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Relationships Between Closure Operations and Choice Functions – Equivalent Descriptions of a Family of Functional Dependencies

Nghia D. Vu*

Abstract

The family of functional dependencies plays an important role in the relational database. The main goal of this paper is to investigate closure operations and choice functions. They are equivalent descriptions of family of functional dependencies. The main properties of and relationship between closure operations and choice functions are presented in this paper.

1 Introduction

The motivation of this study is equivalent descriptions of family of functional dependencies (FDs). FDs play a significant role in the implementations of relational database model, which was defined by E.F Codd. However, relational database is still one of the most powerful databases. One of the most important branches in the theory of relational database is that dealing with the design of database schemes. This branch is based on the theory of FDs and constraints. Armstrong observed that FDs give rise to closure operations on the set of attributes. And he shows that closure operation is an equivalent description of family of FDs, that is, the family of all FDs satisfying Armstrong axiom stated in next section. That the family of FDs can be described by closure operations on the attributes' set plays a very important role in theory of relational database. Because this representation was successfully applied to find many properties of FDs, studying those properties of closure operations is indirect way of finding that of the family of FDs. Besides closure operations, there are some other representations of family of FDs. Such as, the closed sets of a closure form a semilattice. And the semilattice with greatest elements gives an equivalent description of FDs. The closure operations, and other equivalent descriptions of family of FDs have been studied widely by Armstrong [Ar], Beeri, Dowd, Fagin and Statman [BDFS], and H. Mannila and K.J.Raiha [MR]. More, see [DK2], [DHLM], [DT3], and [Li]. Studying equivalent descriptions of family of FDs helps us to understand deeper the family FDs and widens the

*Institute of Information Technology of Vietnam, Department of Database Management System, 18 Hoang Quoc Viet, Hanoi, Vietnam, email: nghiavu@cse.buffalo.edu

study of it. Closure operation is widely known and considered the representation of family of functional dependencies most studied. Among equivalent descriptions of functional dependencies, the properties of choice functions are not developed well enough in contrast to those of closure operations. Moreover, a closure operation can be derived from a choice function and vice versa. Thus, by studying properties of choice function satisfying reverse inclusion was studied in connection with the theory of rational behavior of individuals and groups. For the study on choice function and relationship between closure operations and choice functions, see [DHLM] and [Li].

For relation schemes $s =< U, F >$ and $t =< U, V >$, where U is a set of attributes and F and V sets of FDs over U, we are always able to build a closure $L_1(A)$ on F, for every A is a set of attributes on U. However, if we build $L_2(L_1(A))$ on V, we find out that a meet-semilattice can not be formed from this computation. That is, we can not form a relation scheme from this computation. We are going to show in this paper what condition that provide to build the composition $L_1(L_2(A))$ such that a relation scheme can be formed from this composition. In other words, what is necessary and sufficient conditions that make sure $L_1(L_2(A))$ is a closure. We find this result through the studies of choice functions. Besides that, many properties of choice functions will be studies in depth. The interaction of choice functions and closure operations also are investigated widely in this paper. We also study the relationship between choice functions and FDs. Those results can be used to build many algorithm problems related to choice functions and closure operation and family of FDs.

Direct product of decomposition of a closure operation plays an important role in the theory and practice of relational database. If we consider a relation of database as a matrix, a row contains the data of one individual, the estimation of the minimum cardinality of rows of such matrix is very valuable in practice of relational database. The studies of estimation of the minimum cardinality of rows for direct product of decomposition of a closure operation can be found variously in [DFK], [Li], [DK2]. In this paper we present the new notion and properties of direct product of decomposition of choice function.

In the next section some necessary definitions and facts about relational database, some equivalent descriptions of family of functional dependencies besides choice function and closure operation theory are given.

The result of this paper is presented in the third section. They are organized into six parts as follows.

Part 1 represents necessary and sufficient condition of composition of choice functions to be a choice function. The studies of composition of closure operations has been shown through those of choice functions. The main result of this paper will be presented in depth in Part 1.

The direct product of decomposition of a choice function is in Part 2.

It will be proposed in Part 3 to study some fundamental properties of a composition of closure operations and choice functions. We are giving important properties of intersection, union, and composition of choice functions, which will be fully investigated in depth in Part 1.

In Part 4, we show relationship between and interactive properties of closure operations and choice functions. In this section we consider the closure for which choice function defined in next section satisfies some additional properties.

In Part 5, we are presenting a class of special choice functions, which is very useful in the studying of combinatorial problems related to choice functions and closure operations.

Part 6 gives some special relationship between choice functions and family of FDs, which helps us intensively into algorithm problems on building choice functions and closure operations. Since the theoretical result presented here are preliminary. Thus many open problems in the studies of choice functions and closure operation will be shown in this paper.

2 Basic Definitions

Let us give some formal definitions that are used in the next sections. Those well-known concepts in relational database given in this section can be found in [Ar, BB, BDFS, DK2, DT1, and Ul].

A relational database system of the scheme $R(a_1, ..., a_n)$ is considered as a table, where columns correspond to the attributes ai 's while the row are $n-$tuples of relation r. Let X and Y be nonempty sets of attributes in R. We say that instance r of R satisfies the FD if two tuples agree on the values in attributes X, they must also agree on the values in attributes Y. Here is the formal mathematical definition of FDs.

Definition 2.1. *Let $U = \{a_1, ..., a_n\}$ be a nonempty finite set of attributes. A functional dependency is a statement of the form $A \to B$, where $A, B \subseteq U$. The FD A B holds in a relation $R = \{h_1, ..., h_m\}$ over U if $\forall h_i, h_j \in R$ we have $h_i(a) = h_j(a)$ for all $a \in A$ implies $h_i(b) = h_j(b)$ for all $b \in B$. We also say that R satisfies the FD $A \to B$.*

Let F_R be a family of all FDs that hold in R.

Definition 2.2. *Then $F = F_R$ satisfies.*
(1) $A \to A \in F$,
(2) $(A \to B \in F, B \to C \in F) \Longrightarrow (A \to C \in F)$,
(3) $(A \to B \in F, A \subseteq C, D \subseteq B) \Longrightarrow (C \to D \in F)$,
(4) $(A \to B \in F, C \to D \in F) \Longrightarrow (A \cup C \to B \cup D \in F)$.

A family of FDs satisfying (1)-(4) is called an f-family over U.

Clearly, F_R is an f-family over U. It is known [Ar] that if F is an arbitrary f-family, then there is a relation R over U such that $F_R = F$.

Given a family F of FDs over U, there exits a unique minimal f-family F^+ that contains F. It can be seen that F^+ contains all FDs which can be derived from F by the rules (1)-(4).

Definition 2.3. *A relation scheme s is a pair $< U, F >$, where U is a set of attributes, and F is a set of FDs over U.*

Denote $A^+ = \{a : A \to \{a\} \in F^+\}$. A^+ is called the closure of A over s. It is clear that $A \to B \in F^+$ iff $B \subseteq A^+$. Clearly, if $s =< U, F >$ is a relation scheme, then there is a relation R over U such that $F_R = F+$ (see, [Ar]).

Definition 2.4. *Let U be a nonempty finite set of attributes and $P(U)$ its power set. A map $L : P(U) \to P(U)$ is called a closure operation (closure for short) over U if it satisfies the following conditions:*

(1)$A \subseteq L(A)$, (Extensiveness Property)

(2)$A \subseteq B$ implies $L(A) \subseteq L(B)$, (Monotonicity Property)

(3) $L(L(A)) = L(A)$. (Closure Property)

Let $s =< U, F >$ be a relation scheme. Set $L(A) = \{a : A \to \{a\} \in F^+\}$, we can see that L is a closure over U.

Theorem 2.1. *[Ar] If F is a f-family and if $L_F = \{a : a \in U$ and $A \to \{a\} \in F\}$, then L_F is a closure. Inversely, if L is a closure, there exists only a f-family F over U such that $L = L_F$, and $F = \{A \to B : A, B \subseteq U, B \subseteq L(A)\}$.*

Let $L \subseteq P(U)$. L is called a meet-irreducible family over U (sometimes it is called a family of members which are not intersection of two other members) if $A, B, C \in L$, then $A = BC$ implies $A = B$ or $A = C$.

