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On regular languages determined by nondeterministic directable automata*

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Abstract

It is known that the languages consisting of directing words of deterministic and nondeterministic automata are regular. Here these classes of regular languages are studied and compared. By introducing further three classes of regular languages, it is proved that the 8 classes considered form a semilattice with respect to intersection.

1 Introduction

We recall that an input word of an automaton is called *directing* or *synchronizing* if it brings the automaton from every state into the same state. An automaton is *directable* if it has a directing word. The directable automata and directing words have been studied from different points of view (see [2, 3, 5, 6, 7, 8, 10, 12, 13], for example). For nondeterministic (n.d.) automata, the directability can be defined in several ways. We study here three notions of directability which are defined in [7] as follows. An input word w of an n.d. automaton $\mathcal{A}$ is

(1) D1-directing if the set of states aw in which $\mathcal{A}$ may be after reading w consists of the same single state c whatever the initial state a is;

(2) D2-directing if the set aw is independent of the initial state a;

(3) D3-directing if there exists a state c included in all sets aw.

We mention that D1-directability of complete n.d. automata was already studied by Burkhard [1], where he gave an exact exponential bound for the length of minimum-length D1-directing words of complete n.d. automata. In [5], classes of languages consisting of directing words of different types of n.d. automata were studied. Here, we extend our investigations to three further classes of languages and present some of their properties. The next

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section provides general preliminaries, the formal definitions of the above language classes and some earlier results. Finally, Section 3 presents some new properties of the language families considered, in particular, it is proved that they constitute a semilattice with respect to intersection.

2 Preliminaries

Let X be a finite nonempty alphabet. As usual the set of all (finite) words over X is denoted by X^* and the empty word by ε. The length of a word w is denoted by $|w|$.

By a (deterministic) *automaton* we mean a triplet $\mathcal{A} = (A, X, \delta)$, where A is a finite nonempty set of *states*, X is the *input alphabet*, and $\delta : A \times X \to A$ is the *transition function*. This function can be extended to $A \times X^*$ in the usual way. By a *recognizer* we mean a system $\mathbf{A} = (A, X, \delta, a_0, F)$, where (A, X, δ) is an automaton, $a_0 (\in A)$ is the *initial state*, and $F (\subseteq A)$ is the set of *final states*. The *language recognized* by $\mathbf{A}$ is the set

$$L(\mathbf{A}) = \{w \in X^* : \delta(a_0, w) \in F\}.$$

A language is called *recognizable*, or *regular*, if it is recognized by some recognizer. Sometimes, we say that the recognizer $\mathbf{A}$ *accepts* the language $L(\mathbf{A})$.

An automaton $\mathcal{A} = (A, X, \delta)$ can also be defined as a unary algebra $\mathcal{A} = (A, X)$ for which each input letter x is realized as the unary operation $x^{\mathcal{A}} : A \to A,\ a \mapsto \delta(a, x)$. Now, nondeterministic automata can be introduced as generalized automata in which the unary operations are replaced by binary relations. Therefore, by a *nondeterministic (n.d.) automaton* we mean a system $\mathcal{A} = (A, X)$ where A is a finite nonempty set of *states*, X is the *set of the input signs (or letters)*, and each sign $x (\in X)$ is realized as a binary relation $x^{\mathcal{A}} (\subseteq A \times A)$ on A. For any $a \in A$ and $x \in X$, we define $ax^{\mathcal{A}} = \{b \in A : (a, b) \in x^{\mathcal{A}}\}$. Thus, $ax^{\mathcal{A}}$ is the set of states into which $\mathcal{A}$ may enter from state a by reading the input letter x. For any $C \subseteq A$ and $x \in X$, we set $Cx^{\mathcal{A}} = \bigcup \{ax^{\mathcal{A}} : a \in C\}$. This transition can be extended to arbitrary $w \in X^*$ and $C \subseteq A$. $Cw^{\mathcal{A}}$ is obtained inductively by

(1) $C\varepsilon = C$,

(2) $Cw^{\mathcal{A}} = (Cv^{\mathcal{A}})x^{\mathcal{A}}$ for $w = vx$, $x \in X$, $w \in X^*$.

An n.d. automaton $\mathcal{A} = (A, X)$ is called *complete*, or *c.n.d. automaton*, if $ax^{\mathcal{A}} \neq \emptyset$, for all $a \in A$ and $x \in X$.

