, t_{mj}) \in \varrho$ with t_{jj} as j-th component ($j \in \{1, \dots, m\}$), which we represent as columns of an $(m \times m)$-matrix. If not all rows are different then we can add some further, say $r - m$, columns, which we denote by $(t_{1j}, \dots, t_{mj}) \in \varrho$, $j = m+1, \dots, r$, such that now all rows of the corresponding matrix A are pairwise different (this is possible because ϱ is a reduced relation):

$$A = \begin{pmatrix} t_{11} & \dots & t_{1m} & \dots & t_{1r} \\ \vdots & \ddots & \vdots & \dots & \vdots \\ t_{m1} & \dots & t_{mm} & \dots & t_{mr} \end{pmatrix}.$$

Let k be the maximal height of all trees t_{ij} ($i = 1, \dots, m$, $j = 1, \dots, r$). By Proposition 2.5 there exists a $2r$-ary operation $h \in \langle\!\langle e \rangle\!\rangle_{\mathsf{S}}$ such that

$$h(a_1, \dots, a_r, t_{i1}, \dots, t_{ir}) = a_i \tag{2.6.2}$$

for every $i \in \{1,\dots,m\}$ and $a_1,\dots,a_r \in T$. In fact, in 2.5 take $n = r$, and let $f_{b_1,\dots,b_r} = e_i^r$ for $(b_1,\dots,b_r) = (t_{i1},\dots,t_{ir})$, $i \in \{1,\dots,m\}$, and arbitrary in $\langle\!\langle e \rangle\!\rangle_{\mathsf{S}}$ otherwise (e.g. $f_{b_1,\dots,b_r} := e_1^r$). Consequently, the operation f defined by

$$f(a_1,\dots,a_r) := h(a_1,\dots,a_r,a_1,\dots,a_r) \tag{2.6.3}$$

(i.e. $f = h(e_1^r,\dots,e_r^r,e_1^r,\dots,e_r^r)$) also belongs to $\langle\!\langle e \rangle\!\rangle_{\mathsf{S}} \subseteq \langle\!\langle \mathrm{Pol}\,\varrho \rangle\!\rangle_{\mathsf{S}} = \mathrm{Pol}\,\varrho$.

Applying f to the matrix A row-wise we obtain (cf. 2.6.2 and 2.6.3) the m-tuple $(t_{11},\dots,t_{mm})$; it must belong to ϱ because all columns of A are in ϱ and f preserves ϱ. □

3 Further research and remarks

3.1. Connections between closure operators. For $F \subseteq \mathrm{Op}(T)$, let $\mathrm{PR}(F)$ denote the set which contains F and all tree functions $h = \mathrm{PR}(g_\sigma)_{\sigma\in\Sigma}$ with $g_\sigma \in F$. Further let $\mathrm{PR}^n(F) := \bigcup_{i=1}^n \mathrm{PR}^i(F)$, where $\mathrm{PR}^1(F) := \mathrm{PR}(F)$ and $\mathrm{PR}^{i+1}(F) := \mathrm{PR}(\mathrm{PR}^i(F))$ $(i \in \mathbb{N}_+)$. Then

$$\mathrm{PR}^*\,F := \bigcup_{i=1}^{\infty} \mathrm{PR}^i(F)$$

is the least PR-closed set of tree functions containing F. The mapping $F \mapsto \mathrm{PR}^*\,F$ is a closure operator as well as $F \mapsto \langle F \rangle$ and $F \mapsto \langle\!\langle F \rangle\!\rangle_{\mathsf{PR}}$. By definition (cf. 1.3) we have

$$\begin{aligned} \langle \langle\!\langle F \rangle\!\rangle_{\mathsf{PR}} \rangle &= \langle\!\langle F \rangle\!\rangle_{\mathsf{PR}} \qquad & \langle\!\langle \langle F \rangle \rangle\!\rangle_{\mathsf{PR}} &= \langle\!\langle F \rangle\!\rangle_{\mathsf{PR}} \\ \mathrm{PR}^*\,\langle\!\langle F \rangle\!\rangle_{\mathsf{PR}} &= \langle\!\langle F \rangle\!\rangle_{\mathsf{PR}} \qquad & \langle\!\langle \mathrm{PR}^*\,F \rangle\!\rangle_{\mathsf{PR}} &= \langle\!\langle F \rangle\!\rangle_{\mathsf{PR}}\,. \end{aligned}$$

