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Kleene Theorems for skew formal power series*

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Abstract

We investigate the theory of skew (formal) power series introduced by Droste, Kuske [5, 6], if the basic semiring is a Conway semiring. This yields Kleene Theorems for skew power series, whose supports contain finite and infinite words. We then develop a theory of convergence in semirings of skew power series based on the discrete convergence. As an application this yields a Kleene Theorem proved already by Droste, Kuske [5].

1 Introduction and preliminaries

The purpose of our paper is to investigate the skew formal power series introduced by Droste, Kuske [5, 6]. These skew formal power series are a clever generalization of the ordinary power series and are defined as follows.

Let A be a semiring and $\varphi : A \to A$ be an endomorphism of this semiring. Then Droste, Kuske [5] define the *φ-skew product* $r \odot_\varphi s$ of two power series $r, s \in A^{\Sigma^*}$, Σ an alphabet, by

$$(r \odot_\varphi s, w) = \sum_{uv=w} (r, u)\varphi^{|u|}(s, v)$$

for all $w \in \Sigma^*$. They denote the structure $(A^{\Sigma^*}, +, \odot_\varphi, 0, 1)$ by $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ and prove the following result.

Theorem 1 (Droste, Kuske [5]). *The structure $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is a semiring.*

They call $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ the *semiring of skew (formal) power series* (*over* Σ^*).

In the sequel, we often denote $\odot_\varphi$ simply by $\cdot$ or concatenation and A, φ and Σ denote a semiring, an endomorphism $\varphi : A \to A$ and an alphabet, respectively.

The paper consists of this and four more sections. In this section we give a survey on the results achieved by this paper and then define the necessary algebraic structures: starsemirings, Conway semirings, semimodules, starsemiring-omegasemimodule pairs, Conway semiring-semimodule pairs, complete semiring-semimodule pairs and quemirings. These algebraic structures, due to Elgot [8], Bloom, Ésik [2] and Ésik, Kuich [9] give an algebraic basis for the theory of power

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series, whose supports contain finite and infinite words. At the end of this section we refer to some examples for these algebraic structures.

In Section 2 we prove that the semiring of skew power series over a Conway semiring is again a Conway semiring. Moreover, we prove two isomorphisms of certain semirings defined in connection with Conway semirings.

In Section 3, the results of Section 2 are applied to finite automata. A Kleene Theorem over quemirings defined by skew power series over Conway semirings and the usual Kleene Theorem over Conway semirings are shown.

In Section 4, we consider a semiring-semimodule pair defined by skew power series and prove that under certain conditions this pair is complete. This gives rise to another Kleene Theorem that is then applied to a tropical semiring and yields a result already achieved by Droste, Kuske [5].

In the last section we develop a theory of convergence in semirings of skew power series based on the discrete convergence. We show that important equations, which hold in Conway semirings, are valid under certain conditions also in semirings of skew power series over an arbitrary semiring. As an application this yields then another Kleene Theorem proved already by Droste, Kuske [5].

We assume that the reader of this paper is familiar with the theory of semirings as given in Sections 1–4 of Kuich, Salomaa [14]. Familiarity with Ésik, Kuich [9, 10, 11] is desired.

Recall that a *starsemiring* is a semiring A equipped with a star operation $^* : A \to A$. The *Conway identities* are the *sum-star equation* and the *product-star equation*

$$\begin{aligned}(a+b)^* &= (a^*b)^*a^* \\ (ab)^* &= 1 + a(ba)^*b.\end{aligned}$$

A *Conway semiring* is a starsemiring satisfying the Conway equations. Note that any Conway semiring satisfies the *star fixed point equations*

$$\begin{aligned}aa^* + 1 &= a^* \\ a^*a + 1 &= a^*,\end{aligned}$$

as well as the equations

$$\begin{aligned}a(ba)^* &= (ab)^*a \\ (a+b)^* &= a^*(ba^*)^*.\end{aligned}$$

Suppose that A is a semiring and V is a commutative monoid written additively. We call V a (left) A-*semimodule* if V is equipped with a (left) action

$$\begin{aligned}A \times V &\to V \\ (s, v) &\mapsto sv\end{aligned}$$

subject to the following rules:

$$\begin{aligned} s(s'v) &= (ss')v \\ (s+s')v &= sv + s'v \\ s(v+v') &= sv + sv' \\ 1v &= v \\ 0v &= 0 \\ s0 &= 0, \end{aligned}$$

for all $s, s' \in A$ and $v, v' \in V$. When V is an A-semimodule, we call (A, V) a *semiring-semimodule pair*.

Suppose that (A, V) is a semiring-semimodule pair such that A is a starsemiring and A and V are equipped with an omega operation ${}^\omega : A \to V$. Then we call (A, V) a *starsemiring-omegasemimodule pair*. Following Bloom, Ésik [2], we call a starsemiring-omegasemimodule pair (A, V) a *Conway semiring-semimodule pair* if A is a Conway semiring and if the omega operation satisfies the *sum-omega equation* and the *product-omega equation*:

$$\begin{aligned} (a+b)^\omega &= (a^*b)^\omega + (a^*b)^*a^\omega \\ (ab)^\omega &= a(ba)^\omega, \end{aligned}$$

for all $a, b \in A$. It then follows that the *omega fixed-point equation* holds, i.e.,

$$aa^\omega = a^\omega,$$

for all $a \in A$.