The notion of the directability of deterministic automata can be generalized to n.d. automata in several ways. The following three definitions are taken from [7]. Let $\mathcal{A} = (A, X)$ be an n.d. automaton. For any word $w \in X^*$ we consider the following three conditions:

(D1) $(\exists c \in A)(\forall a \in A)(aw^{\mathcal{A}} = \{c\})$;

(D2) $(\forall a, b \in A)(aw^{\mathcal{A}} = bw^{\mathcal{A}})$;

(D3) $(\exists c \in A)(\forall a \in A)(c \in aw^{\mathcal{A}})$.

If w satisfies condition (Di), then w is called a Di-*directing word* of $\mathcal{A}$ ($i = 1, 2, 3$). For every i, $i = 1, 2, 3$, the set of Di-directing words of $\mathcal{A}$ is denoted by $\mathrm{D}_i(\mathcal{A})$, and $\mathcal{A}$ is called Di-*directable* if $\mathrm{D}_i(\mathcal{A}) \neq \emptyset$. It is proved (see [7]) that $\mathrm{D}_i(\mathcal{A})$ is recognizable, for every n.d. automaton $\mathcal{A}$ and i, $i = 1, 2, 3$. The classes of Di-directable n.d. automata and c.n.d. automata are denoted by $\mathbf{Dir}(i)$ and $\mathbf{CDir}(i)$, respectively.

Now, we can define the following classes of languages: For $i = 1, 2, 3$, let

$$\mathcal{L}_{\mathrm{ND(i)}} = \{\mathrm{D}_i(\mathcal{A}) : \mathcal{A} \in \mathbf{Dir}(i)\} \quad \text{and} \quad \mathcal{L}_{\mathrm{CND(i)}} = \{\mathrm{D}_i(\mathcal{A}) : \mathcal{A} \in \mathbf{CDir}(i)\}.$$

Finally, let $\mathbf{D}$ denote the class of directable deterministic automata, and for any $\mathcal{A} \in \mathbf{D}$, let $\mathrm{D}(\mathcal{A})$ be the set of directing words of $\mathcal{A}$. Moreover, let

$$\mathcal{L}_{\mathrm{D}} = \{\mathrm{D}(\mathcal{A}) : \mathcal{A} \in \mathbf{D}\}.$$

Since all of the languages occuring in the definitions above are recognizable, the defined classes are subclasses of the class of the regular languages.

In what follows, we need the following definition. For any language $L \subseteq X^*$, let us denote by $P_r(L)$ the set of all prefixes of the words in L, *i.e.*, $P_r(L) = \{u : u \in X^* \ \&\ (\exists v \in X^*)(uv \in L)\}$.

Now, we recall some results from [5] and [7] which are used in the following section.

Lemma 1 ([7]). *For any n.d. automaton $\mathcal{A} = (A, X)$, $\mathrm{D}_2(\mathcal{A})X^* = \mathrm{D}_2(\mathcal{A})$. If $\mathcal{A}$ is complete, then $X^*\mathrm{D}_1(\mathcal{A}) = \mathrm{D}_1(\mathcal{A})$, $X^*\mathrm{D}_2(\mathcal{A})X^* = \mathrm{D}_2(\mathcal{A})$, and $X^*\mathrm{D}_3(\mathcal{A})X^* = \mathrm{D}_3(\mathcal{A})$.*

Proposition 1 ([5]). *For a language $L \subseteq X^*$, $L \in \mathcal{L}_{\mathrm{D}}$ if and only if $L \neq \emptyset$, L is regular, and $X^*LX^* = L$.*

Proposition 2 ([5]). *$\mathcal{L}_{\mathrm{CND(2)}} = \mathcal{L}_{\mathrm{D}}$, $\mathcal{L}_{\mathrm{CND(3)}} = \mathcal{L}_{\mathrm{D}}$, $\mathcal{L}_{\mathrm{CND(1)}} \cap \mathcal{L}_{\mathrm{ND(2)}} = \mathcal{L}_{\mathrm{D}}$, and $\mathcal{L}_{\mathrm{CND(1)}} \cap \mathcal{L}_{\mathrm{ND(3)}} = \mathcal{L}_{\mathrm{D}}$.*

Furthermore, we need the following proper inclusions from [5].

Remark 1 ([5]). *The following proper inclusions are valid:*

(a) $\mathcal{L}_{\mathrm{D}} \subset \mathcal{L}_{\mathrm{CND(1)}} \subset \mathcal{L}_{\mathrm{ND(1)}}$,

(b) $\mathcal{L}_{\mathrm{D}} \subset \mathcal{L}_{\mathrm{ND(2)}}$,

(c) $\mathcal{L}_{\mathrm{D}} \subset \mathcal{L}_{\mathrm{ND(3)}}$.