Let $I \subseteq P(U), U \in I$, and $A, B \in I \Rightarrow A \cap B \in I$. I is called a meet-semilattice over U. Let $M \subseteq P(U)$.

Denote $M^+ = \{\cap M' : M' \subseteq M\}$. We say that M is a generator of I if $M^+ = I$. Note that $U \in M^+$ but not in M, by convention it is the intersection of the empty collection of sets. Denote $N = \{A \in I : A \neq \cap\{A' \in I : A \subset A'\}\}$. In [DK2] it is proved that N is the unique minimal generator of I.

It can be seen that N is a family of members which are not intersections of two other members.

Let L be a closure operation over U. Denote $Z(L) = \{A : L(A) = A\}$ and $N(L) = \{A \in Z(L) : A \neq \cap\{A' \in Z(L) : A \subset A'\}\}$. $Z(L)$ is called the family of closed sets of L. We say that $N(L)$ is the minimal generator of L.

It is shown [DK2] that if N is a meet-irreducible family then there is a closure L such that N is the minimal generator of it.

Theorem 2.2. *[Ar] There is an on-to-one correspondence between meet- irreducible families and f-families on U.*

Theorem 2.3. *[DK2] There is a 1-1 correspondence between meet-irreducible families and meet-semilattices on U.*

Definition 2.5. *Let $M \subseteq P(U)$. M is called a Sperner system over U if $A, B \in M$, then A is not a subset of B.*

Definition 2.6. *Let U be a nonempty finite set of attributes. A family $M = \{(A, \{a\}) : A \subset U, a \in U\}$ is called a maximal family of attributes over R iff the following conditions are satisfied:*

(1) $a \notin A$,

(2) For all $(B, \{b\}) \in M, a \notin B$ *and* $A \subseteq B$ *imply*$A = B$.

(3) $\exists(B, \{b\}) \in M : a \notin B, a \neq b$, *and* $L_a \cup B$ *is a Sperner system over* R, *where* $L_a = \{A : (A, \{a\}) \in M\}$.

Remark 2.1.

- It is possible that there are $(A, \{a\}), (B, \{b\}) \in M$ such that $a \neq b$, but $A = B$.
- It can be seen that by (1) and (2) for each$a \in U, L_a$ is a Sperner system over U. It is possible that L_a is an empty Sperner system.
- Let U be a nonempty finite set of attribute and $P(U)$ its power set. According to Definition 2.6 we can see that given a family $Y \subseteq P(U) \times P(U)$ there is a polynomial time algorithm deciding whether Y is a maximal family of attribute over U.

Let L be a closure over R. Denote $Z(L) = \{A : L(A) = A\}$ and $M(L) = \{(A, \{A\}) : A \notin A, A \in Z(L)$ and $B \in Z(L), A \subseteq B, A \notin B$ imply $A = B\}$.

$Z(L)$ is called the family of closed sets of L. It can be seen that for each $(A, \{a\}) \in M(L).A$is a maximal closed set which doesn't contain a.

It is possible that there are $(A, \{a\}), (B, \{b\}) \in M(L)$ such that $a \neq b$, but $A = B$.

The following theorem which shows that closure operations and maximal families of attributes determine each other uniquely.

Theorem 2.4. *[DT4] Let*L *be a closure operation over* U. *Then* $M(L)$ *is a maximal family of attributes over* U. *Conversely, if* M *is a maximal family of attributes over*U, *then there exists exactly one closure operation* L *over* U *so that* $M(L) = M$, *where for all* $B \in P(U)$

$$H(B) = \begin{cases} \bigcap\limits_{\mathrm{B} \subseteq A} \mathrm{A} \ \textit{if}\ \exists A \in \mathrm{L(M)} : \mathrm{B} \subseteq A, \\ \qquad \mathrm{R}\ \textit{otherwise}, \end{cases}$$

and $L(M) = \{a : (a, \{a\}) \in M\}$.

Now, we introduce the following concept.

Definition 2.7. *Let*$Y \in P(U) \times P(U)$. *We say that* Y *is a minimal family over*U *if the following conditions are satisfied:*

(1) $\forall(A, B), (A', B') \in Y : A \subset B \subseteq U, A \subset A'$ *implies* $B \subset B', A \subset B'$ *implies* $B \subseteq B'$,

(2) Put $U(Y) = \{B : (A, B) \in Y\}$. *For each* $B \in U(Y)$ *and* C *such that* $C \subset B$ *and there is no* $B' \in U(Y) : C \subset B' \subset B$, *there is an* $A \in L(B) : A \subseteq C$, *where* $L(B) = \{A : (A, B) \in Y\}$.

Remark 2.2.

- $U \in U(Y)$.
- From $A \subset B'$ implies $B \subseteq B'$ there is no a $B' \in U(Y)$ such that $A \subset B' \subset B$ and $A = A'$ implies $B = B'$.
- Because $A \subset A'$ implies $B \subset B'$ and $A = A'$ implies $B = B'$, we can be see that $L(B)$ is a Sperner system over R and by (2) $L(B) \neq \emptyset$.

Let I be a meet-semilattice over R. Put $M^*(I) = \{(A, B) : \exists C \in I$ such that $A \subset C, A \neq \cap\{C : C \in I, A \subset C\}, B = \cap\{C : C \in I, A \subset C\}\}$. Set $M(I) = \{(A, B) \in M^*(I) :$ there does not exist $(A', B) \in M^*(I)$ such that $A' \subset A\}$.

Theorem 2.5. *[DT4] Let I be a meet-semilattice over U. Then$M(I)$is a minimal family over$U$. Conversely, if Y is a minimal family over U, then there is exactly one meet-semilattice I so that $M(I) = Y$, where $I = \{C \subseteq R : \forall (A, B) \in Y : A \subseteq C$ implies $B \subseteq C\}$.*

Let Z be an intersection semilattice on U and suppose that $H \subset U, H \not\subset Z$hold and $Z \cup \{H\}$ is also closed under intersection. Consider the sets A satisfying $A \in Z, H \subset A$. The intersection of all of these sets is in Z therefore it is different form H. Denote it by $L(H)$. $H \subset L(H)$ is obvious. Let $H(Z)$ denote the set of all pairs $(H, L(H))$ where $H \subset U, H \notin Z$, but $Z \cup \{H\}$ is closed under intersection. The following theorem characterize the possible sets $H(Z)$:

Theorem 2.6. *[DK1] The set $\{(A_i, B_i)|i = 1, ..., m\}$ is equal to $H(Z)$ for some intersection semilattice Z iff the following conditions are satisfied:*

$A_i \subset B_i \subseteq U, A_i \neq B_i,$

$A_i \neq A_j$ *implies either* $B_i \subseteq A_j$, *or* $A_j \subseteq B_i$,

$A_i \subseteq B_j$ *implies* $B_i \subseteq B_j$,

for any i and $C \subset U$ satisfying $A_i \subset C \subset B_i (A_i \neq C \neq B_i)$.

There is a j such that either $C = A_j$ or $A_j \subset C, B_j \not\subset C, C \not\subset B_j$ all hold.

The set of pair (A_i, B_i) satisfying those condition above is called an extension. Its definition is not really beautiful but it is needed in some application. On the other hand it is also an equivalent notion to the closures:

Theorem 2.7. *[DK1] $Z \to H(Z)$ is a bijection between the set of intersection semilattices and the set of extensions.*

Definition 2.8. *Let U be a nonempty finite set of attributes and $P(U)$ its power set. A map $C : P(U) \to P(U)$ is called a choice function, if every $A \in P(U)$, then $C(A) \subseteq A$.*

U is interpreted as a set of alternatives, A as a set of alternatives given to the decision-maker to choose the best and $C(A)$ as a choice of the best alternatives among A.

Let L be a closure operation, we define C and H associated with L as follows:

$$C(A) = U - L(U - A), \qquad (*)$$

and

$$H(A) = A \cap L(U - A). \qquad (**)$$

We can easily prove that $C(A)$ and $H(A)$ are two choice functions. And we name $C(A)$ choice function - I (for short, CF-I), and $H(A)$ choice function - II (for short, CF-II).