Recall that a *complete monoid* is a commutative monoid $(M, +, 0)$ equipped with all sums $\sum_{i \in I} m_i$ such that

$$\begin{aligned} \sum_{i \in \emptyset} &= 0 \\ \sum_{j \in \{1\}} m &= m \\ \sum_{i \in \{1,2\}} m_i &= m_1 + m_2 \\ \sum_{j \in J} \sum_{i \in I_j} m_i &= \sum_{i \in \cup_{j \in J} I_j} m_i, \end{aligned}$$

where in the last equation it is assumed that the sets I_j are pairwise disjoint. A *complete semiring* is a semiring A which is also a complete monoid satisfying the distributive laws

$$\begin{aligned} s(\sum_{i \in I} s_i) &= \sum_{i \in I} ss_i \\ (\sum_{i \in I} s_i)s &= \sum_{i \in I} s_i s, \end{aligned}$$

for all $s \in A$ and for all families $s_i,\ i \in I$ over A. Ésik, Kuich [9] define a *complete semiring-semimodule pair* to be a semiring-semimodule pair (A, V) such that A is a complete semiring, V is a complete monoid and an *infinite product operation*

$$(s_1, s_2, \ldots) \quad \mapsto \quad \prod_{j\geq 1} s_j$$

is given mapping infinite sequences over A to V with

$$\begin{aligned} s(\sum_{i\in I} v_i) &= \sum_{i\in I} s v_i \\ (\sum_{i\in I} s_i) v &= \sum_{i\in I} s_i v, \end{aligned}$$

for all $s \in A$, $v \in V$, and for all families $s_i,\ i \in I$ over A and $v_i,\ i \in I$ over V and with the following three conditions:

$$\begin{aligned} \prod_{i\geq 1} s_i &= \prod_{i\geq 1}(s_{n_{i-1}+1} \cdot \ldots \cdot s_{n_i}) \\ s_1 \cdot \prod_{i\geq 1} s_{i+1} &= \prod_{i\geq 1} s_i \\ \prod_{j\geq 1}\sum_{i_j\in I_j} s_{i_j} &= \sum_{(i_1,i_2,\ldots)\in I_1\times I_2\times\ldots}\ \prod_{j\geq 1} s_{i_j}, \end{aligned}$$

where in the first equation $0 = n_0 \leq n_1 \leq n_2 \leq \ldots$ and $I_1, I_2, \ldots$ are arbitrary index sets. Suppose that (A, V) is complete. Then we define

$$\begin{aligned} s^* &= \sum_{i\geq 0} s^i \\ s^\omega &= \prod_{i\geq 1} s, \end{aligned}$$

for all $s \in A$. This turns (A, V) into a starsemiring-omegasemimodule pair. By Ésik, Kuich [9], each complete semiring-semimodule pair is a Conway semiring-semimodule pair. Observe that, if (A, V) is a complete semiring-semimodule pair, then $0^\omega = 0$.

A *star-omega semiring* is a semiring A equipped with unary operations * and $^\omega : A \to A$. A star-omega semiring A is called *complete* if (A, A) is a complete semiring-semimodule pair, i. e., if A is complete and is equipped with an infinite product operation that satisfies the three conditions stated above.

Consider a starsemiring-omegasemimodule pair (A, V). Then, following Conway [4], we define, for all $n \geq 0$, the operation $^* : A^{n\times n} \to A^{n\times n}$ by the following inductive definition. When $n = 0$, M^* is the unique 0×0-matrix, and when $n = 1$, so that $M = (a)$, for some a in A, $M^* = (a^*)$. Assuming that $n > 1$, let us write M as

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1}$$

where a is 1×1 and d is $(n-1) \times (n-1)$. We define

$$M^* = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{2}$$

where $\alpha = (a + bd^*c)^*$, $\beta = a^*b\delta$, $\gamma = d^*c\alpha$, $\delta = (d + ca^*b)^*$.

Following Bloom, Ésik [2], we define a matrix operation $^\omega : A^{n\times n} \to V^{n\times 1}$ on a starsemiring-omegasemimodule pair (A, V) as follows. When $n = 0$, M^ω is the unique element of V^0, and when $n = 1$, so that $M = (a)$, for some $a \in A$, $M^\omega = (a^\omega)$. Assume now that $n > 1$ and write M as in (1). Then

$$M^\omega = \begin{pmatrix} (a + bd^*c)^\omega + (a + bd^*c)^*bd^\omega \\ (d + ca^*b)^\omega + (d + ca^*b)^*ca^\omega \end{pmatrix}. \tag{3}$$

Following Ésik, Kuich [11], we define matrix operations $^{\omega_k} : A^{n\times n} \to V^{n\times 1}$, $0 \le k \le n$, as follows. Assume that $M \in A^{n\times n}$ is decomposed into blocks a, b, c, d as in (1), but with a of dimension $k \times k$ and d of dimension $(n-k) \times (n-k)$. Then

$$M^{\omega_k} = \begin{pmatrix} (a + bd^*c)^\omega \\ d^*c(a + bd^*c)^\omega \end{pmatrix} \tag{4}$$

Observe that $M^{\omega_0} = 0$ and $M^{\omega_n} = M^\omega$.