By Proposition 2, $\mathcal{L}_{\mathrm{CND(3)}} = \mathcal{L}_{\mathrm{CND(2)}} = \mathcal{L}_{\mathrm{D}}$, and thus, we shall investigate the remaining 5 classes and three more defined as follows. Languages $L \subseteq X^*$ satisfying $X^*L = L$ are called *ultimate definite* (*cf.* [9] or [11]), and we shall consider the subclass $\mathcal{U}$ which consists of all the regular ultimate definite languages. The second class, denoted by $\mathcal{L}'$, contains all the nonempty regular languages satisfying $P_r(L)LX^* = L$. Finally, we shall also consider the class $\mathcal{L}_{\mathrm{ND(1)}} \cap \mathcal{L}_{\mathrm{ND(3)}}$.

3 Some observations on languages of directing words of n.d. automata

First we consider the classes $\mathcal{U}$ and $\mathcal{L}_{\mathrm{ND}(1)}$. It is known (see [5]) that $\mathcal{L}_{\mathrm{CND}(1)} \subset \mathcal{U}$. $\mathcal{L}_{\mathrm{CND}(1)} \subset \mathcal{L}_{\mathrm{ND}(1)}$ by Remark 1. The following assertion shows that $\mathcal{L}_{\mathrm{CND}(1)}$ is the intersection of these two wider classes.

Proposition 3. $\mathcal{L}_{\mathrm{CND}(1)} = \mathcal{L}_{\mathrm{ND}(1)} \cap \mathcal{U}$.

Proof. As we mentioned, $\mathcal{L}_{\mathrm{CND}(1)}$ is contained in both $\mathcal{U}$ and $\mathcal{L}_{\mathrm{ND}(1)}$. Therefore, it is sufficient to show that $\mathcal{L}_{\mathrm{ND}(1)} \cap \mathcal{U} \subseteq \mathcal{L}_{\mathrm{CND}(1)}$. For this reason, let $L \in \mathcal{L}_{\mathrm{ND}(1)} \cap \mathcal{U}$. Then, there exists a nondeterministic D1-directable automaton $\mathcal{A} = (A, X)$ such that $L = D_1(\mathcal{A})$. We show that $\mathcal{A}$ is a complete n.d. automaton. In order to obtain a contradiction, let us assume that there are $a' \in A$ and $x \in X$ such that $a'x^{\mathcal{A}} = \emptyset$. Let $p \in L$ be arbitrary and consider the word xp. Since $L \in \mathcal{U}$, we have $X^*L = L$, and therefore, $xp \in L$, *i.e.*, xp is a D1-directing word. Thus, there exists a state $\bar{a} \in A$ such that $a(xp)^{\mathcal{A}} = \{\bar{a}\}$, for all $a \in A$. In particular, $a'(xp)^{\mathcal{A}} = \{\bar{a}\}$ which is a contradiction. Consequently, $\mathcal{A}$ is a complete n.d. automaton, and thus, $L \in \mathcal{L}_{\mathrm{CND}(1)}$. $\square$

Using Propositions 1 and 2, by the same argument as in the proof of Proposition 3, one can prove the following statement.

Proposition 4. $\mathcal{L}_{\mathrm{ND}(2)} \cap \mathcal{U} = \mathcal{L}_{\mathrm{D}}$ *and* $\mathcal{L}_{\mathrm{ND}(3)} \cap \mathcal{U} = \mathcal{L}_{\mathrm{D}}$.

By the definitions, one can easily prove the following:

Lemma 2. *If* $L \in \mathcal{L}_{\mathrm{ND}(3)}$, *then* $P_r(L)L = L$ *and* $LP_r(L) = L$.

Lemma 3. *If* $L \in \mathcal{L}_{\mathrm{ND}(1)}$, *then* $P_r(L)L = L$.

Now, we show that $\mathcal{L}_{\mathrm{ND}(1)}$ and $\mathcal{L}_{\mathrm{ND}(3)}$ are incomparable. To this aim, let us consider the following examples.

Example 1. Let us define the n.d. automaton $\mathcal{A} = (\{1,2\}, \{x,y\})$ by $x^{\mathcal{A}} = \{(1,1),(1,2),(2,1),(2,2)\}$ and $y^{\mathcal{A}} = \{(1,2),(2,2)\}$.

Then, $\mathcal{A}$ is D1-directable and $\mathrm{D}_1(\mathcal{A}) = X^*y$. Now, let us suppose that $X^*y \in \mathcal{L}_{\mathrm{ND}(3)}$. Since $y, xy \in X^*y$ and $x \in P_r(X^*y)$, by Lemma 2, we have that $yx \in X^*y$ which is a contradiction. Therefore, $\mathcal{L}_{\mathrm{ND}(1)} \not\subseteq \mathcal{L}_{\mathrm{ND}(3)}$.