Theorem 2.8. *The relationship like (*) is considered as a 1-1 correspondence between closures and choice functions, which satisfies the following two conditions: For every* $A, B \subseteq U$,

(1) If $C(A) \subseteq B \subseteq A$, *then* $C(A) = C(B)$ *(Out Casting Property),*

(2) If $A \subseteq B$, *then* $C(A) \subseteq C(B)$ *(Monotonicity Property).*

Theorem 2.9. *The relationship like (**) is considered as a 1-1 correspondence between closures and choice functions, which satisfies the following two conditions: For every* $A, B \subseteq U$,

(1) If If $H(A) \subseteq B \subseteq A$, *then* $H(A) = H(B)$ *(Out Casting Property),*

(2) If $A \subseteq B$, *then* $H(B) \cap A \subseteq H(A)$ *(Heredity Property).*

We also note that both C and H uniquely determine the closure L as the following

$$L(A) = U - C(U - A) \text{ and } H(A) = A \cup L(U - A).$$

For every $A \subseteq U$, the sets $C(A)$ and $H(A)$ form a partition of A, that is, $C(A) \cup H(A) = A$, and $C(A) \cap H(A) = \emptyset$.

Theorem 2.10. *There is a 1-1 correspondence between CFs - I and closure operations on* U.

Theorem 2.11. *There is a 1-1 correspondence between CFs - II and closure operations on* U.

3 Results

3.1 Properties of and Relationship between Composition of Closure Operations and CFs - I and - II

First of all, we are giving the formal definition of composition of functions.

Definition 3.1. *Let f and g be two functions (e.g. closure operations, CFs - I, or II) on U, and we determine a map T as a composition of f and g the following:*

$$T(X) = f(g(X)) = f.g(X) = fg(X) \text{ for every } X \subseteq U.$$

In this section we are going to answer two questions. The first one is: given many CFs - I (or II), what can be said about the composition of those CFs - I (or II). In other words, what is necessary and sufficient conditions that provide that composition to be a CF - I (or II). The second one is: what is the relationship between that composition of CFs - I(or II) and that of closure operations. And if we can find the necessary and sufficient conditions that provide that composition of closure operations to be closure operation through those of CFs - I(or II).

With the same questions, however, first we are going to investigate problems with two choice functions. For convenient, we show the results of CFs - II. We will soon see that

Theorem 3.1. *Let H_1 and H_2 be CFs-II on U, then composition H_1H_2 and H_2H_1 are a CFs-II on U, and $H_1H_2 = H_2H_1 = H_1 \cap H_2$.*

However, to achieve this result, we necessarily prove those following Lemmas and Propositions. First we need to prove the following Proposition:

Proposition 3.1. *Let H_1 and H_2 be CFs - II on U, then for all $X \subseteq U$,*

$$H_1(X) \cap H_2(X) \text{ is a CF - II on } U.$$

To prove $H_1 \cap H_2$ is a CF - II, we need to prove the following.

Lemma 3.1. *Let L_1 and L_2 be closure operations on U, then for all $X \subseteq U$,*

$$L_1(X) \cap L_2(X) \text{ is a closure operation on } U.$$

Proof. Assume L_1 and L_2 be two closure operations on U, then for all $X \subseteq U$, it is easy to obtain that $X \subseteq L_1(X) \cap L_2(X)$ since $X \subseteq L_1(X)$ and $X \subseteq L_2(X)$. Now, to prove the Monotonicity Property of $L_1(X) \cap L_2(X)$, for every $X \subseteq Y$, we have $L_1(X) \subseteq L_1(Y)$ and $L_2(X) \subseteq L_2(Y)$. Therefore, $L_1(X) \cap L_2(X) \subseteq L_1(Y) \cap L_2(Y)$, so $L_1 \cap L_2$ satisfies Monotonicity Property. Then, we have to prove Closure Property of $L_1 \cap L_2$. We always have $X \subseteq L_1(X) \cap L_2(X) \subseteq L_1(X)$. Using Monotonicity Property of L_1, we attain $L_1(X) \subseteq L_1(L_1(X) \cap L_2(X)) \subseteq L_1(L_1(X)) = L_1(X)$. That means $L_1(X) = L_1(L_1(X) \cap L_2(X))$. Similarly, we attain that $L_2(X) = L_2(L_1(X) \cap L_2(X))$. Therefore, $L_1(X) \cap L_2(X) = L_1(L_1(X) \cap L_2(X)) \cap (L_2(L_1(X) \cap L_2(X))$. That is, $L_1 \cap L_2$ satisfies Closure Property, so $L_1 \cap L_2$ is a closure on U. The proof is completed. ☐

Now we are moving on proving Proposition 3.1.

Proof of Proposition 3.1. Assume H_1 and H_2 be CFs - II on U, then for all $X \subseteq U$, we have $H_1(X) = X \cap L_1(U - X)$, and $H_2(X) = X \cap L_2(U - X)$, with L_1 and L_2 two closure operations corresponding to H_1 and H_2 respectively. Thus $H_1(X) \cap H2(X) = (X \cap L_1(U-X)) \cap (X \cap L_2(U-X)) = X \cap L_1(U-X) \cap L_2(U-X)$. However, due to Lemma 3.1, $L_1(U-X) \cap L_2(U-X)$ is a closure operation, that is, there exists a closure operation L_3 such that $L_3(U-X) = L_1(U-X) \cap L_2(U-X)$. Thus,$C_1(X) \cap C_2(X) = X \cap L_3(U-X) = C_3(X)$, with C_3 is a CF - II corresponding to L_3. The proof is completed. □

Before proving Theorem 3.1, we need to prove the follows.

Lemma 3.2. *Let H_1 and H_2 be CFs - II on U, then*
1) $H_1H_2 = H_2H_1H_2$.
2) $H_2H_1 = H_1H_2H_1$

Proof. Assume H_1 and H_2 be CFs - II on U. Then for all $X \subseteq U, H_1(X) = X \cap L_1(U - X)$ and $H_2(X) = X \cap L_2(U - X)$, with L_1 and L_2 two closure operations corresponding to H_1 and H_2 respectively. $H_1H_2(X) = H_1(H_2(X)) = X \cap L_2(U - X) \cap L_1(U - X \cap L_2(U - X)) \subseteq X$. Due to Heredity Property of CFs - II for H_2, we obtain $H_2(X) \cap H_1H_2(X) \subseteq H_2(H_1H_2(X))$. By using $H_1H_2(X) = H_1(H_2(X)) \subseteq H_2(X)$, we attain $H_1H_2(X) \subseteq H_2(H_1H_2(X)) \subseteq H_1H_2(X)$. Hence $H_1H_2(X) = H_2(H_1H_2(X))$, that is, $H_1H_2 = H_2H_1H_2$. Similarly, we obtain $H_2H_1 = H_1H_2H_1$.The proof is completed. □

Lemma 3.3. *Let H_1 and H_2 be CFs - II on U, then following is equivalence:*
1)$H_1 \subseteq H_2$
2)$H_1H_2 = H_1$

Proof.
$(1 \to 2)$. Assume H_1 and H_2 be CFs-II on U and $H_1 \subseteq H_2$. Since H_1 is a CF-II, H_1 must satisfy Out Casting property: if $H_1(X) \subseteq Y \subseteq X$, then $H_1(X) = H_1(Y)$. Therefore, we have $H_1 \subseteq H_2$ or $H_1(X) \subseteq H_2(X) \subseteq X$ for every $X \subseteq U$, so $H_1(H_2(X)) = H_1(X)$ or we conclude that $H_1H_2 = H_1$.

$(2 \to 1)$. Assume H_1 and H_2 be CFs - II on U and $H_1H_2 = H_1$. Since H_1 and H_2 are CFs - II, according to Definition of choice function, we have $H_1H_2 \subseteq H_2$, but $H_1H_2 = H_1$, so we have $H_1 \subseteq H_2$. The proof is completed. □

Easily, we obtain the following Corollary.

