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Hexagonal polyomino weak (1,2)-achievement games

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Abstract

A version of polyomino achievement games is studied, in which the first player marks one cell and the second player marks two cells at each move. All polyominos but one on an infinite 2-dimensional hexagonal board are characterized to be weak winners or losers.

Keywords: polyomino, animal, achievement game

1 Introduction

Achievement games for polyominos have been introduced by Frank Harary [Gar, Ha1, Ha2, HH, Ha3]. They are generalizations of the well known game Tic-Tac-Toe, where the target shape can be some predetermined set of polyominos. The type of the board can vary as well. It can be a tiling of the plane by triangles [BH3] or hexagons [BH2]. The board can also have higher dimensional [HW, DS], it can be a Platonic solid [BH1] or the hyperbolic plane [Bod]. For further results see [HHS, HS1, HS2, HS3].

A *polyomino* or *animal* is a finite set of cells such that both the polyomino and its complement are connected through edges. We only consider polyominos up to congruence, that is, the location and direction of the polyomino on the board is not important.

In a *polyomino weak achievement game,* or *A-achievement game,* two players alternately mark previously unmarked cells of the board using their own colors. The first player (the maker) wins if he can mark a set of cells congruent to a given polyomino. The second player (the breaker) wins if she can prevent the maker from achieving the given polyomino. In a $(1,2)$-achievement game [Plu, Sie] the first player marks a single cell while the second player marks two cells at each move.

A polyomino is called a (*weak*) *winner* if the first player can always win the (weak) achievement game with the given polyomino. Otherwise the polyomino is called a *loser*.

In this paper we classify all animals but one as winners or losers in the weak $(1,2)$-achievement game on an infinite hexagonal board.

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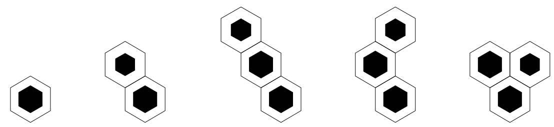
2 Animals with at most three cells

There are five animals with fewer than four cells:

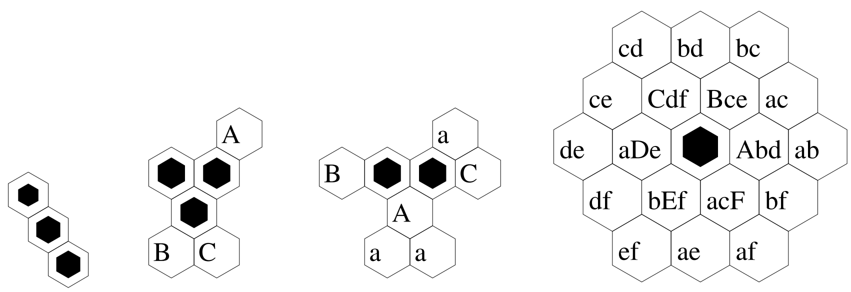


P_1 P_2 P_3 P_4 P_5

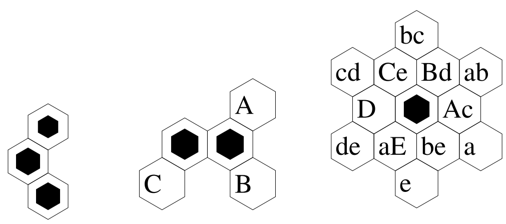
It is clear that P_1 and P_2 are winners.

Proposition 1. *The animals P_3 and P_4 are winners.*

Proof. The following are winning strategies for these animals:



$P_3 = s_0$ s_1 s_2 s_3



$P_4 = s_0$ s_1 s_2

Cells with a solid block represent the marks of the maker. Cells with letters in them represent empty cells. The games start at situation s_3 and s_2 respectively and end at situation s_0. It is easy to see that no matter how the breaker marks two empty cells there is a letter which was not present in those two cells. Then the maker can mark the cell containing the capitalized version of this missing letter and achieve a situation with a smaller number. This situation is determined by the cells containing the missing letter.

The following figure shows the flowcharts of these games:

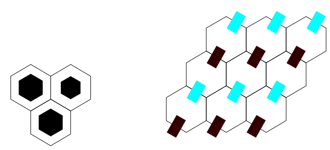
$$\begin{tikzcd} s_3 \arrow[r] & s_2 \arrow[r] \arrow[rr, bend left] & s_1 \arrow[r] & s_0 \end{tikzcd} \qquad \begin{tikzcd} s_2 \arrow[r] & s_1 \arrow[r] & s_0 \end{tikzcd}$$

□

Proposition 2. *The animal P_5 is a loser.*

Proof. To show that an animal is a loser we can pave the board twice, using pairs of cells. We choose the two pavings such that every position of the given animal on the board contains a full pair of cells either from one of the pavings or from the other. The existence of such double paving allows the breaker to win by marking the two cells which are the pairs of the maker's mark in the two pavings. None of these cells can already be marked by the maker but it is possible that they are already marked by the breaker. In that case the breaker can mark any other cell.