Example 2. Let $\mathcal{A} = (\{1,2\}, \{x,y\})$ be the n.d. automaton for which $x^{\mathcal{A}} = \{(1,2),(2,1),(2,2)\}$ and $y^{\mathcal{A}} = \{(1,1)\}$.

Now, $\mathcal{A}$ is D3-directable and $x, x^2y \in \mathrm{D}_3(\mathcal{A})$ while $xy \notin \mathrm{D}_3(\mathcal{A})$. Let us suppose that $\mathrm{D}_3(\mathcal{A}) \in \mathcal{L}_{\mathrm{ND}(1)}$. Then, there exists an n.d. automaton $\mathcal{B} = (B, X)$ such that $\mathrm{D}_3(\mathcal{A}) = \mathrm{D}_1(\mathcal{B})$. In this case, x and x^2y are D1-directing words of $\mathcal{B}$, and thus, there are states $c, d \in B$ such that $bx^{\mathcal{B}} = \{c\}$, for all $b \in B$, in particular $cx^{\mathcal{B}} = \{c\}$, and $b(x^2y)^{\mathcal{B}} = \{d\}$ for all $b \in B$. Then, it is easy to see that $b(xy)^{\mathcal{B}} = \{d\}$, for all $b \in B$, and hence, $xy \in \mathrm{D}_1(\mathcal{B}) = \mathrm{D}_3(\mathcal{A})$ must hold, which is a contradiction since $xy \notin \mathrm{D}_3(\mathcal{A})$. Consequently, $\mathcal{L}_{\mathrm{ND}(3)} \not\subseteq \mathcal{L}_{\mathrm{ND}(1)}$.

Regarding the class $\mathcal{L}'$ defined by property $P_r(L)LX^* = L$, where $L \subseteq X^*$ is a nonempty regular language, the following assertion is valid.

Proposition 5. $\mathcal{L}' = \mathcal{L}_{\mathrm{ND}(2)} \cap \mathcal{L}_{\mathrm{ND}(3)}$.

Proof. To prove the inclusion $\mathcal{L}_{\mathrm{ND}(2)} \cap \mathcal{L}_{\mathrm{ND}(3)} \subseteq \mathcal{L}'$, let us suppose that $L \in \mathcal{L}_{\mathrm{ND}(2)} \cap \mathcal{L}_{\mathrm{ND}(3)}$. Since both classes, $\mathcal{L}_{\mathrm{ND}(2)}$ and $\mathcal{L}_{\mathrm{ND}(3)}$, contain nonempty regular languages (*cf.* [7]), L is nonempty and regular. Since $L \in \mathcal{L}_{\mathrm{ND}(2)}$, by Lemma 1, $LX^* = L$. On the other hand, by Lemma 2, from $L \in \mathcal{L}_{\mathrm{ND}(3)}$ it follows that $P_r(L)L = L$. Therefore, $P_r(L)LX^* = L$, and thus, $L \in \mathcal{L}'$.

In order to prove the inclusion $\mathcal{L}' \subseteq \mathcal{L}_{\mathrm{ND}(2)} \cap \mathcal{L}_{\mathrm{ND}(3)}$, let $L \in \mathcal{L}'$. Then, L is a nonempty regular language with $P_r(L)LX^* = L$. Since L is regular, there exists a minimal recognizer (A, X, δ, a_0, F) recognizing L. By our assumption, $LX^* = L$, and hence, by the minimality of the recognizer, we have that $F = \{f\}$ for some $f \in A$. Now, let us define the new n.d. automaton $\mathcal{B} = (B, X)$ for which $B = \{a_0 q^{\mathcal{A}} : q \in P_r(L)\}$ and the transitions are defined as follows. For every $b \in B$ and $x \in X$, let

$$bx^{\mathcal{B}} = \begin{cases} bx^{\mathcal{A}} & if bx^{\mathcal{A}} \in B, \\ \emptyset & otherwise. \end{cases}$$

Now, we prove that $\mathcal{B}$ is both D2-directable and D3-directable, moreover, $L = \mathrm{D}_2(\mathcal{B}) = \mathrm{D}_3(\mathcal{B})$. For this purpose, let us observe that if $p \in L$, then $a_0(qp)^{\mathcal{B}} = \{f\}$, for every $q \in P_r(L)$ since $P_r(L)L = L$. Consequently, p is simultaneously a D2-directing and a D3-directing word of $\mathcal{B}$, moreover, $L \subseteq \mathrm{D}_2(\mathcal{B})$ and $L \subseteq \mathrm{D}_3(\mathcal{B})$.

