The lexicographic decision function

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Abstract

In this paper the lexicographic decision process is presented in a unified way. We construct a lexicographic decision function using a universal preference function and a unary function. This construction incorporates the different outranking approaches,the lexicographic decision process and the utility based decision making models. Finally we consider the connection of the lexicographic decision method and the Arrow paradox.

1   Introduction

In this section we describe the concept of the lexicographic method. We use the terminology of P.C. Fishburn, see [6]. The general concept of a finite lexicographic order involves a set $I = \{1, 2, \ldots, n\}$ and an order relation $\prec_i$ on a nonempty set X_i for each $i \in I$. We let $\sim_i$ denote the symmetric complement of $\prec_i$ so that $x_i \sim_i y_i$ if and only if $(x_i \prec_i y_i$ or $y_i \prec_i x_i)$ does not hold. With $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, y precedes x lexicographically under the natural order $<$ on I and with respect to the $\prec_i$ or $x <^L y$ for short, iff $\{i : i \in I$ and $(x_i \prec_i y_i$ or $y_i \prec_i x_i)\}$ is nonempty, and $x_i \prec_i y_i$ for the first (smallest) i in this set.

For this reason a lexicographic order $<^L$ is also referred to as an order by first difference.

An example of a lexicographic order arises from the alphabetical order of words in a dictionary or lexicon. To show this let $I = \{1, 2, \ldots, n\}$, let $X_i = A = \{\emptyset, a, b, \ldots, z\}$ with $\emptyset \prec_i a \prec_i b \prec_i \ldots \prec_i z$ for each i , take n as large as the longest listed word , and let the English word $\alpha_1 \alpha_2 \ldots \alpha_m$ with $m \leq n$ correspond to $(\alpha_1, \alpha_2, \ldots, \alpha_m, \emptyset, \ldots, \emptyset)$ in A^n. Then $<^L$ on the subset of A^n which corresponds to the "legitimate " words orders these words in their natural alphabetical order. For example, "as" precedes "ask" since $(a, s, \emptyset, \ldots, \emptyset) <^L (a, s, k, \emptyset, \ldots, \emptyset)$ which is to say that $a \sim_1 a$, $s \sim_2 s$, $\emptyset \prec_3 k$.

In multicriteria decision making the idea of the lexicographic decision consists of a hierarchy or ordered set of attributes or criteria. Decision alternatives are examined initially on the basis of the first or most important criterion, If more than one alternative is "best" or "satisfactory" on this basis, then these are compared under

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the second most important criterion and so forth. The principle of order by first difference says that one alternative is "better" than another iff the first is "better" than the second on the most important criterion on which they differ.

So let x and y be two alternatives (actions) and $c_1, c_2, \ldots, c_n$ be different criteria, x_i and y_i are the utilities (evaluations) of x and y. We identify x and y with their evaluation vector $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$. Then $\prec_i$ is the order relation according to c_i on the set of alternatives.

We say that $x_i \sim_i y_i$ iff the alternatives x and y are indifferent according to the c_i, and we say that $x <^L y$ iff $x_i \sim_i y_i$ for $i \in \{1, 2, \ldots, k-1\}$ where $0 \leq k-1 \leq n-1$ and $x_k \prec_k y_k$ in other words the alternative y is preferred to the alternative x, according to the criteria c_k.