It is still an open question how the iterations of the operators PR^* or PR and $\langle . \rangle$ behave in general. E.g. do we have

$$\langle\!\langle F \rangle\!\rangle_{\mathsf{PR}} = \mathrm{PR}^* \langle \dots \langle \mathrm{PR}^* \langle \mathrm{PR}^* \langle F \underbrace{\rangle\rangle \rangle \dots \rangle}_{n} \tag{3.1.1}$$

$$\text{or} \quad \langle\!\langle F \rangle\!\rangle_{\mathsf{PR}} = \mathrm{PR} \langle \dots \langle \mathrm{PR} \langle \mathrm{PR} \langle F \underbrace{\rangle\rangle \rangle \dots \rangle}_{n} \tag{3.1.2}$$

for a fixed finite number n of iterations of PR^* or PR and $\langle . \rangle$? Or, if the answer is negative, for which F does this iteration stabilize after a finite number of steps?

For instance, if $F = \mathrm{PREC}_\Sigma$ (cf. 1.4b), then obviously $\langle\!\langle F \rangle\!\rangle_{\mathsf{PR}} = F$. On the other hand, if $F = F_{\mathrm{base}}$ (cf. 1.4b), then $\langle\!\langle F \rangle\!\rangle_{\mathsf{PR}} = \bigcup_{n=1}^{\infty} \langle \mathrm{PR} \rangle^n(F) = \bigcup_{n=1}^{\infty} \langle \mathrm{PR}^* \rangle^n(F)$ where $\langle \mathrm{PR} \rangle^n(F)$ and $\langle \mathrm{PR}^* \rangle^n(F)$ is an abbreviation of the right side of 3.1.2 and 3.1.1, respectively. Here, in general, we really need the union over all $n \in \mathbb{N}_+$ (the first union reflects the Grzegorczyk-hierarchy).

Analogous questions arise with the closure $\langle\!\langle F \rangle\!\rangle_{\mathsf{S}}$ instead of $\langle\!\langle F \rangle\!\rangle_{\mathsf{PR}}$.

3.2. Specialization to $\mathbb{N}$. Let us specialize the signature Σ to the signature of the natural numbers $\langle \mathbb{N}; \mathrm{succ}, 0\rangle$ with successor function $n \mapsto \mathrm{succ}\, n$ ($\mathrm{succ}\, n = n+1$) and the constant 0. Then $\Sigma = \{\mathrm{succ}, 0\}$ and the set T of trees over σ (cf. 1.2) can be identified with $\mathbb{N}$. Primitive recursion (as defined in 1.3) is just the usual primitive recursion for operations on natural numbers. This case was studied in detail in, e.g., [Pét57]. In [Sem02] Theorem 2.1 has been proved for this particular signature. Then an order ideal in $\mathbb{N}$ is either a principal ideal or the whole set $\mathbb{N}$. Thus (cf. Remark 2.2), in order to describe PR-closed clones of the form $\mathrm{Pol}\, \varrho$, one can restrict to relations ϱ that are the product of principal ideals $I_j = \{0, 1, \dots, a_j\}$ of $\mathbb{N}$. Moreover, in [Sem02], it was proved that if Q is a set of finitary reduced relations such that $\langle\!\langle \mathrm{Pol}\, Q \rangle\!\rangle_{\mathsf{PR}} = \mathrm{Pol}\, Q$, then $\langle\!\langle \mathrm{Pol}\, \varrho \rangle\!\rangle_{\mathsf{PR}} = \mathrm{Pol}\, \varrho$ for each relation $\varrho \in Q$. Further, the partially ordered set of clones of the form $\mathrm{Pol}\, Q$, where Q is as above, is isomorphic to the lattice of all subsets of $\mathbb{N}$ ordered by inclusion. In addition, it was shown what happens if one considers only recursive or primitive recursive polymorphisms instead of all possible polymorphisms.

It is still unknown how these results can be generalized to arbitrary tree functions.

3.3. Generalization to infinitary relations. Theorem 2.1 shows that PR-closed classes of tree functions characterizable by a finitary relation have a very simple structure (cf. 2.2).