To prove the inclusion $\mathrm{D}_2(\mathcal{B}) \subseteq L$, let $p \in \mathrm{D}_2(\mathcal{B})$ be arbitrary. Then there exists a set H of states of $\mathcal{B}$ such that $bp^{\mathcal{B}} = H$, for all $b \in B$. But, $fp^{\mathcal{B}} = \{f\}$, and therefore, $H = \{f\}$, which results that $p \in L$.

For verifying $\mathrm{D}_3(\mathcal{B}) \subseteq L$, let $p \in \mathrm{D}_3(\mathcal{B})$ be arbitrary. Since $p \in \mathrm{D}_3(\mathcal{B})$ and $fp^{\mathcal{B}} = \{f\}$, we have $f \in bp^{\mathcal{B}}$, for all $b \in B$. Then, by the definition of $\mathcal{B}$, $bp^{\mathcal{B}} = \{f\}$, for all $b \in B$. In particular, $a_0 p^{\mathcal{B}} = \{f\}$, so that $a_0 p^{\mathcal{A}} = f$, proving $p \in L$.

Consequently, we have proved that $L \in \mathcal{L}_{\mathrm{ND}(2)}$ and $L \in \mathcal{L}_{\mathrm{ND}(3)}$, and therefore, $L \in \mathcal{L}_{\mathrm{ND}(2)} \cap \mathcal{L}_{\mathrm{ND}(3)}$. ☐

Regarding the above proof, let us observe that the constructed automaton $\mathcal{B}$ is also D1-directable, and $L = \mathrm{D}_1(\mathcal{B})$. By this observation, one can prove the next statement in the same way as Proposition 5.

Proposition 6. $\mathcal{L}' = \mathcal{L}_{\mathrm{ND}(2)} \cap \mathcal{L}_{\mathrm{ND}(1)}$.

The next corollary follows from Propositions 5 and 6.

Corollary 1. $\mathcal{L}' = (\mathcal{L}_{\mathrm{ND}(1)} \cap \mathcal{L}_{\mathrm{ND}(3)}) \cap \mathcal{L}_{\mathrm{ND}(2)}$.

Since $\mathcal{L}_{\mathrm{ND}(1)}$ and $\mathcal{L}_{\mathrm{ND}(3)}$ are incomparable with respect to set inclusion, $\mathcal{L}_{\mathrm{ND}(1)} \cap \mathcal{L}_{\mathrm{ND}(3)}$ is a proper subclass of both $\mathcal{L}_{\mathrm{ND}(1)}$ and $\mathcal{L}_{\mathrm{ND}(3)}$. Moreover, by Corollary 1, $\mathcal{L}' \subseteq \mathcal{L}_{\mathrm{ND}(1)} \cap \mathcal{L}_{\mathrm{ND}(3)}$ and $\mathcal{L}' \subseteq \mathcal{L}_{\mathrm{ND}(2)}$. Both inclusions are proper. To verify this observation, let us consider the following examples.

Example 3. Let the n.d. automaton $\mathcal{A} = (\{1,2\}, X)$ be defined by $X = \{x, y\}$, $x^{\mathcal{A}} = \{(2,1),(2,2)\}$, and $y^{\mathcal{A}} = \{(1,1),(2,1)\}$.

Then, y is a D1- and D3-directing word, and $L = y\{y\}^* = \mathrm{D}_1(\mathcal{A}) = \mathrm{D}_3(\mathcal{A})$. Now, if $L \in \mathcal{L}'$, then $P_r(L)LX^* = L$ must hold, which is a contradiction since $y^k x \notin L$, for every integer $k \geq 1$. Therefore, $\mathcal{L}' \subset \mathcal{L}_{\mathrm{ND}(1)} \cap \mathcal{L}_{\mathrm{ND}(3)}$.

Example 4. Let the n.d. automaton $\mathcal{A} = (\{1,2\}, X)$ be defined by $X = \{x, y\}$, $x^{\mathcal{A}} = \{(1,2),(2,2)\}$, and $y^{\mathcal{A}} = \{(2,1)\}$.

Then, $\mathcal{A}$ is D2-directable and $\mathrm{D}_2(\mathcal{A}) = xX^* \cup X^*y^2X^*$. Now, if $\mathrm{D}_2(\mathcal{A}) \in \mathcal{L}'$, then since $y \in P_r(\mathrm{D}_2(\mathcal{A}))$ and $x \in \mathrm{D}_2(\mathcal{A})$, $yx \in \mathrm{D}_2(\mathcal{A})$ must hold, which is a contradiction. Consequently, $\mathcal{L}' \subset \mathcal{L}_{\mathrm{CND}(2)}$.