The lexicographic decision method is a well adaptable method. It can arrange data of arbitrary scales, and it is suitable to evaluate a set of considerable alternatives. This method does not require the weight of criteria and in spite of its simlicity always arranges the alternatives, Rapcsák [18]. Some decision procedures have lexicographic decision rules to prevent ties, Temesi [20]. Sequential screening procedures illustrate another common application of the lexicographic idea. Candidates or alternatives are first screened under a given criterion (perhaps with the use of a test or an interview) and separated into "rejects" and "others". In terms of $\prec_1$ of the set of candidates, $x \sim y$ whenever both x and y are "rejects" or "others", with $x \prec_1 y$ when x is a "reject" and y is an "other". The "others" are then screened further by the second criterion or test and sorted into two groups. Of course the "rejects" from the first stage may not be tested for the second stage, but that is of no importance from the viewpoint of the lexicographic rule except from the standpoint of efficiency that it may promote.This process may continue through several more stages, perhaps including a ranking of all candidates who survive to the last stage. Another aspect of the using of lexicographic decision method is to avoid the intransitivity of preference. If $\prec_i$ is a weak order for every i then $<^L$ is a weak order, if $\prec_i$ is a linear order for every i then $<^L$ is also a linear order, but when $\prec_i$ is a partial order for every i it does not follow that $<^L$ is a partial order, even if $<^L$ includes cycles. If $\prec_i \neq \emptyset$ then the lexicographic aggeregation preserves transitivity, Fishburn [6], Solymosi [19]. About a general concept of the preference cycles and its representation, see Dombi,Vincze [4].

In the evaluation of alternatives, according to the c_k criteria the values x_k, and y_k would be numerical values or categories. In the case of categories the lexicographic decision can be characterized with weighted criteria. We will prove that there exists a weighted representation of lexicographic decision method on the real numbers. This yields a universal form: PROMETHEE, ELECTRE and utility are special cases of it, see Dombi[3].

It is important to note that the solution of many MCDM problems requires the application of two or three decision methods. For example when the groups of criteria needs different aggregation procedures. In our model we can give different decision making methods by changing the parameters.

We construct a weighted method to get the decision function of the lexicographic decision method. We choose the weights in such a way, that a range of alternatives

by c_k criteria could not be changed by $c_{k+1}, c_{k+2}, \ldots, c_n$ criteria.

Finally we compare the conditions of the lexicographic decision method and the Arrow impossibility theorem.

The main aspect of our motivation is that the mentioned non-compensatory property arrange the criteria by their importancy and hence is the dictator in this decision model. So the dictatorship is an essential precept in this method.

In our paper we suppose that among the alternatives there are no two lexicographically equal.

2 The construction of the lexicographic decision function.

The lexicographic decision method is a seldom occuring theme in publications. For its numerical representation we could not find solution. It may follow from the negative results in this logic, for example the lexicographic order of the plane:

Theorem 1. *There does not exist any continuous $f(x,y)$ function, such that:*

$$(x,y) <^L (v,z) \text{ iff } f(x,y) < f(v,z).$$

Proof. Let the values x, x_1, x_2, y_1, y_2 be such that $x_1 < x < x_2$ and $y_1 < y_2$. We suppose that there exists continuous $f(x,y)$ function, for which:

$$(x,y) <^L (v,z) \text{ iff } f(x,y) < f(v,z).$$

Then for the mentioned values it is true, that:

$$(x,y_2) <^L (x_2,y_1) <^L (x_2,y_2) \text{ iff } f(x,y_2) <^L f(x_2,y_1) <^L f(x_2,y_2).$$

Because $f(x,y)$ is continuous, it is continuous at the point (x_2, y_2).

Let ε be an arbitrarily fixed positive value such that

$$\varepsilon < f(x_2,y_2) - f(x_2,y_1).$$

Then there exists a value δ, such that:

$$\text{if } |(x_2,y_2) - (x,y_2)| < \delta \text{ then } f(x_2,y_2) - f(x,y_2) < \varepsilon.$$

but

$$f(x_2,y_2) - f(x,y_2) > f(x_2,y_2) - f(x_2,y_1) > \varepsilon$$

which contradicts that $f(x,y)$ is continuous. $\qquad\square$

2.1   The preference and the modifier functions

In the introduction we shown the lexicographical decision concept. In this section we construct a lexicographical decision function. For the construction we use a general preference function $p(x, y)$ and a $\tau(x)$ modifier, (or threshold) function, which are the following, according to Dombi [3]:

$$p(x, y) = (y - x + 1)/2$$

$$\tau(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1/2, \\ 1/2 & \text{if } \quad x = 1/2, \\ 1 & \text{if } 1/2 < x \leq 1. \end{cases}$$