Suppose that (A, V) is a semiring-semimodule pair and consider $T = A \times V$. Define on T the operations

$$\begin{aligned} (s, u) \cdot (s', v) &= (ss', u + sv) \\ (s, u) + (s', v) &= (s + s', u + v) \end{aligned}$$

and constants $0 = (0, 0)$ and $1 = (1, 0)$. Equipped with these operations and constants, T satisfies the equations

$$(x + y) + z = x + (y + z) \tag{5}$$
$$x + y = y + x \tag{6}$$
$$x + 0 = x \tag{7}$$
$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \tag{8}$$
$$x \cdot 1 = x \tag{9}$$
$$1 \cdot x = x \tag{10}$$
$$(x + y) \cdot z = (x \cdot z) + (y \cdot z) \tag{11}$$
$$0 \cdot x = 0. \tag{12}$$

Elgot[8] also defined the unary operation ¶ on T: $(s, u)¶ = (s, 0)$. Thus, ¶ selects the "first component" of the pair (s, u), while multiplication with 0 on the right selects the "second component", for $(s, u) \cdot 0 = (0, u)$, for all $u \in V$. The new operation satisfies:

$$
\begin{aligned}
x\P \cdot (y+z) &= (x\P \cdot y) + (x\P \cdot z) && (13)\\
x &= x\P + (x \cdot 0) && (14)\\
x\P \cdot 0 &= 0 && (15)\\
(x+y)\P &= x\P + y\P && (16)\\
(x \cdot y)\P &= x\P \cdot y\P. && (17)
\end{aligned}
$$

Note that when V is idempotent, also

$$x \cdot (y+z) = x \cdot y + x \cdot z$$

holds.

Elgot[8] defined a *quemiring* to be an algebraic structure T equipped with the above operations $\cdot, +, \P$ and constants $0, 1$ satisfying the equations (5)–(12) and (13)–(17). A morphism of quemirings is a function preserving the operations and constants. It follows from the axioms that $x\P\P = x\P$, for all x in a quemiring T. Moreover, $x\P = x$ iff $x \cdot 0 = 0$.

When T is a quemiring, $A = T\P = \{x\P \mid x \in T\}$ is easily seen to be a semiring. Moreover, $V = T0 = \{x \cdot 0 \mid x \in T\}$ contains 0 and is closed under $+$, and, furthermore, $sx \in V$ for all $s \in A$ and $x \in V$. Each $x \in T$ may be written in a unique way as the sum of an element of $T\P$ and a sum of an element of $T0$ as $x = x\P + x \cdot 0$. Sometimes, we will identify $A \times \{0\}$ with A and $\{0\} \times V$ with V. It is shown in Elgot [8] that T is isomorphic to the quemiring $A \times V$ determined by the semiring-semimodule pair (A, V).

Suppose now that (A, V) is a starsemiring-omegasemimodule pair. Then we define on $T = A \times V$ a *generalized star operation*:

$$(s, v)^{\otimes} = (s^*, s^{\omega} + s^* v) \tag{18}$$

for all $(s, v) \in T$. Note that the star and omega operations can be recovered from the generalized star operation, since s^* is the first component of $(s, 0)^{\otimes}$ and s^{ω} is the second component. Thus:

$$
\begin{aligned}
(s^*, 0) &= (s, 0)^{\otimes}\P\\
(0, s^{\omega}) &= (s, 0)^{\otimes} \cdot 0.
\end{aligned}
$$

Observe that, for $(s, 0) \in A \times \{0\}$, $(s, 0)^{\otimes} = (s^*, 0) + (0, s^{\omega})$.

Suppose now that T is an (abstract) quemiring equipped with a generalized star operation $^{\otimes}$. As explained above, T as a quemiring is isomorphic to the quemiring $A \times V$ associated with the semiring-semimodule pair (A, V), where $A = T\P$ and $V = T0$, an isomorphism being the map $x \mapsto (x\P, x \cdot 0)$. It is clear that a generalized star operation $^{\otimes} : T \to T$ is determined by a star operation $^* : A \to A$ and an omega operation $^{\omega} : A \to V$ by (18) iff

$$
\begin{aligned}
x^{\otimes}\P &= (x\P)^{\otimes}\P && (19)\\
x^{\otimes} \cdot 0 &= (x\P)^{\otimes} \cdot 0 + x^{\otimes}\P \cdot x \cdot 0 && (20)
\end{aligned}
$$

hold. Indeed, these conditions are clearly necessary. Conversely, if (19) and (20) hold, then for any $x¶ \in T¶$ we may define

$$(x¶)^* = (x¶)^{\otimes} ¶ \tag{21}$$
$$(x¶)^\omega = (x¶)^{\otimes} \cdot 0\,. \tag{22}$$

It follows that (18) holds. The definition of star and omega was forced.

Let us call a quemiring equipped with a generalized star operation $^{\otimes}$ a *generalized starquemiring*. Morphisms of generalized starquemirings preserve the quemiring structure and the $^{\otimes}$ operation.

We now refer to some examples for the algebraic structures defined in this section. All the following semiring-semimodule pairs are complete. Hence, they are starsemiring-omegasemimodule pairs and Conway semiring-semimodule pairs, and by (18) give rise to a generalized starquemiring.

(i) The pair $(\mathfrak{P}(\Sigma^*), \mathfrak{P}(\Sigma^\omega))$, where Σ is an alphabet and $\mathfrak{P}$ denotes the power set, is a complete semiring-semimodule pair. The first component of this pair is the set of formal languages over finite words over Σ, the second component is the set of formal languages over infinite words over Σ. (See Ésik, Kuich [10], Example 3.2.)