By the definition of $\mathcal{L}'$ and Proposition 1, we obviously have that $\mathcal{L}_{\mathrm{D}} \subseteq \mathcal{L}'$. For proving that this inclusion is proper, let us consider the following example.

Example 5. Let $\mathcal{A} = (\{1,2\}, X)$, where $X = \{x, y\}$, $x^{\mathcal{A}} = \{(2,2)\}$, and $y^{\mathcal{A}} = \{(1,2),(2,2)\}$.

Then, $\mathrm{D}_1(\mathcal{A}) = \mathrm{D}_2(\mathcal{A}) = \mathrm{D}_3(\mathcal{A}) = yX^*$. By Proposition 5, $yX^* \in \mathcal{L}'$. Let us suppose now that $yX^* \in \mathcal{L}_{\mathrm{D}}$. Then, by Proposition 1, $xy \in yX^*$ must hold, which is a contradiction. Therefore, $yX^* \notin \mathcal{L}_{\mathrm{D}}$, and thus, $\mathcal{L}_{\mathrm{D}} \subset \mathcal{L}'$.

Summarizing, we obtain the following result.

Theorem 1. *If $|X| \geq 2$, then the 8 classes under consideration constitute a semilattice with respect to intersection.*

The semilatice of these classes is depicted in Figure 1.

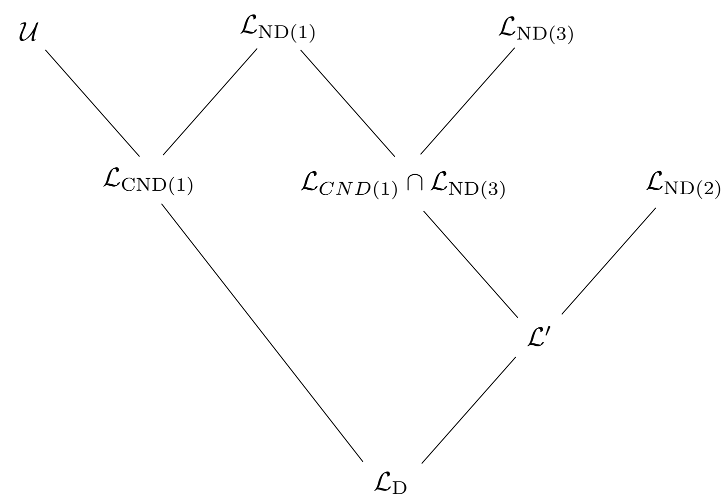

Figure 1: Semilattice of the classes considered.

Let $\mathcal{A} = (A, X)$ be an n.d. automaton and $x \in X$. Then, x is called a *complete input sign* if $ax^{\mathcal{A}} \neq \emptyset$, for all $a \in A$.

The following statement shows that the languages belonging to $\mathcal{L}_{\mathrm{ND}(2)}$ can be decomposed into a particular form.

Proposition 7. *If $L \in \mathcal{L}_{\mathrm{ND}(2)}$, then L is a disjoint union of regular languages L_1 and L_2 where at least one of L_1 and L_2 is nonempty, furthermore,*

(1) *$L_1 \in \mathcal{L}_{\mathrm{D}}$ or $L_1 = \emptyset$,*

and

(2) *$L_2 = P_r(L_2)L_2Y^*$ and $L_2 = Y^*L_2Y^*$, where $Y \subseteq X$ denotes the set of complete input symbols of $\mathcal{A}$, or $L_2 = \emptyset$.*

Proof. Let $L \in \mathcal{L}_{\mathrm{ND}(2)}$ be arbitrary. Then, there exists a D2-directable n.d. automaton $\mathcal{A} = (A, X)$ such that $L = D_2(\mathcal{A})$, *i.e.*, L consists of the D2-directing words of $\mathcal{A}$. Let us classify now the D2-directing words of $\mathcal{A}$ as follows. Let

$$\begin{aligned} L_1 &= \{p : p \in L \;\&\; ap^{\mathcal{A}} = \emptyset, \text{ for all } a \in A\}, \\ L_2 &= \{p : p \in L \;\&\; ap^{\mathcal{A}} \neq \emptyset, \text{ for some } a \in A\}. \end{aligned}$$

Obviously, $L_1 \cap L_2 = \emptyset$ and $L_1 \cup L_2 = L$, furthermore, one of the languages L_1 and L_2 is nonempty.