Let $A = \{a_1, a_2, \cdots, a_m\}$ be the set of alternatives. Let $C = \{c_1, c_2, \cdots, c_n\}$ be the set of the criteria, ordered by importance. Let x_{ij} denote the evaluation (utility) of c_j criteria in the case of choosing a_i as an alternative , $0 \leq x_{ij} \leq 1$. The decision situation can be described with the following decision matrix:

	c_1	c_2	$\ldots$	c_n
a_1	x_{11}	x_{12}	$\ldots$	x_{1n}
a_2	x_{21}	x_{22}	$\ldots$	x_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_m	x_{m1}	x_{m2}	$\ldots$	x_{mn}

2.1.1   Properties of the preference function

Let $p(x, y) = P(y - x)$ consider as the function of $y - x$, and let $0 \leq x, y \leq 1$. Then $y - x \in [-1, 1]$. We get that:

$$sign(y - x) = \begin{cases} -1 & \text{if } 0 \leq p(x, y) < 1/2 \\ 0 & \text{if } \quad p(x, y) = 1/2 \\ 1 & \text{if } 1/2 < p(x, y) \leq 1 \end{cases}$$

Then

$$P(sign(y - x)) = \begin{cases} 0 & \text{if } 0 \leq p(x, y) < 1/2 \\ 1/2 & \text{if } \quad p(x, y) = 1/2 \\ 1 & \text{if } 1/2 < p(x, y) \leq 1 \end{cases}$$

As decsribed in the introduction we identify the alternatives with its evaluation n-tuples, so we let

$$a_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \quad \text{and} \quad a_j = (x_{j1}, x_{j2}, \ldots, x_{jn}).$$

To order the alternatives a_i, and a_j with respect to criteria c_k, we set $x = x_{ik}$ and $y = x_{jk}$ in the preference function $p(x, y)$.

2.1.2 The composition of the preference and the modifier function.

Definition 1. *We can define for every (a_i, a_j) pair the $p^*(a_i, a_j)$ preference n-tuple in the following manner. Let*

$$p^*(a_i, a_j) = (\varepsilon_{ij}^1, \varepsilon_{ij}^2, \ldots, \varepsilon_{ij}^n) \quad for \quad \varepsilon_{ij}^k = \tau(p(x_{ik}, x_{jk})).$$

Then

$$\tau(p(x_{ik}, x_{jk})) = \begin{cases} 0 & \text{if} & x_{ik} > x_{jk} \\ 1/2 & \text{if} & x_{ik} = x_{jk} \\ 1 & \text{if} & x_{ik} < x_{jk} \end{cases}$$

The indicators ε_{ij}^k can be considered as the elements of a pairwise comparison matrix with respect to the c_k criterion.

c_k	a_1	a_2	$\ldots$	a_m
a_1	ε_{11}^k	ε_{12}^k	$\ldots$	ε_{1m}^k
a_2	ε_{21}^k	ε_{22}^k	$\ldots$	ε_{2m}^k
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_m	ε_{m1}^k	ε_{m2}^k	$\ldots$	ε_{mm}^k

All the elements in the main diagonal equal to 0.5. As mentioned before, we suppose, that among the alternatives there are no two lexicographically equal, so for each pair (a_i, a_j) $a_i = (x_{i1}, x_{i2}, \ldots, x_{in})$, $a_j = (x_{j1}, x_{j2}, \ldots, x_{jn})$, there exist k_1 and k_2 such that $x_{ik_1} < x_{jk_1}$ and $x_{jk_2} < x_{ik_2}$.

2.2 The lexicographic decision function

The main result of this paper is the following Theorem:

Theorem 2. *Let $A = \{a_1, a_2, \cdots, a_m\}$ be the set of alternatives. Let $C = \{c_1, c_2, \cdots, c_n\}$ be the set of criteria, ordered by importancy. Let x_{ij} denote the evaluation (utility) of criterion c_j in the case of choosing a_i as an alternative, $0 \leq x_{ij} \leq 1$. The decision situation can be described with the decision matrix:*

	c_1	c_2	$\ldots$	c_n
a_1	x_{11}	x_{12}	$\ldots$	x_{1n}
a_2	x_{21}	x_{22}	$\ldots$	x_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_m	x_{m1}	x_{m2}	$\ldots$	x_{mn}

Let $p(x, y)$ be the preference function and $\tau(x)$ be the modifier, (or threshold) function as we defined in section 2.1.