Corollary 3.1. *If H is a CF - II on U, then $HH = H$.*

Proof of Theorem 3.1. Assume H_1 and H_2 be CFs - II on U. Then for all $X \subseteq U, H_2(X) \subseteq X$. Due to Heredity Property of CF - II for H_1, we obtain $H_1(X) \cap H_2(X) \subseteq H_1(H_2(X))$. Besides that, $H_1(H_2(X)) \subseteq H_2(X) \subseteq X$, we obtain $H_1 \cap H_2(X) \subseteq H_1H_2(X) \subseteq X$. By Proposition 3.1, $H_1(X) \cap H_2(X)$ is a CF - II. Using

Out Casting Property for $H_1 \cap H_2$, we achieve $H_1 \cap H_2(H_1H_2(X)) = H_1 \cap H_2(X)$ or $H_1(H_1H_2(X)) \cap H_2(H_1H_2(X)) = H_1 \cap H_2(X)$. Due to Corollary 3.1, we obtain $H_1(H_1H_2(X)) = H_1H_2(X)$, and Lemma 3.2, we obtain $H_1H_2(X) = H_2H_1H_2(X)$. Therefore, we attain that $H_1H_2(X) = H_1 \cap H_2(X)$, that is $H_1H_2 = H_1 \cap H_2$.That means H_1H_2 is a CF-II. Similarly, we obtain $H_2H_1 = H_1 \cap H_2$ and H_2H_1 is a CF-II. The proof is completed. □

We can generalize Theorem 3.1 by the following

Generalization 3.1. Let H_i be CFs - II on U with $i = 1 \rightarrow n$, then $H_{i1}H_{i2}....H_{i(n-1)}H_{in}$ is a CFs - II on U, and

$$H_{i1}H_{i2}...H_{in} = \bigcap_{i=1}^{n} H_i$$

with $\{H_{i1}, H_{i2}., ..., H_{i(n-1)}, H_{in}\}$ be permutations of $\{H_1, H_2, .., H_{(n-1)}, H_n\}$.

Thus, for CFs - II, a composition of CFs - II is always a CF - II. Now we move on the composition of CFs - I before investigating on closure operations.

Theorem 3.2. *Let C_1 and C_2 be CFs - I on U. A composition of C_1 and C_2, denoted as C_1C_2, is a CF - I if and only if*

$$C_1C_2C_1 = C_1C_2.$$

However, before proving Theorem 3.2, we need to have the following Lemmas.

Lemma 3.4. *Let C_1 and C_2 be CF-Is on U. Then*

1) $C_1C_2 \subseteq C_1$,
2) $C_1C_2 \subseteq C_2$,
3) $C_2C_1 \subseteq C_1$,
4) $C_2C_1 \subseteq C_2$.

Due to Definition of choice function, and Monotonicity Property of CFs - II, clearly we obtain that Lemma.

Lemma 3.5. *Let C_1 and C_2 be CFs - I on U, then following is equivalence:*

1) $C_1 \subseteq C_2$,
2) $C_1C_2 = C_1$.

Proof. The proof of this Lemma is similar to that of Lemma 3.3. □

Easily, we obtain the following Corollary.

Corollary 3.2. *If C is a CF - I on U, then $CC = C$.*

Now we move on proving Theorem 3.2.

Proof Theorem 3.2. Assume C_1 and C_2 be CFs-I on U and the composition C_1C_2 also a CF - I. Due to Lemma 3.4, we have $C_1C_2(X) \subseteq C_1(X) \subseteq X$. Due to Out Casting Property of composition C_1C_2, we have $C_1C_2C_1(X) = C_1C_2(X)$.

Inversely, assume C_1 and C_2 be CFs-I on U and the composition C_1C_2 satisfying that $C_1C_2C_1 = C_1C_2$. For all $X \subseteq U$, it is clear to obtain that $C_1(C_2(X)) \subseteq C_2(X) \subseteq X$. It means C_1C_2 is a choice function. Now, to prove the Monotonicity Property of the composition C_1C_2. For every $X \subseteq Y$, using Monotonicity Property of C_1 and C_2, we have $C_2(X) \subseteq C_2(Y)$, then $C_1(C_2(X)) \subseteq C_1(C_2(Y))$. That means that C_1C_2 satisfies Monotonicity Property. Then, we have to prove Out Casting Property of the composition C_1C_2. For all X and $Y \subseteq U, C_1C_2(X) \subseteq Y \subseteq X$, we need to prove that $C_1C_2(X) = C_1C_2(Y)$. Using Monotonicity Property of C_1, we obtain that $C_2(C_1C_2(X)) \subseteq C_2(Y) \subseteq C_2(X)$. Applying Monotonicity Property of C_1 once again, we have $C_1(C_2C_1C_2(X)) \subseteq C_1(C_2(Y)) \subseteq C_1(C_2(X))$. However, $C_1C_2C_1 = C_1C_2$. Therefore, $C_1C_2C_2(X) \subseteq C_1C_2(Y) \subseteq C_1C_2(X)$. Due to Corollary 3.2, we obtain that $C_1C_2(X) \subseteq C_1C_2(Y) \subseteq C_1C_2(X)$ That means $C_1C_2(X) = C_1C_2(Y)$. That is, C_1C_2 satisfies Out Casting Property, so C_1C_2 is a CF-I on U. The proof is completed. □

We generalize the Theorem above.

Generalization 3.2. Let $C_1, C_2, ..,$ and C_n be CFs-I on U. A composition of $C_1, C_2, ...,$and C_n, denoted as $C_1C_2....C_{n-1}C_n$, is a CF-I if and only if

$$C_1C_2...C_{n-1}C_nC_1C_2....C_{n-1} = C_1C_2...C_{n-1}C_n.$$

Proof. We prove this Generalization by induction. It is obvious for $n = 1$. The Theorem 3.2 proves the case that $n = 2$.

For $n = k$, we assume that $C_1, C_2, ..,$ and C_k be CFs-I on U, and the composition of $C_1, C_2, ...,$ and C_k, denoted as $C_1C_2....C_{k-1}C_k$, is a CF - I. We need to prove that, for $n = k + 1$, the composition $C_1C_2....C_kC_{k+1}$ is a CF - I iff $C_1C_2...C_kC_{k+1}C_1C_2....C_k = C_1C_2...C_kC_{k+1}$. Surely, since $C_1C_2....C_{k-1}C_k$ is a CF - I, by using Theorem 3.2, we obtain that the composition $C_1C_2....C_kC_{k+1}$ is a CF - I iff $C_1C_2...C_kC_{k+1}C_1C_2....C_k = C_1C_2...C_kC_{k+1}$. The proof is completed. □

Now we move on the relationship between the composition of closure operations and CFs - I. We have the following Theorem.

Theorem 3.3. *Let L_1 and L_2 be closure operations and C_1 and C_2 be CF-Is corresponding to L_1 and L_2 respectively on U. The following are equivalent:*
1) C_1C_2 is a CF-I,
2)L_1L_2 is a closure operation.

Proof. $(1 \to 2)$. Assume L_1 and L_2 be closure operations, and C_1 and C_2 be CF-Is corresponding to L_1 and L_2 respectively on U and C_1C_2 is a closure operation. Then for all $X \subseteq U$, we have $L_1(X) = U - C_1(U - X)$, and $L_2(X) = U - C_2(U - X)$. Thus, $L_1L_2(X) = L_1(L_2(X)) = U - C_1(U - (U - C_2(U - X))) = U - C_1(C_2(U - X)) = U - C_1C_2(U - X)$. However, C_1C_2 is a closure operation. Therefore, there exists a closure operation C_3 such that $C_3(U-X) = C_1C_2(U-X)$. Thus, $L_1L_2(X) = U - C_3(U-X) = L_3(X)$, with L_3 a closure operation corresponding to C_3. That is L_1L_2 is a closure operation. The proof is completed.

$(2 \to 1)$. Assume L_1 and L_2 be closure operations and C_1 and C_2 be CF-Is corresponding to L_1 and L_2 respectively on U and L_1L_2 is a closure operation. Then for all $X \subseteq U$, we have $C_1(X) = U - L_1(U-X)$, and $C_2(X) = U - L_2(U-X)$. Thus, $C_1C_2(X) = C_1(C_2(X)) = U - L_1(U-(U-L_2(U-X))) = U - L_1(L_2(U-X)) = U - L_1L_2(U-X)$. However, L_1L_2 is a closure operation. Therefore, there exists a closure operation L_3 such that $L_3(U-X) = L_1L_2(U-X)$. Thus, $C_1C_2(X) = U - L_3(U-X) = C_3(X)$, with C_3 a choice function-I corresponding to L_3. That is C_1C_2 is a CF-I. The proof is completed. □

It can be seen that.