(ii) The pair $(\mathbb{N}^\infty\langle\langle\Sigma^*\rangle\rangle, \mathbb{N}^\infty\langle\langle\Sigma^\omega\rangle\rangle)$, where $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ denotes the complete semiring of nonnegative integers augmented by ∞ with the usual operations, is a complete semiring-semimodule pair. The first component of this pair is the set of power series with coefficients in $\mathbb{N}^\infty$ over the finite words over Σ, the second component is the set of power series with coefficients in $\mathbb{N}^\infty$ over the infinite words over Σ. This pair is used if ambiguities of the formal languages in (i) are considered. (See Ésik, Kuich [10], Example 3.3.)

(iii) The pair $(\mathbb{R}^\infty_{\max,q}\langle\langle\Sigma^*\rangle\rangle, \mathbb{R}^\infty_{\max,q}\langle\langle\Sigma^\omega\rangle\rangle)$ is a complete semiring-semimodule pair. It is defined before Corollary 30.

(iv) The clock languages of Bouyer, Petit [3] give rise to a complete semiring-semimodule pair. (See Ésik, Kuich [11].)

2 Skew power series over Conway semirings

Let A be a starsemiring. Then, for $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$, we define $r^* \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$, called the *star of* r by

$$\begin{aligned}(r^*, \varepsilon) &= (r, \varepsilon)^*\,,\\ (r^*, w) &= (r, \varepsilon)^* \cdot \sum_{uv=w,\ u\neq\varepsilon} (r, u)\varphi^{|u|}(r^*, v)\,.\end{aligned}$$

Moreover, we define $r^+ \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ by $r^+ = rr^*$. We prove now the result that the structure $\langle A^{\Sigma^*}, +, \odot_\varphi, *, 0, 1\rangle$, again denoted by $A_\varphi\langle\langle\Sigma^*\rangle\rangle$, is a Conway semiring if A is a Conway semiring. The proof of this result is a generalization of the proofs of Theorems 2.19, 2.20, 2.21 of Aleshnikov, Boltnev, Ésik, Ishanov, Kuich, Malachowskij [1]

Theorem 2. *Let A be a Conway semiring, $\varphi : A \to A$ be an endomorphism and Σ be an alphabet. Then the sum-star equation holds in $A_\varphi\langle\langle\Sigma^*\rangle\rangle$.*

Proof. Let $r, s \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$. Then we prove by induction on the length of $w \in \Sigma^*$ that $((r+s)^*, w) = ((r^*s)^*r^*, w)$. The case $w = \varepsilon$ is clear. Assume now $w \neq \varepsilon$. Then we obtain $((r+s)^*, w) = ((r+s)^*, \varepsilon) \sum_{uv=w,\ u\neq\varepsilon} (r+s, u)\varphi^{|u|}((r+s)^*, v) = ((r+s)^*, \varepsilon) \sum_{uv=w,\ u\neq\varepsilon} (r, u)\varphi^{|u|}((r+s)^*, v) + ((r+s)^*, \varepsilon) \sum_{uv=w,\ u\neq\varepsilon} (s, u)\varphi^{|u|}((r+s)^*, v)$. We call the first and second of these terms L_1 and L_2, respectively. Moreover, we obtain

$$
\begin{aligned}
&((r^*s)^*r^*, w) = \textstyle\sum_{w_1w_2=w} ((r^*s)^*, w_1)\varphi^{|w_1|}(r^*, w_2) = \\
&((r^*s)^*, \varepsilon)(r^*, w) + \textstyle\sum_{w_1w_2=w,\ w_1\neq\varepsilon} ((r^*s)^*, w_1)\varphi^{|w_1|}(r^*, w_2) = \\
&((r^*s)^*, \varepsilon)(r^*, w) + \\
&\quad ((r^*s)^*, \varepsilon) \textstyle\sum_{w_1w_2=w} \sum_{u_1v_1=w_1,\ u_1\neq\varepsilon} (r^*s, u_1)\varphi^{|u_1|}((r^*s)^*, v_1)\varphi^{|w_1|}(r^*, w_2) = \\
&((r^*s)^*, \varepsilon)(r^*, w) + ((r^*s)^*, \varepsilon) \textstyle\sum_{w_1w_2=w} \sum_{u_1v_1=w_1,\, u_1\neq\varepsilon} \sum_{w_3w_4=u_1} \\
&\quad (r^*, w_3)\varphi^{|w_3|}(s, w_4)\varphi^{|u_1|}((r^*s)^*, v_1)\varphi^{|w_1|}(r^*, w_2) = \\
&((r^*s)^*, \varepsilon)(r^*, w) + ((r^*s)^*, \varepsilon) \cdot \\
&\quad \textstyle\sum_{w_1w_2=w} \sum_{u_1v_1=w_1,\ u_1\neq\varepsilon} (r^*, \varepsilon)(s, u_1)\varphi^{|u_1|}((r^*s)^*, v_1)\varphi^{|w_1|}(r^*, w_2) + \\
&\quad ((r^*s)^*, \varepsilon) \textstyle\sum_{w_1w_2=w} \sum_{u_1v_1=w_1} \sum_{w_3w_4=u_1,\, w_3\neq\varepsilon} \\
&\quad (r^*, w_3)\varphi^{|w_3|}(s, w_4)\varphi^{|u_1|}((r^*s)^*, v_1)\varphi^{|w_1|}(r^*, w_2).
\end{aligned}
$$

We call the first, second and third of these terms R_1, R_2 and R_3, respectively.