Then there exists w_k weights $k = 1, 2, \ldots, n$ such that the real numbers:

$$l_i = \frac{1}{m} \sum_{j=1}^{m} \tau(\sum_{k=1}^{n} w_k \tau(p(x_{ik}, x_{jk}))), \ i = 1, 2, \ldots, m$$

satisfy that

$$l_i < l_j \quad \text{if and only if} \quad a_i >^L a_j.$$

So we construct the lexicographic decision function with the help of a weighting system. This function is non compensatory. This we give in the following. Next we give the weighting system.

Let the weight of c_i criterion be:

$$w_i = 1/2^i + 1/(n2^n)$$

It can be verified, that:

$$\sum_{k=1}^{n} w_k = 1.$$

The lexicographic decision function is constructed with the following function composition:

$$\tau\left(\sum_{k=1}^{n} w_k \tau(p(x_{ik}, x_{jk}))\right) = \begin{cases} 0 & \text{if} & a_i >^L a_j \\ 1 & \text{if} & a_i <^L a_j \end{cases}$$

Since $\varepsilon_{ij}^k = \tau(p(x_{ik}, x_{jk}))$, we denote

$$\varepsilon_{ij} = \tau\left(\sum_{k=1}^{n} w_k \tau(p(x_{ik}, x_{jk}))\right).$$

The following matrix provides the pairwise comparison matrix with respect to the weighted system of criteria $(c_1, w_1; c_2, w_2; \ldots; c_n, w_n)$

(C, w)	a_1	a_2	$\ldots$	a_m
a_1	ε_{11}	ε_{12}	$\ldots$	ε_{1m}
a_2	ε_{21}	ε_{22}	$\ldots$	ε_{2m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_m	ε_{m1}	ε_{m2}	$\ldots$	ε_{mm}

All the elements in the main diagonal equal always to 0.5.

Normalizing the lexicographic decision function we get real l_i in the interval [0,1].

$$l_i = \frac{1}{m}\sum_{j=1}^{m} \tau\left(\sum_{k=1}^{n} w_k \tau(p(x_{ik}, x_{jk}))\right), \ i = 1, 2, \ldots, m$$

so that:

$$l_i < l_j \quad \text{iff} \quad a_i >^L a_j.$$

This sequence of real numbers is constructed in such a way, that for alternative a_i we aggregate the preferences between a_i and a_j for $j = 1, 2, \ldots, i-1, i+1, \ldots, m$.

This is the main idea of the global preference construction of the PROMETHEE method.

To prove the correctness of the construction, first we prove the correctness of the weighting.

Lemma 1. *Let $\varepsilon_{ij}^k = \tau(p(x_{ik}, x_{jk})$ as we defined it in 2.1.2. Then the following statements are true:*

(1) $\min\limits_{i,j} \sum\limits_{k=1}^{n} w_k \varepsilon_{ij}^k = 1/2 + 1/(n2^n)$ *if* $a_i <^L a_j$, *and it is minimal if*

$$(\varepsilon_{ij}^1, \varepsilon_{ij}^2, \ldots, \varepsilon_{ij}^n) = (1, 0, \ldots, 0).$$

(2) $\max\limits_{i,j} \sum\limits_{k=1}^{n} w_k \varepsilon_{ij}^k = 1/2 - 1/(n2^n)$ *if* $a_i >^L a_j$, *and it is maximal if*

$$(\varepsilon_{ij}^1, \varepsilon_{ij}^2, \ldots, \varepsilon_{ij}^n) = (0, 1, \ldots, 1).$$

Proof (of Lemma 1).