Let us suppose that $L_1 \neq \emptyset$. It is easy to see that L_1 is regular. Now, if $p \in L_1$, then $ap^{\mathcal{A}} = \emptyset$, for all $a \in A$. Thus also $a(qpr)^{\mathcal{A}} = \emptyset$, for all $q, r \in X^*$ and $a \in A$. Therefore, $X^*L_1X^* = L_1$, and by Proposition 1, we obtain that $L_1 \in \mathcal{L}_{\mathrm{D}}$ if $L_1 \neq \emptyset$.

The regularity of L_2 can be concluded by the fact that $L_2 = L \setminus L_1$. Let us observe that $Y = \emptyset$ implies $L_2 = \emptyset$.

Now, let us suppose that $L_2 \neq \emptyset$ and let $p \in L_2$ and $q \in P_r(L_2)$. Then, there exists an $r \in X^*$ with $qr \in L_2$. Since $qr \in L_2$, $a(qr)^{\mathcal{A}} \neq \emptyset$, for all $a \in A$. Therefore, $aq^{\mathcal{A}} = A_a \neq \emptyset$, for all $a \in A$. Furthermore, since $p \in L_2$, we have that there exists a nonempty set H of states such that $A'p^{\mathcal{A}} = H$, for every nonempty subset A' of A. In particular, $A_ap^{\mathcal{A}} = H$, for all $a \in A$. Consequently, $a(qp)^{\mathcal{A}} = (aq^{\mathcal{A}})p^{\mathcal{A}} = A_ap^{\mathcal{A}} = H$, for all $a \in A$, and hence, $qp \in L_2$. On the other hand, since Y is the set of complete input signs, $L_2Y^* = L_2$.

To prove the second equality, let $q \in Y^*$ and $p \in L_2$ be arbitrary words. From $p \in L_2$ it follows again that there exists a nonempty set H of states such that $A'p^{\mathcal{A}} = H$, for all nonempty subsets A' of A. On the other hand, since $q \in Y^*$, $aq^{\mathcal{A}} \neq \emptyset$, for all $a \in A$. Consequently, $H = aq^{\mathcal{A}}p^{\mathcal{A}} = a(qp)^{\mathcal{A}}$, for all $a \in A$, and thus, $Y^*L_2 = L_2$. The validity of the equality $L_2Y^* = L_2$ is obvious, and hence, $Y^*L_2Y^* = L_2$. □

Now, we study the representation of the languages of $\mathcal{L}_{\mathrm{ND}(2)}$ which have the form $L = MX^*$, where M is a regular prefix code. For this reason, we recall some notions.

Let $\emptyset \neq M \subseteq X^+$. Then, M is said to be a *prefix code* over X if $M \cap MX^+ = \emptyset$. A prefix code $M \subseteq X^+$ is said to be *maximal* if, for any $u \in X^*$, there exists $v \in X^*$ such that $uv \in MX^*$. Finally, a prefix code M is called *regular* if M is a regular language. Note that any $L \in \mathcal{L}_{\mathrm{ND}(2)}$ can be represented as $L = MX^*$ such that $M = L \setminus LX^+$ and M is a prefix code because $LX^* = L$.

Proposition 8. *Let $M \subseteq X^+$ be a regular prefix code that is not maximal. Let $L = MX^*$. Then, $L \in \mathcal{L}_{\mathrm{ND}(2)}$ if and only if $P_r(M)M \subseteq L$.*

Proof. To prove the necessity, let us assume $L \in \mathcal{L}_{\mathrm{ND}(2)}$. Then, there exists an n.d. automaton $\mathcal{A} = (A, X)$ such that $L = D_2(\mathcal{A})$. Let $u \in P_r(M)$ and $w \in M$. Since $u \in P_r(M)$, there exists $v \in X^*$ such that $uv \in M \subseteq L$. Hence, for any $a, b \in A$, $a(uv)^{\mathcal{A}} = b(uv)^{\mathcal{A}}$. Suppose $a(uv)^{\mathcal{A}} = \emptyset$ for any $a \in A$. Then, for any $a \in A$ and $z \in X^*$, $a(z(uv))^{\mathcal{A}} = \emptyset$. This yields that $zuv \in L$, for all $z \in X^*$, and hence, M is a maximal prefix code, which is a contradiction. Therefore, $a(uv)^{\mathcal{A}} \neq \emptyset$, and thus, $au^{\mathcal{A}} \neq \emptyset$, for all $a \in A$. Consequently, $a(uw)^{\mathcal{A}} = b(uw)^{\mathcal{A}}$ for any $a, b \in A$ since $w \in M \subseteq L$. Thus, $uw \in L$.