Generalization 3.3. Let L_i be closure operations and $\{C_i\}$ be CFs - I corresponding to L_i respectively on U, with $i = 1 \to n$. The following are equivalent:

1) $L_1L_2...L_n$ is a closure operation
2) $C_1C_2...C_n$ is a CF-I

And we also have $L_1L_2...L_n(X) = U - C_1C_2...C_n(U-X)$ and $C_1C_2...C_n(X) = U - L_1L_2...L_n(U-X)$.

Through Theorem 3.2 and 3.3, it is easy to obtain the following Theorem.

Theorem 3.4. *Let L_1 and L_2 be closure operations on U. A composition of L_1 and L_2, denoted as L_1L_2, is a closure operation if and only if*

$$L_1L_2L_1 = L_1L_2.$$

For generalization, we have the same conclusion for following Generalization as Generalization 3.2.

Generalization 3.4. Let $L_1, L_2, .., $ and L_n be closure operations on U. A composite function of $L_1, L_2, ...,$ and L_n, denoted as $L_1L_2....L_{n-1}L_n$, is a closure operation if and only if

$$L_1L_2...L_{n-1}L_nL_1L_2....L_{n-1} = L_1L_2...L_{n-1}L_n.$$

3.2 Direct Product of CFs - I and - II

The direct product of closure operations plays very important role in theory of relational database, especially in combinatorial problems. Plenty of properties related to direct product of closure operation can be found in [DFK] and [Li]. By relationship and interaction between closure operations and choice functions, we introduce the new definitions of direct product of choice function-Is as well as -IIs. First of all, we have the following.

Theorem 3.5. *[Li] Let L_1 and L_2 be closure operations on the disjoint ground sets U_1 and U_2 respectively. The direct product of closure operations $L_1 \times L_2$ is defined as following*

$$(L_1 \times L_2)(X) = L_1(X \cap U_1) \cup L_2(X \cap U_2), \qquad X \subseteq U_1 \cup U_2.$$

Then $(L_1 \times L_2)(X)$ *is a closure operations on* $U_1 \cup U_2$.

Here we give the Generalization of above Theorem.

Generalization 3.5. Let $\{L_i|\ i = 1 \to n\}$ be closure operations on the disjoint ground sets U_i respectively. The direct product of those closure operations $L_1 \times L_2 \times ... \times L_n$ is defined as following

$$(L_1 \times L_2 \times \ldots \times L_n)(X) = \bigcup_{i=1}^{n} L_i(X \cap U_i)$$

with $X \subseteq U_1 \cup U_2 \cup ... \cup U_n$.

Then $(L_1 \times L_2 \times ... \times L_n)(X)$ is a closure operation on $U_1 \cup U_2 \cup ... \cup U_n$.

Theorem 3.6. *Let* C_1 *and* C_2 *be CFs - I on the disjoint ground sets* U_1 *and* U_2 *respectively. The direct product of CFs - I,* $C_1 \times C_2$*, is defined as following*

$$(C_1 \times C_2)(X) = C_1(X \cap U_1) \cup C_2(X \cap U_2), \qquad X \subseteq U_1 \cup U_2.$$

Then $(C_1 \times C_2)(X)$ *is a CF - I on* $U_1 \cup U_2$.

Proof. For all $X \subseteq U_1 \cup U_2, (C_1 \times C_2)(X) = C_1(X \cap U_1) \cup C_2(X \cap U_2) \subseteq (X \cap U_1) \cup (X \cap U_2) \subseteq X \cap (U_1 \cup U_2) = X$. Thus, $(C_1 \times C_2)(X) \subseteq X$. For every X and $Y \subseteq U_1 \cup U_2$ and $X \subseteq Y$, then $X \cap U_1 \subseteq Y \cap U_1$, and $X \cap U_2 \subseteq Y \cap U_2$. By using Monotonicity Property of C_1 and C_2, we obtain $C_1(X \cap U_1) \subseteq C_1(Y \cap U_1)$ and $C_2(X \cap U_2) \subseteq C_2(Y \cap U_2)$. Hence $C_1(X \cap U_1) \cup C_2(X \cap U_2) \subseteq C_1(Y \cap U_1) \cup C_2(Y \cap U_2)$, that is, $(C_1 \times C_2)(X) \subseteq (C_1 \times C_2)(Y)$ or $(C_1 \times C_2)$ satisfies Monotonicity Property. Now we need to show that $(C_1 \times C_2)(X)$ satisfies the Out Casting Property also. That is, for every $X, Y \subseteq U_1 \cup U_2$ and $(C_1 \times C_2)(X) = C_1(X \cap U_1) \cup C_2(X \cap U_2) \subseteq Y \subseteq X$, we need to show that $(C_1 \times C_2)(X) = (C_1 \times C_2)(Y)$. Since $Y \subseteq X$, we have $(C_1 \times C_2)(Y) \subseteq (C_1 \times C_2)(X)$. And it is obvious that $C_1(X \cap U_1) \subseteq C_1(X \cap U_1) \cup C_2(X \cap U_2) \subseteq Y$. Thus, we have $C_1(X \cap U_1) \cap U_1 \subseteq Y \cap U_1$ or $C_1(X \cap U_1) \subseteq Y \cap U_1$. Using Monotonicity Property of C_1, we have $C_1(C_1(X \cap U_1)) \subseteq C_1(Y \cap U_1)$ or $C_1(X \cap U_1) \subseteq C_1(Y \cap U_1)$ due to Corollary 3.2. Similarly, we obtain $C_2(X \cap U_1) \subseteq C_2(Y \cap U_1)$. Therefore $C_1(X \cap U_1) \cup C_2(X \cap U_2) \subseteq C_1(Y \cap U_1) \cup C_2(Y \cap U_2)$ or $(C_1 \times C_2)(X) \subseteq (C_1 \times C_2)(Y)$. Hence $(C_1 \times C_2)(X) = (C_1 \times C_2)(Y)$. The proof is completed. □

Generalization 3.6. Let $\{C_i|\ i = 1 \to n\}$ be CFs - I with on the disjoint ground sets $\{U_i\}$ respectively. The direct product of CFs - I, $C_1 \times C_2 \times ... \times C_n$, is defined as following

$$(C_1 \times C_2 \times \ldots \times C_n)(X) = \bigcup_{i=1}^{n} C_i(X \cap U_i)$$

with $X \subseteq U_1 \cup U_2 \cup ... \cup U_n$.

Then $(C_1 \times C_2 \times ... \times C_n)(X)$ is a CF - I on $U_1 \cup U_2 \cup ... \cup U_n$.

Theorem 3.7. *Let H_1 and H_2 be CFs - II on the disjoint ground sets U_1 and U_2 respectively. The direct product of CFs - II, $H_1 \times H_2$, is defined as following*

$$(H_1 \times H_2)(X) = H_1(X \cap U_1) \cup H_2(X \cap U_2), \qquad X \subseteq U_1 \cup U_2.$$

Then $(H_1 \times H_2)(X)$ is a CF - II on $U_1 \cup U_2$.

Proof. For all $X \subseteq U_1 \cup U_2, (H_1 \times H_2)(X) = H_1(X \cap U_1) \cup H_2(X \cap U_2) \subseteq (X \cap U_1) \cup (X \cap U_2) \subseteq X \cap (U_1 \cup U_2) = X$. Thus, $(H_1 \times H_2)(X) \subseteq X$. For every X and $Y \subseteq U_1 \cup U_2$ and $X \subseteq Y$, we need to prove that $(H_1 \times H_2)$ satisfies Heredity Property. Since $X \subseteq Y$, we have $X \cap U_1 \subseteq Y \cap U_1$, and $X \cap U_2 \subseteq Y \cap U_2$. By using Heredity Property of H_1 and H_2, we obtain $H_1(Y \cap U_1) \cap (X \cap U_1) \subseteq H_1(X \cap U_1)$ or $H_1(Y \cap U_1) \cap X \subseteq H_1(X \cap U_1)$. Similarly, we have $H_2(Y \cap U_2) \cap X \subseteq H_2(X \cap U_2)$. Hence, $(H_1(Y \cap U_1) \cap X) \cup (H_2(Y \cap U_2) \cap X) \subseteq H_1(X \cap U_1) \cup H_2(X \cap U_2)$, then $(H_1(Y \cap U_1) \cup H_2(Y \cap U_2)) \cap X \subseteq H_1(X \cap U_1) \cup H_2(X \cap U_2)$ that is, $(H_1 \times H_2)(Y) \cap X \subseteq (H_1 \times H_2)(X)$ or $(H_1 \times H_2)$ satisfies Heredity Property.