(1) If $a_i <^L a_j$ and $\varepsilon_{ij}^k = \tau(p(x_{ik}, x_{jk}))$ then a preference n-tuple

$$(\varepsilon_{ij}^1, \varepsilon_{ij}^2, \ldots, \varepsilon_{ij}^n) = (1/2, 1/2, \ldots, 1/2, 1, \varepsilon_{ij}^{t+2}, \ldots, \varepsilon_{ij}^n) \text{ for } 0 \le t < n$$

has minimal non-zero element, if $\varepsilon_{ij}^{t+2} = \varepsilon_{ij}^{t+3} = \ldots = \varepsilon_{ij}^n = 0$.

In this case:
$$\sum_{k=1}^{n} w_k \varepsilon_{ij}^k = 1/2 + (t/2 + 1)[1/(n2^n)].$$

It is minimal if $t = 0$. Then $(\varepsilon_{ij}^1, \varepsilon_{ij}^2, \ldots, \varepsilon_{ij}^n) = (1, 0, \ldots, 0)$ and the minimum is equal to $1/2 + 1/(n2^n)$.

(2) If $a_i >^L a_j$ then a preference n-tuple

$$(\varepsilon_{ij}^1, \varepsilon_{ij}^2, \ldots, \varepsilon_{ij}^n) = (1/2, 1/2, \ldots, 1/2, 0, \varepsilon_{ij}^{t+2}, \ldots, \varepsilon_{ij}^n) \quad \text{for} \quad 0 \le k < n$$

has minimal zero element, if $\varepsilon_{ij}^{t+2} = \varepsilon_{ij}^{t+3} = \ldots = \varepsilon_{ij}^n = 1$.

Then
$$\sum_{k=1}^{n} w_k \varepsilon_{ij}^k = 1/2 - (t/2 + 1)[1/(n2^n)].$$

This is maximal, if $t = 0$ and the maximum is $1/2 - 1/(n2^n)$.

Then
$$(\varepsilon_{ij}^1, \varepsilon_{ij}^2, \ldots, \varepsilon_{ij}^n) = (0, 1, \ldots, 1).$$

$\square$

Proof (of Theorem 2). By Lemma 1 we get for the weighted sum that:

$$0 \le \sum_{k=1}^{n} w_k \varepsilon_{ij}^k < 1/2 \quad if \quad a_i >^L a_j$$
$$1/2 < \sum_{k=1}^{n} w_k \varepsilon_{ij}^k \le 1 \quad if \quad a_i <^L a_j$$

Applying the modifier (or threshold) function $\tau(x)$ for this weighted sum, we obtain:

$$\tau(\sum_{k=1}^{n} w_k \tau(p(x_{ik}, x_{jk}))) = \left\{ \begin{array}{llll} 0 & if & a_i >^L a_j, \\ 1 & if & a_i <^L a_j. \end{array} \right.$$

So $\tau(\sum_{k=1}^{n} w_k \tau(p(x_{ik}, x_{jk})))$ gives the lexicographic preference ordering between alternatives. So with this construction we get a decision function. Then we get:

$$\sum_{j=1}^{m} \tau(\sum_{k=1}^{n} w_k \tau(p(x_{ik}, x_{jk}))) = \left| \left\{ a_j : a_i <^L a_j \right\} \right| + \frac{1}{2}.$$

To transform this number to the [0,1] interval, we get the real values:

$$l_i = (1/m)\sum_{j=1}^{m} \tau \left(\sum_{k=1}^{n} w_k \tau(p(x_{ik}, x_{jk})) \right)$$

for which

$$l_i < l_j \quad \text{if and only if} \quad a_i >^L a_j.$$

$\square$

2.3 The lexicographic decision method as the limit of decision methods

Using the mentioned $\tau(x)$ threshold function, we construct the lexicographic decision function. This form is the general form of decision functions (for examples of PROMETHEE, ELECTRE and utility). In this formulation the form of the general modifier (threshold) function is, see Dombi [3]:

$$\tau_{p_1 p_2}(x) = \left\{ \begin{array}{llll} 0 & if & 0 \leq x < p_1, \\ (x - p_1)/(p_2 - p_1) & if & p_1 \leq x \leq p_2, \\ 1 & if & p_2 < x \leq 1. \end{array} \right.$$