Eventually, we obtain

$$
\begin{aligned}
&R_2 = ((r+s)^*, \varepsilon) \textstyle\sum_{u_1z=w,\ u_1\neq\varepsilon} (s, u_1)\varphi^{|u_1|}((r^*s)^*r^*, z) = \\
&((r+s)^*, \varepsilon) \textstyle\sum_{u_1z=w,\ u_1\neq\varepsilon} (s, u_1)\varphi^{|u_1|}((r+s)^*, z) = L_2
\end{aligned}
$$

and

$$
\begin{aligned}
&R_1 + R_3 = ((r^*s)^*, \varepsilon)(r^*, \varepsilon) \textstyle\sum_{uv=w,\ u\neq\varepsilon} (r, u)\varphi^{|u|}(r^*, v) + \\
&\quad ((r^*s)^*, \varepsilon) \textstyle\sum_{w_1w_2=w} \sum_{u_1v_1=w_1} \sum_{w_3w_4=u_1} (r^*, \varepsilon) \cdot \\
&\quad \textstyle\sum_{u_2v_2=w_3,\ u_2\neq\varepsilon} (r, u_2)\varphi^{|u_2|}(r^*, v_2)\varphi^{|w_3|}(s, w_4)\varphi^{|u_1|}((r^*s)^*, v_1)\varphi^{|w_1|}(r^*, w_2) = \\
&((r+s)^*, \varepsilon) \textstyle\sum_{uv=w,\ u\neq\varepsilon} (r, u)\varphi^{|u|}(r^*, v) + \\
&\quad ((r+s)^*, \varepsilon) \textstyle\sum_{u_2z=w,\ u_2\neq\varepsilon} (r, u_2)\varphi^{|u_2|}((r^*s)^+r^*, z) = \\
&((r+s)^*, \varepsilon) \textstyle\sum_{u_2z=w,\ u_2\neq\varepsilon} (r, u_2)\varphi^{|u_2|}((r^*s)^*r^*, z) = \\
&((r+s)^*, \varepsilon) \textstyle\sum_{u_2z=w,\ u_2\neq\varepsilon} (r, u_2)\varphi^{|u_2|}((r+s)^*, z) = L_1\,.
\end{aligned}
$$

Hence, $L_1 + L_2 = R_1 + R_2 + R_3$ and the sum-star equation holds in $A_\varphi\langle\langle\Sigma^*\rangle\rangle$. □

Theorem 3. *Let A be a Conway semiring, $\varphi : A \to A$ be an endomorphism and Σ be an alphabet. Then, for $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$, the following equation is satisfied:*

$$r^* = \varepsilon + rr^*\,.$$

Proof. We prove by induction on the length of $w \in \Sigma^*$ that $(r^*, w) = (\varepsilon + rr^*, w)$. The case $w = \varepsilon$ is clear. Assume now $w \neq \varepsilon$. Then we obtain

$$\begin{aligned}
&(\varepsilon + rr^*, w) = \sum_{w_1 w_2 = w} (r, w_1)\varphi^{|w_1|}(r^*, w_2) = \\
&(r, \varepsilon)(r^*, w) + \sum_{w_1 w_2 = w,\ w_1 \neq \varepsilon} (r, w_1)\varphi^{|w_1|}(r^*, w_2) = \\
&(r, \varepsilon)(r^*, \varepsilon) \sum_{uv = w,\ u \neq \varepsilon} (r, u)\varphi^{|u|}(r^* v) + \sum_{w_1 w_2 = w,\ w_1 \neq \varepsilon} (r, w_1)\varphi^{|w_1|}(r^*, w_2) = \\
&(r^+, \varepsilon) \sum_{uv = w,\ u \neq \varepsilon} (r, u)\varphi^{|u|}(r^* v) + \sum_{w_1 w_2 = w,\ w_1 \neq \varepsilon} (r, w_1)\varphi^{|w_1|}(r^*, w_2) = \\
&(r^*, \varepsilon) \sum_{uv = w,\ u \neq \varepsilon} (r, u)\varphi^{|u|}(r^* v) = (r^*, w)\,.
\end{aligned}$$

□

Theorem 4. *Let A be a Conway semiring, $\varphi : A \to A$ be an endomorphism and Σ be an alphabet. Then, for $r, s \in A_\varphi\langle\langle \Sigma^* \rangle\rangle$, the following equation is satisfied:*

$$r(sr)^* = (rs)^* r\,.$$

Proof. We prove by induction on the length of $w \in \Sigma^*$ that $(r(sr)^*, w) = ((rs)^* r, w)$. The case $w = \varepsilon$ is clear. Assume now $w \neq \varepsilon$. Then we obtain