The following figure shows P_5 and its double paving:



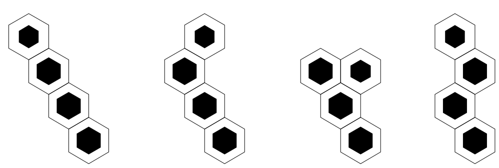
P_5

The reader can easily verify that the double paving has the required properties.

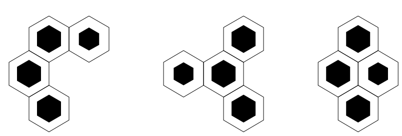
□

3 Animals with four cells

An animal containing a smaller losing animal is a loser itself. Hence the only possibility to find winners with four cells is to consider animals that are created from P_3 or P_4 by adding a single cell. There are eight such animals:



P_6 P_7 P_8 P_9



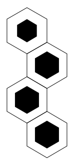
P_{10} P_{11} P_{12}

Proposition 3. *The animals P_8 and P_{12} are losers.*

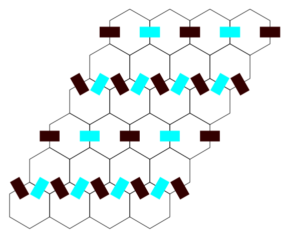
Proof. All of these contain the loser P_5 as a subset and therefore they are losers themselves. □

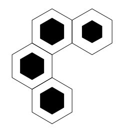
Proposition 4. *The animals P_9, P_{10} and P_{11} are losers.*

Proof. These animals are losers because of the strategies based on the following double pavings:

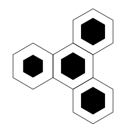


P_9

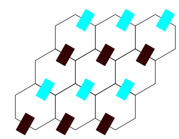




P_{10}



P_{11}

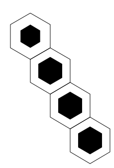


□

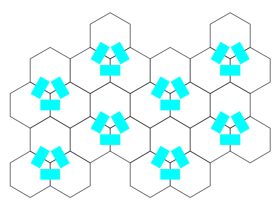
Proposition 5. *The animal P_6 is a loser.*

Proof. To show that an animal is a loser we can pave the board using triples of cells. We choose the paving such that every position of the given animal on the board contains at least two cells from one of the triples of the paving. The existence of such paving allows the breaker to win by marking the two cells which are in the same triple as the maker's mark.

The following figure shows the animal and its paving by triples:



P_6

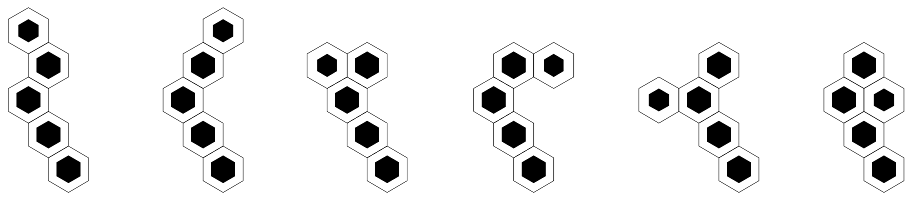


□

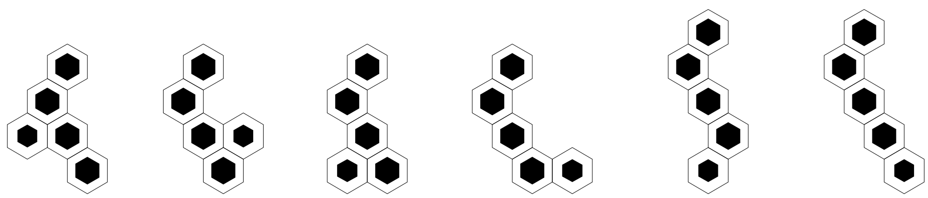
Animal P_7 remains a mystery. Playing the game suggests that it is a loser. Full analysis of the game tree shows that the maker wins if the breaker is required to mark cells that share an edge with an already marked cell. This shows that P_7 is a *paving winner*, that is, there is no winning strategy for the breaker based on a paving. We suspect that it is also a *pair partition winner*, that is, the breaker does not have a winning strategy based on a pairing even if disconnected pairs of cells are allowed.