This function is linear in the interval $[p_1, p_2]$. Taking the limit of this function we obtain:

$$\lim_{p_1 \to \frac{1}{2}, p_2 \to \frac{1}{2}} \tau_{p_1 p_2}(x) = \tau(x).$$

So we get the lexicographic decision method as the limits of decision methods. These methods may be compensatorical or non compensatorical.

3   The lexicographic decision method, and the Arrow paradox

As mentioned in the introduction, the concept of arranging the criteria according to their importance and the lexicographic decision method is dictatorical. Because of this there may connections between the lexicographic rule, and Arrow's impossibility theorem. But conditions of Arrow's impossibility theorem are applied to the voting situation, and so the lexicographic decision situation should be applied to a voting situation.

Let the evaluation of alternatives with respect to criterion c_i be $x_{1i}, x_{2i}, \ldots, x_{ni}$. Let their order be $x_{1i}^* < x_{2i}^* < \ldots < x_{ni}^*$, and set $x_{ki}^* = \frac{k}{n}$, so we get simply an ordering on alternatives by c_i. To transform the voting situation to multicriteria decision situation we map the individuals to criteria. In this section the profile is a weak order on the alternatives based on a criteria (or individual). The social welfare function is a decision function which aggregates the criterion (or individual) ordering. Let R be the set of all possible weak orders on the set of alternatives. We say that an individual is a dictator if its preferences become automatically social prefrences.

The axioms and conditions of the Arrow paradox are the following, see Hwang, Lin [13]:

Axiom 1 (The preference relation is strongly complete). *For all a_i and a_j either a_i 'is preferred or indifferent to' a_j or a_j 'is preferred or indifferent to' a_i.*

Axiom 2 (The preference relation is transitive). *For all a_i and a_j and a_k: a_i 'is preferred or indifferent to' a_j and a_j 'is preferred or indifferent to' a_k imply a_i 'is preferred or indifferent to' a_k.*

Condition 1 (Universal domain). *The social welfare function(decision function) f is defined for all possible profiles of individual(criteria).*

Condition 2 (The weak Pareto concept). *If $a_k, a_l \in A$ and $a_k \prec_i a_l$ for $i = 1, 2, \ldots, n$ then $a_k <^L a_l$.*

Condition 3 (Independence from irrelevant alternatives). *$R^{(a_i, a_j)} = F^{(a_i, a_j)}(p^{(a_i, a_j)})$, for every pair $(a_i, a_j) \in AxA$, where $R^{(a_i, a_j)}, F^{(a_i, a_j)}, p^{(a_i, a_j)}$ are the contraction of the social preference ordering, the social welfare function (i.e. the social decision function), and the p profile, to the pair (a_i, a_j).*

Condition 4 (Non-dictatorship). *There is no dictator in the society, i.e.there is no individual that whenever he prefers a_i to a_j for any a_i and a_j society does likewise regardless of the preferences of other individuals.*

Theorem 3 (General possibility theorem(Arrow)). *If there are at least two individuals, and three alternatives, which the members of the society are free to order in any way,(condition 1.) then every social welfare function satisfying condition 2 and 3 and yielding a social ordering satisfying axioms I. and II. must be dictatorical.*

It means that if a given social welfare function satisfies conditions 1-4, then a contradiction arises.

It can be seen that the lexicographic decision function satisfies Axioms 1-2 and Conditions 1-3.

We now consider the formulation in which there are preference orders $\prec_i$ on the set of alternatives for each criteria along with holistic order $<^L$ on A, see Fishburn [5],[6], May [15], Plott [17].

We shall refer to an $n+1$ tuple $(\prec_1, \prec_2, \ldots, \prec_n, <^L)$ of weak orders on A as a situation. Then we consider the possibility that any one of a number of potential situation might arise.