$$\begin{aligned}
&(r(sr)^*, w) = \sum_{w_1 w_2 = w} (r, w_1)\varphi^{|w_1|}((sr)^*, w_2) = \\
&(r, \varepsilon)((sr)^*, w) + \sum_{w_1 w_2 = w,\ w_1 \neq \varepsilon} (r, w_1)\varphi^{|w_1|}((sr)^*, w_2) = \\
&(r, \varepsilon)((sr)^*, \varepsilon) \sum_{uv = w,\ u \neq \varepsilon} (sr, u)\varphi^{|u|}((sr)^*, v) + \\
&\quad \sum_{w_1 w_2 = w,\ w_1 \neq \varepsilon} (r, w_1)\varphi^{|w_1|}((sr)^*, w_2) = \\
&(r(sr)^*, \varepsilon) \sum_{uv = w,\ u \neq \varepsilon} \sum_{w_3 w_4 = u} (s, w_3)\varphi^{|w_3|}(r, w_4)\varphi^{|u|}((sr)^*, v) + \\
&\quad \sum_{w_1 w_2 = w,\ w_1 \neq \varepsilon} (r, w_1)\varphi^{|w_1|}((sr)^*, w_2) = \\
&(r(sr)^*, \varepsilon) \sum_{uv = w,\ u \neq \varepsilon} (s, \varepsilon)(r, u)\varphi^{|u|}((sr)^*, v) + \\
&\quad (r(sr)^*, \varepsilon) \sum_{uv = w} \sum_{w_3 w_4 = u,\ w_3 \neq \varepsilon} (s, w_3)\varphi^{|w_3|}(r, w_4)\varphi^{|u|}((sr)^*, v) + \\
&\quad \sum_{w_1 w_2 = w,\ w_1 \neq \varepsilon} (r, w_1)\varphi^{|w_1|}((sr)^*, w_2) = \\
&((rs)^+, \varepsilon) \sum_{uv = w,\ u \neq \varepsilon} (r, u)\varphi^{|u|}((sr)^*, v) + \\
&\quad (r(sr)^*, \varepsilon) \sum_{uv = w} \sum_{w_3 w_4 = u,\ w_3 \neq \varepsilon} (s, w_3)\varphi^{|w_3|}(r, w_4)\varphi^{|u|}((sr)^*, v) + \\
&\quad \sum_{w_1 w_2 = w,\ w_1 \neq \varepsilon} (r, w_1)\varphi^{|w_1|}((sr)^*, w_2) = \\
&((rs)^*, \varepsilon) \sum_{uv = w,\ u \neq \varepsilon} (r, u)\varphi^{|u|}((sr)^*, v) + \\
&\quad (r(sr)^*, \varepsilon) \sum_{w_3 z = w,\ w_3 \neq \varepsilon} (s, w_3)\varphi^{|w_3|}(r(sr)^*, z)
\end{aligned}$$

and

$$\begin{aligned}
&((rs)^* r, w) = \sum_{w_1 w_2 = w} ((rs)^*, w_1)\varphi^{|w_1|}(r, w_2) = \\
&((rs)^*, \varepsilon)(r, w) + \sum_{w_1 w_2 = w,\ w_1 \neq \varepsilon} ((rs)^*, w_1)\varphi^{|w_1|}(r, w_2) = \\
&((rs)^*, \varepsilon)(r, w) + \\
&\quad \sum_{w_1 w_2 = w} ((rs)^*, \varepsilon) \sum_{uv = w_1,\ u \neq \varepsilon} (rs, u)\varphi^{|u|}((rs)^*, v)\varphi^{|w_1|}(r, w_2) = \\
&((rs)^*, \varepsilon)(r, w) + \sum_{w_1 w_2 = w} ((rs)^*, \varepsilon) \cdot \\
&\quad \sum_{uv = w_1,\ u \neq \varepsilon} \sum_{w_3 w_4 = u} (r, w_3)\varphi^{|w_3|}(s, w_4)\varphi^{|u|}((rs)^*, v)\varphi^{|w_1|}(r, w_2) = \\
&((rs)^*, \varepsilon)(r, w) + \sum_{w_1 w_2 = w} ((rs)^*, \varepsilon) \cdot \\
&\quad \sum_{uv = w_1,\ u \neq \varepsilon} (r, \varepsilon)(s, u)\varphi^{|u|}((rs)^*, v)\varphi^{|w_1|}(r, w_2) + \sum_{w_1 w_2 = w} ((rs)^*, \varepsilon) \cdot \\
&\quad \sum_{uv = w_1} \sum_{w_3 w_4 = u,\ w_3 \neq \varepsilon} (r, w_3)\varphi^{|w_3|}(s, w_4)\varphi^{|u|}((rs)^*, v)\varphi^{|w_1|}(r, w_2) = \\
&((rs)^*, \varepsilon)(r, w) + ((rs)^* r, \varepsilon) \sum_{uz = w,\ u \neq \varepsilon} (s, u)\varphi^{|u|}((rs)^* r, z) + \\
&\quad ((rs)^*, \varepsilon) \sum_{w_3 z = w,\ w_3 \neq \varepsilon} (r, w_3)\varphi^{|w_3|}((sr)^+, z) =
\end{aligned}$$

$$((rs)^*, \varepsilon)(r, w) + ((rs)^*r, \varepsilon) \sum_{uz=w,\ u\neq\varepsilon}(s, u)\varphi^{|u|}((rs)^*r, z) +$$
$$((rs)^*, \varepsilon) \sum_{w_3z=w,\ w_3\neq\varepsilon,\ z\neq\varepsilon}(r, w_3)\varphi^{|w_3|}((sr)^*, z) +$$
$$((rs)^*, \varepsilon)(r, w)\varphi^{|w|}((sr)^+, \varepsilon) =$$
$$((rs)^*r, \varepsilon) \sum_{uz=w,\ u\neq\varepsilon}(s, u)\varphi^{|u|}((rs)^*r, z) +$$
$$((rs)^*, \varepsilon) \sum_{w_3z=w,\ w_3\neq\varepsilon}(r, w_3)\varphi^{|w_3|}((sr)^*, z)\,.$$

Hence, $(r(sr)^*, w) = ((rs)^*r, w)$. □

Corollary 5. *If A is a Conway semiring, $\varphi : A \to A$ is an endomorphism and Σ is an alphabet then $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is again a Conway semiring.*

Proof. The equations of Theorems 3 and 4 hold iff the product-star equation holds. □

Corollary 6 (Bloom, Ésik [2])**.** *If A is a Conway semiring and Σ is an alphabet then $A\langle\langle\Sigma^*\rangle\rangle$ is again a Conway semiring.*

In the next corollary we consider $A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$. Here $\varphi : A^{n\times n} \to A^{n\times n}$ is the pointwise extension of the endomorphism $\varphi : A \to A$. Clearly, the extended φ is again an endomorphism. Note that the set $A^{n\times n}$ of $n \times n$-matrices is equipped with the usual matrix operations addition and multiplication.