However, it was also mentioned in 2.2 that for infinitary relations ϱ, $\langle\!\langle \mathrm{Pol}\, \varrho \rangle\!\rangle_{\mathsf{PR}} = \mathrm{Pol}\, \varrho$ in general does not imply that ϱ is the direct product of order ideals. To give an example, consider the signature $\Sigma = \{\mathrm{succ}, 0\}$ as in 3.2 and the infinitary ($|\mathbb{N}|$-ary) relation $\varrho \subseteq \mathbb{N}^{\mathbb{N}}$ defined by

$$\varrho := \{g \in \mathbb{N}^{\mathbb{N}} \mid g \text{ is a unary primitive recursive function}\}\,.$$

<u>Claim:</u> $\mathrm{Pol}\, \varrho$ *is the set of all primitive recursive functions over* $\mathbb{N}$.

Proof of the claim:
At first recall that an operation $f : \mathbb{N}^n \to \mathbb{N}$ preserves ϱ iff $g_1, \dots, g_n \in \varrho$ implies $f(g_1, \dots, g_n) \in \varrho$ where $f(g_1, \dots, g_n) : \mathbb{N} \to \mathbb{N}$ is the composition defined by

$$f(g_1, \dots, g_n)(x) := f(g_1(x), \dots, g_n(x)) \tag{3.3.1}$$

for $x \in \mathbb{N}$; this is the obvious generalization of 1.1.3 to the infinitary case.

By definition, the composition $f(g_1, \dots, g_n)$ of primitive recursive functions $f, g_1, \dots, g_n$ is again primitive recursive. Thus every primitive recursive $f : \mathbb{N}^n \to \mathbb{N}$ preserves ϱ, i.e., $\mathrm{Pol}\, \varrho$ contains the set of all primitive recursive functions on $\mathbb{N}$.

Conversely, every $f \in \mathrm{Pol}\, \varrho$ is primitive recursive.

Indeed, let $f \in \mathrm{Pol}\, \varrho$ be unary, then $f(e) \in \varrho$ because $e \in \varrho$; thus $f = f(e)$ is also primitive recursive.

Now suppose that $f \in \mathrm{Pol}\, \varrho$ is n-ary, where $n > 1$. In this case, we use the fact (see, for example, [Pét57]) that there exist unary primitive recursive functions g_1, g_2 and a binary primitive recursive function h such that $g_1(h(x_1, x_2)) = x_1$ and

$g_2(h(x_1, x_2)) = x_2$ for all $x_1, x_2 \in \mathbb{N}$. Now define the functions

$$\begin{array}{lll} c_2(x_1,x_2) := h(x_1,x_2), & c_{i+1}(x_1,\dots,x_{i+1}) := h(c_i(x_1,\dots,x_i),x_{i+1}) & \text{for } i>1, \\ l_1(x) := g_1(x), & l_{j+1}(x) := g_1(l_j(x)) & \text{for } j \in \mathbb{N}_+, \\ r_1(x) := g_2(x), & r_{j+1}(x) := g_2(l_j(x)) & \text{for } j \in \mathbb{N}_+. \end{array}$$

Clearly, the functions c_{j+1}, l_j, and r_j are primitive recursive for $j \in \mathbb{N}_+$. It can easily be checked that

$$r_j(c_i(x_1,\dots,x_i)) = x_{i-j+1} \qquad \text{for } j \in \mathbb{N}_+ \text{ and } i > j, \tag{3.3.2}$$

$$l_j(c_{j+1}(x_1,\dots,x_{j+1})) = x_1 \qquad \text{for } j \in \mathbb{N}_+. \tag{3.3.3}$$

Since $f \in \operatorname{Pol}\varrho$ and $l_{n-1}, r_{n-1}, r_{n-2}, \dots, r_1 \in \varrho$ we have

$$g := f(l_{n-1}, r_{n-1}, r_{n-2}, \dots, r_1) \in \varrho\,,$$

i.e., g is primitive recursive. Using 3.3.2 and 3.3.3, we get $f(x_1,\dots,x_n) = g(c_n(x_1,\dots,x_n))$. Hence f is primitive recursive because it is a composition of the primitive recursive functions g and c_n. $\square$

From the above claim we know $\langle\!\langle \operatorname{Pol}\varrho \rangle\!\rangle_{\mathsf{PR}} = \operatorname{Pol}\varrho$. On the other hand, ϱ is not the direct product of order ideals of $\mathbb{N}$.