Theorem 4. *Let us suppose that A contains at least three alternatives, (A is otherwise unlimited) and every n-tuple $(\prec_1, \prec_2, \ldots, \prec_n)$ of weak orders on A appears in at least one situation. Then preferences are lexicographic, iff the following hold for all situations $(\prec_1, \prec_2, \ldots, \prec_n, <^L)$ and $(\prec_1', \prec_2', \ldots, \prec_n', <'^L)$ and all $a_j, a_k \in A$:*
$(a_j \sim_i a_k$ for all $i) \Longrightarrow a_j \sim a_k$; $(a_j \precsim_i a_k$ for all $i)$ & $(a_j \prec_i a_k$ for some $i) \Longrightarrow a_j <^L a_k$,
and $(a_j \prec_i a_k$ iff $a_j \prec_i' a_k)$ & $(a_k \prec_i a_j$ iff $a_k \prec_i' a_j$ for all $i) \Longrightarrow (a_j <^L a_k$ iff $a_k <^L a_j)$ & $(a_k <^L a_j$ iff $a_j <^L a_k)$.

Now we compare the axioms and conditions of the lexicographic decision method, and the Arrow paradox. We shall refer to the lexicographic method and the Arrow Paradox in this comparison by letters L and A, respectively.

1. *Preference completness and transitivity*

   L: The preference is a weak order, so it is strongly complete and transitive.

   A: The preference is strongly complete, and transitive.

2. *Universal domain*

   L: Every n-tuple $(\prec_1, \prec_2, \ldots, \prec_n)$ of weak orders on A appears in at least one situation.

   A: The social welfare function(decision function) f is defined for all possible profiles of individual (criteria).

3. *The Pareto concepts*

   L: *The strong Pareto concept $(a_j \sim_i a_k$ for all $i) \Longrightarrow a_j \sim a_k$; $(a_j \precsim_i a_k$ for all i, & $a_j \prec_i a_k$ for some $i) \Longrightarrow a_j <^L a_k$,*

   A: *The weak Pareto concept If $a_k, a_l \in A$ and $a_k \prec_i a_l$ for $i = 1, 2, \ldots, n$ then $a_k <^L a_l$.*

4. *Independence from irrelevant alternatives*

   L: $(a_j \prec_i a_k$ iff $a_j \prec_i' a_k)$&$(a_k \prec_i a_j$ iff $a_k \prec_i' a_j$ for all $i) \Longrightarrow (a_j <^L a_k$ iff $a_k <^L a_j)$&$(a_k <^L a_j$ iff $a_j <^L a_k)$,

A: $R^{(a_i,a_j)} = F^{(a_i,a_j)}(p^{(a_i,a_j)})$, for every pair $(a_i, a_j) \in AxA$, where $R^{(a_i,a_j)}$, $F^{(a_i,a_j)}$ and $p^{(a_i,a_j)}$ are the contraction of the social preference ordering, the social welfare function (i.e. the social decision function), and the p profile to the pair (a_i, a_j).

It seems, that the conditions used in Arrow's impossibility theorem for a 'social welfare function' are formally similar to the conditions of Theorem 4, or are the same of the condition of this theorem. As it is mentioned above we can set a multicriteria decision situation to a voting situation. Then $\prec_i$ is interpreted as the prefrence order for the ith individual or voter. The Arrowian axioms, and the condition of universal domain and independence from irrelevant alternatives are the same as the conditions of theorem of the lexicographic decision. The difference is that while Arrow's theorem uses strong Pareto concept, the theorem of lexicographic decision method uses weak Pareto concept, and Arrow adds the condition that no individual shall be a 'dictator'. The main result of Arrow's theorem, to be shown that all conditions other than the nondictatorship condition imply that some individual is a dictator.

By deleting specific references to dictators and replacing the weak Pareto concept with the strong Pareto concept, as in Theorem 4., we derive a hierarchy of 'dictators' $\sigma(1), \sigma(2), \ldots, \sigma(n)$,which verifies the existence of lexicographic preferences.

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