Now we need to show that $(H_1 \times H_2)(X)$ satisfies the Out Casting Property also. That is, for every X and $Y \subseteq U_1 \cup U_2$ and $(H_1 \times H_2)(X) = H_1(X \cap U_1) \cup H_2(X \cap U_2) \subseteq Y \subseteq X$, we need to show that $(H_1 \times H_2)(X) = (H_1 \times H_2)(Y)$. It is obvious that $H_1(X \cap U_1) \subseteq H_1(X \cap U_1) \cup H_2(X \cap U_2) \subseteq Y \subseteq X$. Then $H_1(X \cap U_1) \cap U_1 \subseteq Y \cap U_1 \subseteq X \cap U_1$ or $H_1(X \cap U_1) \subseteq Y \cap U_1 \subseteq X \cap U_1$. Using Out Casting Property of H_1, we obtain $H_1(X \cap U_1) = H_1(Y \cap U_1)$. Similarly, we attain $H_2(X \cap U_2) = H_2(Y \cap U_2)$. Therefore $H_1(X \cap U_1) \cup H_2(X \cap U_2) = H_1(Y \cap U_1) \cup H_2(Y \cap U_2)$ or $(H_1 \times H_2)(X) = (H_1 \times H_2)(Y)$. The proof is completed. ☐

Generalization 3.7. Let $\{H_i | \ i = 1 \rightarrow n\}$ be CFs - II with on the disjoint ground sets U_i respectively. The direct product of CFs - II, $H_1 \times H_2 \times ... \times H_n$, is defined as following

$$(H_1 \times H_2 \times \ldots \times H_n)(X) = \bigcup_{i=1}^{n} H_i(X \cap U_i)$$

with $X \subseteq U_1 \cup U_2 \cup ... \cup U_n$.

Then $(H_1 \times H_2 \times \ldots \times H_n)(X)$ is a CF - II on $U_1 \cup U_2 \cup ... \cup U_n$.

3.3 Properties of CFs - I and - II and Closure Operations

Proposition 3.2. *Let C_1 and C_2 be CFs-I on U, then for all $X \subseteq U$,*

$$C_1(X) \cup C_2(X) \text{ is a CF-I on } U.$$

Proof. Assume C_1 and C_2 be CFs-I on U, then for all $X \subseteq U$, it is easy to obtain that $C_1(X) \cup C_2(X) \subseteq X$ since $C_1(X) \subseteq X$ and $C_2(X) \subseteq X$. Now, to prove the Monotonicity Property of $C_1 \cup C_2$, for every $X \subseteq Y$, we have $C_1(X) \subseteq C_1(Y)$ and $C_2(X) \subseteq C_2(Y)$. Therefore, $C_1(X) \cup C_2(X) \subseteq C_1(Y) \cup C_2(Y)$, so $C_1 \cup C_2$ satisfies Monotonicity Property. Then, we have to prove Out Casting Property of $C_1 \cup C_2$. We always have $C_1(X) \subseteq C_1(X) \cup C_2(X) \subseteq Y \subseteq X$. Using Out Casting Property of C_1, we attain $C_1(X) = C_1(Y)$. Similarly, we attain that $C_2(X) = C_2(Y)$ from $C_2(X) \subseteq C_1(X) \cup C_2(X) \subseteq Y \subseteq X$. Therefore, $C_1 \cup C_2(X) = C_1 \cup C_2(Y)$. That is, $C_1 \cup C_2$ satisfies Out Casting Property, so $C_1 \cup C_2$ is a CF-I on U. The proof is completed. □

Proposition 3.3. *Let H_1 and H_2 be CFs-II on U, then for all $X \subseteq U$,*

$$H_1(X) \cup H2(X) \text{ is a CF-II on } U.$$

Proof. Assume H_1 and H_2 be CFs-II on U. Similarly to above proof, for all $X \subseteq U$ it is clear to obtain that $H_1(X) \cup H_2(X) \subseteq X$ since $H_1(X) \subseteq X$ and $H_2(X) \subseteq X$. Now, to prove the Heredity Property of $H_1 \cup H_2$, for every $X \subseteq Y$, we have $H_1(Y) \cap X \subseteq H_1(X)$ and $H_2(Y) \cap X \subseteq H_2(X)$. Therefore, $X \cap (H_1(Y) \cup H_2(Y)) \subseteq H_1(X) \cup H_2(X)$, so $H_1 \cup H_2$ satisfies Heredity Property. For Out Casting Property of $H_1 \cup H_2$, we prove the same as the proof of Proposition 3.2. The proof is completed. □

From Proposition 3.3, we lead to the following Lemmas.

Lemma 3.6. *Let L_1 and L_2 be closure operations on U, then for all $X \subseteq U$,*

$$L_1(X) \cup L_2(X) \text{ is a closure operation on } U.$$

Proof. Assume L_1 and L_2 be closure operations on U, then for all $X \subseteq U$, we have $L_1(X) = X \cup H_1(U - X), L_2(X) = X \cup H_2(U - X)$, with H_1 and H_2 two choice function-IIs corresponding to L_1 and L_2 respectively. Thus $L_1(X) \cup L_2(X) = X \cup H_1(U - X) \cup H_2(U - X)$. However, due to Proposition 3.3, $H_1(U - X) \cup H_2(U - X)$ is a CF - II, that is, there exists a choice function H_3 such that $H_3(U - X) = H_1(U - X) \cup H_2(U - X)$. Thus, $L_1(X) \cup L_2(X) = X \cup H_3(U - X) = L_3(X)$, with L_3 a closure operation corresponding to H_3. The proof is completed. □

Using similar method of above proof, we can achieve two following.

Lemma 3.7. *Let C_1 and C_2 be CFs - I on U, then for all $X \subseteq U$,*

$$C_1(X) \cap C_2(X) \text{ is a CF - I on } U.$$

Proof. Assume C_1 and C_2 be CFs - I on U, then for all $X \subseteq U$, we have $C_1(X) = U - L_1(U - X)$, and $C_2(X) = U - L_2(U - X)$, with L_1 and L_2 two closure operations corresponding to C_1 and C_2 respectively. Thus $C_1(X) \cap C_2(X) = (U - L_1(U - X)) \cap (U - L_2(U - X)) = U - L_1(U - X) \cup L_2(U - X)$. However, due to Lemma 3.6, $L_1(U - X) \cup L_2(U - X)$ is a closure operation, that is, there exists a closure operation L_3 such that $L_3(U - X) = L_1(U - X) \cup L_2(U - X)$. Thus, $C_1(X) \cup C_2(X) = U - L_3(U - X) = C_3(X)$, with C_3 a CF - I corresponding to L_3. The proof is completed. □

Proposition 3.4. *Let H be a CF-II on U. Then for all $X \subseteq U$, we have*

$$H(X) \cap H(Y) \subseteq H(X \cap Y).$$

Proof. For all X and $Y \subseteq U$, due to Monotonicity Property of closure operations, we easily obtain $L(X) \cap L(Y) \subseteq L(X \cup Y)$. Therefore, $L(U - X) \cap L(U - Y) \subseteq L((U - X) \cup (U - Y))$. Using $L((U - X) \cup (U - Y)) = L(U - X \cap Y)$, we have $L(U - X) \cap L(U - Y) \subseteq L(U - X \cap Y)$. Hence, $(X \cap Y) \cap L(U - X) \cap L(U - Y) \subseteq (X \cap Y) \cap L((U - X \cap Y)$ or $H(X) \cap H(Y) \subseteq H(X \cap Y)$. The proof is completed. □