Thus ϱ is a counterexample to the straightforward generalization of Theorem 2.1 to infinitary relations and there arises the problem how this theorem could be extended appropriately to infinitary relations.

3.4. C-closed clones. A clone F of tree functions shall be called C-*closed* if it is PR-closed (i.e. $\langle\!\langle F \rangle\!\rangle_{\mathsf{PR}} = F$) and closed under *iteration*. The latter means (cf. e.g. [Hup78]) that if $f, h, g_1, \dots, g_n$ are n-ary tree functions in F then every tree function $k : T^n \to T$, which is definable by a program of the following form for some $t \in T$, must also belong to F.

$$\begin{array}{rl} \text{WHILE} & f(x_1,\dots,x_n) \neq t \\ \text{DO} & x_1 := g_1(x_1,\dots,x_n); \\ & \vdots \\ & x_n := g_n(x_1,\dots,x_n) \\ \text{OD;} & \\ \text{OUTPUT} & h(x_1,\dots,x_n) \end{array}$$

Note that here we consider only total operations k defined by iteration (while in [Hup78] also partial operations are allowed). The C-closure is stronger than the PR-closure, e.g., let $\langle\!\langle F \rangle\!\rangle_{\mathsf{C}}$ denote the smallest C-closed clone containing F, then, as shown in [Hup78], $\langle\!\langle F_{\text{base}} \rangle\!\rangle_{\mathsf{C}}$ is the set of all computable tree functions (for F_{base} see 1.4b).

Nevertheless, as it was pointed out by the referee, clones of the form $\operatorname{Pol}\varrho$ with ϱ as in Theorem 2.1(iii) are C-closed. Thus Theorem 2.1 can be extended by an additional equivalent condition

(iv) $\langle\!\langle \mathrm{Pol}\,\varrho \rangle\!\rangle_{\mathsf{C}} = \mathrm{Pol}\,\varrho$.

Obviously $\langle\!\langle F \rangle\!\rangle_{\mathsf{S}} \subseteq \langle\!\langle F \rangle\!\rangle_{\mathsf{PR}} \subseteq \langle\!\langle F \rangle\!\rangle_{\mathsf{C}}$. It is an open question to characterize the least and (if it exists) the largest closure which agrees with $\langle\!\langle F \rangle\!\rangle_{\mathsf{PR}}$ (or equivalently with $\langle\!\langle F \rangle\!\rangle_{\mathsf{S}}$, $\langle\!\langle F \rangle\!\rangle_{\mathsf{C}}$) for clones of the form $\mathrm{Pol}\,\varrho$. More precisely, let $\mathcal{K}$ be the class of all closure operators $K : \mathfrak{P}(\mathrm{Op}(T)) \to \mathfrak{P}(\mathrm{Op}(T))$ on tree functions such that $K(\mathrm{Pol}\,\varrho) = \langle\!\langle \mathrm{Pol}\,\varrho \rangle\!\rangle_{\mathsf{PR}}$ for all finitary relations $\varrho \subseteq T^m$ ($m \in \mathbb{N}_+$). Then

$$K_0 : F \mapsto \bigcap_{K \in \mathcal{K}} K(F)$$

is the least closure operator in $\mathcal{K}$, but it is not clear how it could be characterized internally. Moreover, does there exist a largest closure operator in $\mathcal{K}$?

3.5. Further generalizations. The preservation (or invariance) property (cf. 1.1) constitutes a Galois connection between sets of operations and relations. There are many generalizations and modifications of this Galois connection changing the operations and/or relations under consideration (see e.g. [Pös01]). A systematic investigation of operations, their invariant relations and various closures, which are of special interest for computer science, would be desirable. In connection with tree functions and primitive recursion the class of partial tree functions may be of particular interest. Then the C-closure (cf. 3.4) might play the role of the PR-closure in Theorem 2.1. However note that the C-closure still can be extended further: the condition that the operation f used in the iteration program in 3.4 belongs to F can be dropped (without changing the property that k preserves ϱ whenever $h, g_1, \ldots, g_n$ do).

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