Corollary 7. *Let A be a Conway semiring, $\varphi : A \to A$ be an endomorphism, Σ be an alphabet and $n \geq 1$. Then $(A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$ and $A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$ are again Conway semirings.*

Theorem 8. *Let A be a Conway semiring, $\varphi : A \to A$ be an endomorphism, Σ be an alphabet and $n \geq 1$. Then $(A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$ and $A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$ are isomorphic starsemirings.*

Proof. We will prove that $(A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$ and $A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$ are isomorphic by the correspondence of $M \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$ and $M' \in A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$ given by $(M_{ij}, w) = (M', w)_{ij}$, $w \in \Sigma^*$, $1 \leq i, j \leq n$.

We prove only the compatibility of multiplication and star. Let $M_1, M_2 \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$ with corresponding $M_1', M_2' \in A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$, respectively. Then, for all $w \in \Sigma^*$ and $1 \leq i, j \leq n$, we obtain

$$(M_1'M_2', w)_{ij} = (\sum_{uv=w}(M_1', u)\varphi^{|u|}(M_2', v))_{ij} =$$
$$\sum_{uv=w}\sum_{1\leq k\leq n}(M_1', u)_{ik}\varphi^{|u|}((M_2', v)_{kj}) =$$
$$\sum_{1\leq k\leq n}\sum_{uv=w}((M_1)_{ik}, u)\varphi^{|u|}((M_2)_{kj}, v) =$$
$$\sum_{1\leq k\leq n}((M_1)_{ik}(M_2)_{kj}, w) = ((M_1M_2)_{ij}, w)\,.$$

Let now $M \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$ correspond to $M' \in A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$. We assume that M is partitioned as usual into blocks

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{1\times 1}$, $b \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{1\times(n-1)}$, $c \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{(n-1)\times 1}$, $d \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{(n-1)\times(n-1)}$. We first show by induction on the length of $w \in \Sigma^*$ that $((M^*)_{11}, w) = ((a+bd^*c)^*, w)$ and $(M'^*, w)_{11} = ((M'^*, \varepsilon)\sum_{uv=w,\ u\neq\varepsilon}(M', u)\cdot \varphi^{|u|}(M'^*, v))_{11}$ coincide. The case $w = \varepsilon$ is clear. Assume now $w \neq \varepsilon$. Then we obtain

$$\begin{aligned}
&((M^*)_{11}, w) = (a+bd^*c, \varepsilon)^* \textstyle\sum_{uv=w,\ u\neq\varepsilon}(a+bd^*c, u)\varphi^{|u|}((a+bd^*c)^*, v) = \\
&(a+bd^*c, \varepsilon)^* \textstyle\sum_{uv=w,\ u\neq\varepsilon}(a, u)\varphi^{|u|}((a+bd^*c)^*, v) + \\
&\quad (a+bd^*c, \varepsilon)^* \textstyle\sum_{uv=w,\ u\neq\varepsilon}\sum_{z_1z_2=u}(bd^*, z_1)\varphi^{|z_1|}(c, z_2)\varphi^{|u|}((a+bd^*c)^*, v) = \\
&(a+bd^*c, \varepsilon)^* \textstyle\sum_{uv=w,\ u\neq\varepsilon}(a, u)\varphi^{|u|}((a+bd^*c)^*, v) + \\
&(a+bd^*c, \varepsilon)^* \textstyle\sum_{uv=w,\ u\neq\varepsilon}(bd^*, \varepsilon)(c, u)\varphi^{|u|}((a+bd^*c)^*, v) + \\
&(a+bd^*c, \varepsilon)^* \textstyle\sum_{uv=w}\sum_{z_1z_2=u,\ z_1\neq\varepsilon}(bd^*, z_1)\varphi^{|z_1|}(c, z_2)\varphi^{|u|}((a+bd^*c)^*, v)\,.
\end{aligned}$$

We call the first, second and third of these terms L_1, L_2 and L_3, respectively.

Moreover, we obtain

$$\begin{aligned}
&(M'^*, w)_{11} = \textstyle\sum_{1\le i,j\le n}(M'^*, \varepsilon)_{1i}\sum_{uv=w,\ u\neq\varepsilon}(M', u)_{ij}\varphi^{|u|}((M'^*, v)_{j1}) = \\
&\textstyle\sum_{1\le i,j\le n}((M^*)_{1i}, \varepsilon)\sum_{uv=w,\ u\neq\varepsilon}(M_{ij}, u)\varphi^{|u|}((M^*)_{j1}, v) = \\
&((a+bd^*c)^*, \varepsilon)\textstyle\sum_{uv=w,\ u\neq\varepsilon}(a, u)\varphi^{|u|}((a+bd^*c)^*, v) + \\
&((a+bd^*c)^*, \varepsilon)\textstyle\sum_{uv=w,\ u\neq\varepsilon}(b, u)\varphi^{|u|}(d^*c(a+bd^*c)^*, v) + \\
&((a+bd^*c)^*bd^*, \varepsilon)\textstyle\sum_{uv=w,\ u\neq\varepsilon}(c, u)\varphi^{|u|}((a+bd^*c)^*, v) + \\
&((a+bd^*c)^*bd^*, \varepsilon)\textstyle\sum_{uv=w,\ u\neq\varepsilon}(d, u)\varphi^{|u|}(d^*c(a+bd^*c)^*, v)\,.
\end{aligned}$$

We call the first, second, third and fourth of these terms R_1, R_2, R_3 and R_4, respectively.