4 Animals with five cells

The only possibility to find winners with five cells is to consider animals that are created from P_7 by adding a single cell. There are twelve such animals:



P_{13} P_{14} P_{15} P_{16} P_{17} P_{18}



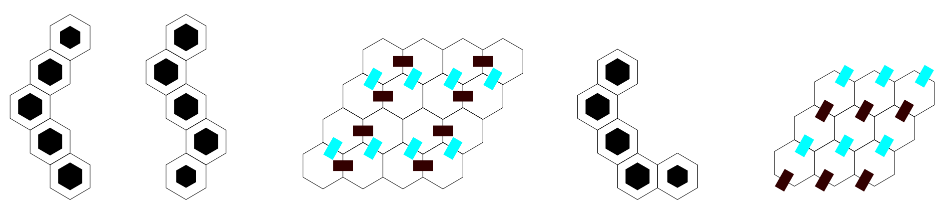
P_{19} P_{20} P_{21} P_{22} P_{23} P_{24}

Proposition 6. *Animals P_{13}, P_{15}, P_{16}, P_{17}, P_{18}, P_{19}, P_{20}, P_{21} and P_{24} are losers.*

Proof. All of these animals contain one of the losers P_5, P_6, P_9, P_{10} or P_{11} as a subset and therefore they are losers themselves. □

Proposition 7. *The animals P_{14}, P_{22} and P_{23} are losers.*

Proof. All these animals are losers because of strategies based on the following double pavings:



P_{14} P_{23} P_{22}

□

5 Main result

Since all animals with five cells are losers, there cannot be any winners with more than four cells. So we have the following main result.

Theorem 8. *The only winning animals in the weak* $(1,2)$*-achievement game on the 2-dimensional infinite hexagonal board are the animals* P_1*,* P_2*,* P_3*,* P_4 *and possibly* P_7*.*

There are several questions to be answered. Is P_7 a winner or a loser? What are the winning pairs, triples or larger sets of animals? What are the handicap numbers of the losers, that is, how many extra marks does the maker need before the game starts to be able to win? What are the winners in the (1,2)-achievement game on triangular, and higher dimensional boards? What are the winners in the (2,3)-achievement game?

References

[Bod] J. Bode, *Strategien für Aufbauspiele mit mosaik-polyominos,* Doctoral dissertation, Technishen Universität Braunschweig.

[BH1] J. Bode, H. Harborth, *Achievement games on Platonic solids,* Bull. Inst. Combin. Appl. 23 (1998), 23–32.

[BH2] J. Bode, H. Harborth, *Hexagonal polyomino achievement,* Discrete Math. **212** (2000), np. 1–2, 5–18.

[BH3] J. Bode, H. Harborth, *Triangle polyomino set achievement,* Congr. Numer. **148** (2001), 97–101.

[DS] E. Deabay, N. Sieben, *Poliomino weak achievement games on 3-dimensional rectangular boards,* preprint.

[Gar] M. Gardner, *Mathematical games,* Sci. Amer. 240 (1979) 18–26.

[Ha1] F. Harary, *Achieving the Skinny animal,* Eureka **42** (1982) 8–14.

[Ha2] F. Harary, *Achievement and avoidance games for graphs,* Ann. Discrete Math. **13** (1982), 111–120.

[Ha3] F. Harary, *Is Snaky a winner?* Geombinatorics **2** (1993), 79–82.

[HH] F. Harary, H. Harborth, *Achievement and avoidance games with triangular animals,* Recreational Math. **18** (1985–86), 110–115.

[HHS] F. Harary, H. Harborth, M. Seeman, *Handicap achievement for polyominoes,* Congr. Numer. **145** (2000), 65–80.

[HW] F. Harary, M. Weisbach, *Polycube achievement games,* J. Recreational Mathematics **15** (1982-83), 241–246.

[HS1] H. Harborth, M Seemann, *Handicap achievement for squares,* To appear.

[HS2] H. Harborth, M Seemann, *Snaky is an edge-to-edge loser,* Geombinatorics **5** (1996), no. 4, 132–136.

[HS3] H. Harborth, M Seemann, *Snaky is a paving winner,* Bull. Inst. Combin. Appl. **19** (1997), 71–78.

[Plu] A. Pluhár, *Generalized Harary games,* Acta Cybernet. **13** (1997), no. 1, 77–83.

[Sie] N. Sieben, *Wild polyomino weak* $(1,2)$-*achievement games,* preprint.

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