In order to prove the sufficiency, let $\mathbf{A'} = (A, X, a_0, \delta, F)$ be the minimal recognizer (deterministic but not necessarily complete) accepting L. Notice that $\mathbf{A'}$ is a trim (*i.e.* accessible and coaccessible, see [4]) and $F = \{f\}$, since M is a prefix code and $L = MX^*$. Consider the n.d. automaton $\mathcal{A} = (A, X)$. Note that $fx^{\mathcal{A}} = \{f\}$ for any $x \in X$. Let $a \in A$ and $w \in L$. Since $\mathbf{A'}$ is trim, there exist $u, v \in X^*$ such that $\{a\} = a_0u^{\mathcal{A}}$ and $a_0(uv)^{\mathcal{A}} = \{f\}$, *i.e.*, $uv \in L$. Consequently, $u \in P_r(M)$ or $u \in MX^*$. If $u \in P_r(M)$, then $uw \in P_r(M)MX^* \subseteq LX^* = L$. If $u \in MX^*$, then $uw \in MX^*X^* = MX^* = L$. Hence, $aw^{\mathcal{A}} = \{f\}$, for all $a \in A$. This means that $w \in \mathrm{D}_2(\mathcal{A})$. Now, let $w \notin L$. In this case, $fw^{\mathcal{A}} = \{f\}$ but $a_0w^{\mathcal{A}} \neq \{f\}$. This means that $w \notin \mathrm{D}_2(\mathcal{A})$. Consequently, $L = \mathrm{D}_2(\mathcal{A})$. This completes the proof of the proposition. □

The above proposition does not always hold for a regular maximal prefix code.

Example 6. Let $X = \{x, y\}$ and let $A = \{1, 2\}$. Moreover, let $\mathcal{A} = (A, X)$ be the following n.d. automaton: $x^{\mathcal{A}} = \{(1, 2), (2, 2)\}$, $y^{\mathcal{A}} = \{(1, 2)\}$.

Then, $L = D_2(A) = (x \cup yx^*y)X^* \in \mathcal{L}_{\mathrm{ND}(2)}$. Let $M = L \setminus LX^+$. Then, $P_r(M)M \subseteq L$ does not hold since $y \in P_r(M)$, $x \in M$ but $yx \notin L = MX^*$.

However, for the class of finite maximal prefix codes, we have the following:

Proposition 9. *Let $\emptyset \neq M \subseteq X^+$ be a finite maximal prefix code. Let $L = MX^*$. Then, $L \in \mathcal{L}_{\mathrm{ND}(2)}$ if and only if $P_r(M)M \subseteq L$.*

Proof. The sufficiency can be proved in the same way as in the proof of the previous proposition. To prove the necessity, let us assume that $L = MX^* \in \mathcal{L}_{\mathrm{ND}(2)}$. Let $\mathcal{A} = (A, X)$ be an n.d. automaton such that $L = \mathrm{D}_2(\mathcal{A})$. Let $u \in P_r(M)$ and $w \in M$. Since M is a finite maximal prefix code, $uw^i \in MX^*$ for some $i, i \geq 1$. There are two cases. First, assume $a(uw^i)^{\mathcal{A}} \neq \emptyset$ for any $a \in A$. In this case, $au^{\mathcal{A}} \neq \emptyset$ for any $a \in A$. Since $w \in M \subseteq L$, $(au^{\mathcal{A}})w^{\mathcal{A}} = (bu^{\mathcal{A}})w^{\mathcal{A}}$ for any

$a, b \in A$. Thus, $a(uw)^{\mathcal{A}} = b(uw)^{\mathcal{A}}$ for any $a, b \in A$. This means that $uw \in L$. Now, assume $a(uw^i)^{\mathcal{A}} = \emptyset$ for any $a \in A$. Suppose that there exists $a \in A$ such that $a(uw)^{\mathcal{A}} \neq \emptyset$. In this case, there exists a nonempty subset H of A such that $(au^{\mathcal{A}})w^{\mathcal{A}} = H \neq \emptyset$. Thus, $Hw^{\mathcal{A}} = H$ holds because $w \in L$. This implies that $a(uw^i)^{\mathcal{A}} = (a(uw^{i-1})^{\mathcal{A}})w^{\mathcal{A}} = H \neq \emptyset$, a contradiction. Consequently, $a(uw)^{\mathcal{A}} = \emptyset$ for any $a \in A$, and hence $uw \in L$. In either case, $uw \in L$, completing the proof of the proposition. □

Example 7. Let $X = \{x, y\}$ and let $M = \{x, yxx, yxy, yy\}$. Then, M is a finite maximal prefix code. Take $y \in P_r(M)$ and $x \in M$. Then, $yx \notin MX^*$. Therefore, $MX^* \notin \mathcal{L}_{\mathrm{ND}(2)}$.

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