Similarly, we obtain the follow

Proposition 3.5. *Let H be a CF-II on U. Then for all $X \subseteq U$, we have $H(X \cup Y) \subseteq H(X) \cup H(Y)$.*

Proof. For all X and $Y \subseteq U$, due to Monotonicity Property of closure operations, we easily obtain $L(X \cap Y) \subseteq L(X) \cap L(Y)$. Therefore, $L((U - X) \cap (U - Y)) \subseteq L(U - X) \cap L(U - Y)$. Using $L((U - X) \cap (U - Y)) = L(U - X \cup Y)$, we have $L(U - X \cup Y) \subseteq L(U - X) \cap L(U - Y)$. Hence, $(X \cup Y) \cap L((U - X \cup Y) \subseteq (X \cup Y) \cap L(U - X) \cap L(U - Y)$ or $H(X \cap Y) \subseteq (X \cap L(U - X) \cap L(U - Y)) \cup (Y \cap L(U - X) \cap L(U - Y)) \subseteq (X \cap L(U - X)) \cup (Y \cap L(U - Y)) = H(X) \cup H(Y)$. The proof is completed. □

Lemma 3.8. *Let H_1 and H_2 be CFs-II on U. Then*
1) $H_1H_2 \subseteq H_2$
2) $H_2H_1 \subseteq H_1$

Since H_1 and H_2 are a CFs-II , it is obvious to have above Lemma.

Lemma 3.9. *Let H_1 and H_2 be CFs - II on U, then*
1) $H_1 \cap H_2 \subseteq H_1H_2$
2) $H_1 \cap H_2 \subseteq H_2H_1$

Proof. Assume H_1 and H_2 be CFs - II on U. Then for all $X \subseteq U, H_2(X) \subseteq X$. Due to Heredity Property of CFs-II, we obtain $H_1(X) \cap H_2(X) \subseteq H_1(H_2(X))$, that is, $H_1 \cap H_2 \subseteq H_1H_2$. Similarly, we achieve $H_1 \cap H_2 \subseteq H_2H_1$. □

Proposition 3.6. *Let H_1 and H_2 be CFs - II on U, then $H_1 \cap H_2 = H_1 \cap H_1H_2 = H_2 \cap H_2H_1$.*

In order to prove this Proposition, we need to have the following Lemma.

Lemma 3.10. *Let H_1 and H_2 be CFs - II on U, then $H_1 \cap H_2 = H_1(H_1 \cap H_2) = H_2(H_1 \cap H_2)$.*

Proof. Assume H_1 and H_2 be CFs - II on U. Then for all $X \subseteq U$, we always have $H_1(X) \cap H_2(X) \subseteq H_2(X)$. Due to Heredity Property of CF-IIs, we obtain $H_1(H_2(X)) \cap H_1(X) \cap H_2(X) \subseteq H_1(H_1(X) \cap H_2(X))$. According to Lemma 3.9, we obtain $H_1(X) \cap H_2(X) \subseteq H_1(H_1(X) \cap H_2(X))$. However, $H_1(H_1(X) \cap H_2(X)) \subseteq H_1(X) \cap H_2(X)$. Hence, $H_1(H_1(X) \cap H_2(X)) = H_1(X) \cap H_2(X)$, that is, $H_1 \cap H_2 = H_1(H_1 \cap H_2)$. Similarly, we achieve $H_1 \cap H_2 = H_2(H_1 \cap H_2)$. The proof is completed. □

Proof of Proposition 3.6. Assume H_1 and H_2 be CFs - II on U. For all $X \subseteq U$ due to Proposition 3.4 and Corollary 3.1, we obtain $H_1(X) \cap H_1(H_2(X)) \subseteq H_1(H_1(X) \cap H_2(X))$. However, $H_1 \cap H_2 = H_1(H_1 \cap H_2)$ according to Lemma 3.10, and $H_1 \cap H_2 \subseteq H_1 H_2$ due to Lemma 3.9. Therefore, $H_1(X) \cap H_1(X) \cap H_2(X) \subseteq H_1(X) \cap H_1(H_2(X)) \subseteq H_1(X) \cap H_2(X)$. Then, $H_1(X) \cap H_1(H_2(X)) = H_1(X) \cap H_2(X)$, that is, $H_1 \cap H_2 = H_1 \cap H_1 H_2$. Similarly, we obtain $H_1 \cap H_2 = H_2 \cap H_2 H_1$. The proof is completed. □

From Proposition 3.6, it is clear to obtain the follow.

Corollary 3.3. *Let H_1 and H_2 be CFs - II on U, then $H_1 \cap H_2 = H_1 \cap H_2(H_1 \cap H_2)$.*

3.4 Interaction between Closure Operations and CFs - I

Let L be a closure and Σ a corresponding full family of FDs. We recall that an FD $X \to Z \in \Sigma$ iff $Z \subseteq L(X)$. In this section, we consider the closures for which CF - I and -II defined in section 0 satisfy some additional properties. We are now going to give some properties.

Proposition 3.7. *Let L and C be a closure operation and a CF-I corresponding to Lrespectively on U. The following are equivalent:*

1) $C(X \cup Y) = C(X) \cup C(Y)$,

2) $L(X \cap Y) = L(X) \cap L(Y)$,

3) $X \to Z$ and $Y \to Z$ are FDs from Σ iff $X \cap Y \to Z$.

Proof. $(1 \to 2)$. Let C satisfies 1). Then for all $X, Y \subseteq U : L(X \cap Y) = U - C(U - X \cap Y) = U - C((U - X) \cup (U - Y)) = U - C(U - X) \cup C(U - Y) = (U - C(U - X)) \cap (U - C(U - Y)) = L(X) \cap L(Y)$. That is, L satisfies 2).

$(2 \to 1)$ Let L satisfies 2). Then for all $X, Y \subseteq U : C(X \cup Y) = U - L(U - X \cup Y) = U - L((U - X) \cap (U - Y)) = U - L(U - X) \cap L(U - Y) = (U - L(U - X)) \cup (U - L(U - Y)) = C(X) \cup C(Y)$. That is, C satisfies 1).

$(2 \leftrightarrow 3)$ Let L satisfies 2). Then for all $X, Y \subseteq U : L(X \cap Y) = L(X) \cap L(Y)$. For $Z \in L(X \cap Y)$ iff $X \cap Y \to Z$. And $Z \in L(X) \cap L(Y)$, that means $Z \in L(X)$ and $Z \in L(Y)$ iff $X \to Z$ and $Y \to Z$. □

Proposition 3.8. *Let L and C be a closure operation and a CF-I corresponding to L respectively on U. The following are equivalent:*

1) $C(X \cap Y) = C(X) \cap C(Y)$,

2) $L(X \cup Y) = L(X) \cup L(Y)$.

Proof. $(1 \to 2)$. Let C satisfies 1). Then for all $X, Y \subseteq U : L(X \cup Y) = U - C(U - X \cup Y) = U - C((U - X) \cap (U - Y)) = U - C(U - X) \cap C(U - Y) = (U - C(U - X)) \cup (U - C(U - Y)) = L(X) \cup L(Y)$. That is, L satisfies 2).

$(2 \to 1)$ Let L satisfies 2). Then for all $X, Y \subseteq U : C(X \cap Y) = U - L(U - X \cap Y) = U - L((U - X) \cup (U - Y)) = U - L(U - X) \cup L(U - Y) = (U - L(U - X)) \cap (U - L(U - Y)) = C(X) \cap C(Y)$. That is, C satisfies 1). □

Proposition 3.9. *Let L_1 and L_2 be closure operations and C_1 and C_2 be CF-Is corresponding to L_1 and L_2 respectively on U. The following are equivalent:*
1) $C_1(X) \cap C_2(X) \subseteq C_1C_2(X)$
2) $L_1L_2(X) \subseteq L_1(X) \cup L_2(X)$

Proof. $(1 \to 2)$. Let C_1 and C_2 satisfy 1). Then for all $X \subseteq U : L_1L_2(X) = U - C_1C_2(U-X) \subseteq U - C_1(U-X) \cap C_2(U-X) = (U - C_1(U-X)) \cup (U - C_2(U-X)) = L_1(X) \cup L_2(X)$. That is, L_1 and L_2 satisfy 2).

$(2 \to 1)$. Let L_1 and L_2 satisfy 2). Then for all $X \subseteq U : C_1(X) \cap C_2(X) = (U - L_1(U-X)) \cap (U - L_2(U-X)) = U - L_1(U-X) \cup L_2(U-X) \subseteq U - L_1L_2(U-X) = C_1C_2(X)$. That is, C_1 and C_2 satisfy 1). □

3.5 Special cases of Choice Function-Is and -IIs

Theorem 3.8. *Let consider a partition $V : \{V_1, V_2, V_3, ..., V_n\}$, that is, $V_i \cap V_j = \emptyset$, with $i \neq j$. Let construct a set*

$$W(A) = A \cap \bigcup_{i=1}^{n} V_i$$

for all $A \subseteq U$. Then, $W(A)$ is a CF-I on U.

Proof. For all $A \subseteq U$, it is clear that $W(A) \subseteq A$. Now we need to prove that W satisfies Monotonicity and Out Casting Property. We have

$$W(A) = A \cap \bigcup_{i=1}^{n} V_i = \bigcup_{i=1}^{n} (A \cap V_i)$$

$$\Rightarrow W(W(A)) = \bigcup_{j=1}^{n} (A \cap \bigcup_{i=1}^{n} V_i) \cap V_j = \bigcup_{j=1}^{n} (\bigcup_{i=1}^{n} (A \cap V_i \cap V_j))$$

$$= \bigcup_{i=1}^{n} (A \cap V_i) = W(A),$$

since $V_i \cap V_j = \emptyset$, for $i \neq j$. For $A \subseteq B$, it is obvious that $A \cap V_i \subseteq B \cap V_i$, then

$$\bigcup_{i=1}^{n} (A \cap V_i) \subseteq \bigcup_{i=1}^{n} (B \cap V_i).$$

Thus, $W(A) \subseteq W(B)$, so W satisfies Monotonicity Property.

To prove Out Casting Property of W, let assume $W(A) \subseteq B \subseteq A$, we have show that $W(A) = W(B)$. Using Monotonicity Property of W, we attain $W(W(A)) \subseteq W(B) \subseteq W(A)$. However, $W(W(A)) = W(A)$, we lead to that $W(A) = W(B)$. The proof is completed. □

We can illustrate $W(A)$ as the sum of all intersections of A and V_i , for $i = 1 \to n$. Here is a property of W.

Proposition 3.10. *Let consider partition of* $V : \{V_1, V_2, V_3 ..., V_n\}$*, that is,* $V_i \cap V_j = \emptyset$*, with* $i \neq j$*, and partition of* $T : \{T_1, T_2, T_3 ..., T_m\}$*, that is,* $T_i \cap T_j = \emptyset$*, with* $i \neq j$*. For all* $A \subseteq U$*, let construct two CF-I as the following:*

$$C_1(A) = A \cap \bigcup_{i=1}^{n} V_i,$$

$$C_2(A) = A \cap \bigcup_{j=1}^{m} T_j.$$

Then, $C_1(A) \cap C_2(A) = C_1C_2(A)$*, and both also are CF-Is.*

Proof. For all $A \subseteq U$, we have

$$C_1(A) \cap C_2(A) = (A \cap \bigcup_{i=1}^{n} V_i) \cap (A \cap \bigcup_{j=1}^{m} T_j) = A \cap (\bigcup_{i=1}^{n} V_i \cap \bigcup_{j=1}^{m} T_j)$$
$$= (A \cap \bigcup_{j=1}^{m} T_j) \cap \bigcup_{i=1}^{n} V_j = C_1C_2(A).$$

However,

$$C_1(A) \cap C_2(A) = A \cap (\bigcup_{i=1}^{n} V_i \cap \bigcup_{j=1}^{m} T_j) = A \cap \bigcup_{i=1}^{n} (\bigcup_{j=1}^{m} T_j \cap V_i).$$

It is easy to see that, for every $x \neq y$,

$$(\bigcup_{j=1}^{m} T_j \cap V_x) \cap (\bigcup_{j=1}^{m} T_j \cap V_y) = \emptyset.$$

That is, $\{(\bigcup T_j \cap V_i) | i = 1 \to n, j = 1 \to m\}$ is a partition. Due to Theorem 3.10, we conclude that $C_1(A) \cap C_2(A)$ as well as $C_1C_2(A)$ is a CF-I. The proof is completed. □

Let us define $W_c(A)$, the complementary set of $W(A)$, as $W_c(A) = A - W(A)$, that is

$$W_c(A) = A - A \cap \bigcup_{i=1}^{n} V_i = (A - A) \cup (A - \bigcup_{i=1}^{n} V_i) = A - \bigcup_{i=1}^{n} V_i = \bigcap_{i=1}^{n} (A - V_i).$$

Since $W(A)$ is a CF-I, and CF-I and CF-II of A form a partition of A, for every $A \subseteq U$, we lead to the following Theorem.

Theorem 3.9. *Let consider partition of* $V : \{V_1, V_2, V_3..., V_n\}$*, that is,* $V_i \cap V_j = \emptyset$*, with* $i \neq j$*. Let construct a set*

$$W_c(A) = \bigcap_{i=1}^{n} (A - V_i)$$

for all $A \subseteq U$*. Then,* $W_c(A)$ *is a CF-II on* U*.*

3.6 Discussion and Open Problems

Given a set of F of functional dependencies over U and the attribute set $X \subseteq U$, so the functional dependencies closure of $X, L(X)$, is the set $\{A \subseteq U | X \to A \in F\}$. It turns out that this set is independent of the underlying attribute set U. We have known that two types of choice function -I and -II associated with L as follows:

$$C(A) = U - L(U - A), \text{ and} H(A) = A \cap L(U - A).$$

Thus, given a set of F of functional dependencies, we define, $X \subseteq U$, choice-I and -II of X as follows:

$$H_F(X) = X \cap \{A \subseteq U | (U - X) \to A \in F\} \quad (1)$$

$$C_F(X) = U - \{A \subseteq U | (U - X) \to A \in F\} \quad (2)$$

It can be seen the following Propositions.

Proposition 3.11. *Let* F *be a set of functional dependencies and* $X \to Y$ *an functional dependency. Then* $X \to Y \in F$ *iff* $Y \not\subset C_F(U - X)$*.*

Proposition 3.12. *Let* F *be a set of functional dependencies and* $X \to Y$ *an functional dependency. Then* $X \to Y \in F$ *and* $Y \notin X$ *iff* $Y \subseteq H_F(U - X)$*.*

Now we move to compute $C_F(X)$ and $H_F(X)$. First of all, we now mention about the Algorithm of computing a closure from a set of functional dependencies and X a set of attributes.

In [BB], we were known the Algorithm to computing closure operation, by using relation between choice functions and closure operation, we can easily build Algorithm to compute choice functions.

Even though we already have an algorithm to compute closure of X, from the Theorem 3.4 above as follows: Let L_1 and L_2 be closure operations on U. A composite function of L_1 and L_2, denoted as L_1L_2, is a closure operation if and only if

$$L_1L_2L_1 = L_1L_2.$$

Open problems are set up as following:

Open Problem 1. Let $s =< U, F >$ and $t =< U, V >$ two relation schemes, where U is a set of attributes and F and V are two different sets of FDs over U. We define F^+ and V^+ be a set of all FDs that can be derived from F and V respectively.

1) Is it possible build a closure L_1 and a closure L_2 from F^+ and V^+ respectively such that $L_1L_2 = L_1L_2L_1$?

2) If so, how can we design L_1L_2 ? In other word, how can we design a relation scheme $w =< U, H >$ from which we can build H^+ , from which we can design the closure $L_1L_2 = L_1L_2L_1$?

3) If so, is it possible to generalize this design for more than two closure operations?

Open Problem 2. A similar problem as above, but for choice -I and -II of X.

Open Problem 3. Algorithm problems related to union and intersection for choice -I and -II and closures.

Open Problem 4. Generalize those theories presented in this paper to mutilvalued dependencies.

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