It is clear that $L_1 = R_1$ and $L_2 = R_3$. Hence, we have only to prove that $L_3 = R_2 + R_4$. We obtain

$$\begin{aligned}
&L_3 = ((a+bd^*c)^*, \varepsilon)\textstyle\sum_{uv=w}\sum_{z_1z_2=u,\ z_1\neq\varepsilon}\sum_{z_3z_4=z_1} \\
&\quad (b, z_3)\varphi^{|z_3|}(d^*, z_4)\varphi^{|z_1|}(c, z_2)\varphi^{|u|}((a+bd^*c)^*, v) = \\
&((a+bd^*c)^*, \varepsilon)\textstyle\sum_{uv=w}\sum_{z_1z_2=u}\sum_{z_3z_4=z_1}(b, z_3)\varphi^{|z_3|}(d^*, \varepsilon)\cdot \\
&\quad \textstyle\sum_{u_1v_1=z_4,\ u_1\neq\varepsilon}\varphi^{|z_3|}(d, u_1)\varphi^{|z_3u_1|}(d^*, v_1)\varphi^{|z_1|}(c, z_2)\varphi^{|u|}((a+bd^*c)^*, v) = \\
&((a+bd^*c)^*, \varepsilon)\textstyle\sum_{uv=w}\sum_{z_1z_2=u}(b, \varepsilon)(d^*, \varepsilon)\cdot \\
&\quad \textstyle\sum_{u_1v_1=z_1,\ u_1\neq\varepsilon}(d, u_1)\varphi^{|u_1|}(d^*, v_1)\varphi^{|z_1|}(c, z_2)\varphi^{|u|}((a+bd^*c)^*, v) + \\
&\quad ((a+bd^*c)^*, \varepsilon)\textstyle\sum_{uv=w}\sum_{z_1z_2=u}\sum_{z_3z_4=z_1,\ z_3\neq\varepsilon}(b, z_3)\varphi^{|z_3|}(d^*, \varepsilon)\cdot \\
&\quad \textstyle\sum_{u_1v_1=z_4,\ u_1\neq\varepsilon}\varphi^{|z_3|}(d, u_1)\varphi^{|z_3u_1|}(d^*, v_1)\varphi^{|z_1|}(c, z_2)\varphi^{|u|}((a+bd^*c)^*, v)\,.
\end{aligned}$$

We call the first and second of these terms L_4 and L_5, respectively.

We now obtain

$$\begin{aligned}
&L_4 = ((a+bd^*c)^*bd^*, \varepsilon)\textstyle\sum_{uv=w}\sum_{z_1z_2=u}\sum_{u_1v_1=z_1,\ u_1\neq\varepsilon} \\
&\quad (d, u_1)\varphi^{|u_1|}(d^*, v_1)\varphi^{|z_1|}(c, z_2)\varphi^{|u|}((a+bd^*c)^*, v) = \\
&((a+bd^*c)^*bd^*, \varepsilon)\textstyle\sum_{uv=w}\sum_{u_1z_3=u,\ u_1\neq\varepsilon} \\
&\quad (d, u_1)\varphi^{|u_1|}(d^*c, z_3)\varphi^{|u|}((a+bd^*c)^*, v) = \\
&((a+bd^*c)^*bd^*, \varepsilon)\textstyle\sum_{u_1z_4=w,\ u_1\neq\varepsilon}(d, u_1)\varphi^{|u_1|}(d^*c(a+bd^*c)^*, z_4) = R_4\,.
\end{aligned}$$

Eventually, we obtain

$$L_5 = ((a+bd^*c)^*,\varepsilon)\sum_{uv=w}\sum_{z_1z_2=u}\sum_{z_3z_4=z_1,\ z_3\neq\varepsilon}$$
$$(b,z_3)\varphi^{|z_3|}(d^*,z_4)\varphi^{|z_1|}(c,z_2)\varphi^{|u|}((a+bd^*c)^*,v) =$$
$$((a+bd^*c)^*,\varepsilon)\sum_{uv=w}\sum_{z_3z_5=u,\ z_3\neq\varepsilon}$$
$$(b,z_3)\varphi^{|z_3|}(d^*c,z_5)\varphi^{|u|}((a+bd^*c)^*,v) =$$
$$((a+bd^*c)^*,\varepsilon)\sum_{z_3z_6=w,\ z_3\neq\varepsilon}(b,z_3)\varphi^{|z_3|}(d^*c(a+bd^*c)^*,z_6) = R_2\,.$$

Hence, $L_3 = L_4 + L_5 = R_4 + R_2$.

Next, we prove by induction on the length of $w\in\Sigma^*$ that the $(1,2)$-blocks of M^* and M'^* correspond to each other: $((M^*)_{12},w) = (M'^*,w)_{12}$. Here we have $((M^*)_{12},w) = ((a+bd^*c)^*bd^*,w)$ and $(M'^*,w)_{12} = ((M'^*,\varepsilon)\sum_{uv=w,\ u\neq\varepsilon}(M',u)\cdot\varphi^{|u|}(M'^*,v))_{12}$. The case $w=\varepsilon$ is clear. Assume now $w\neq\varepsilon$. Then we obtain

$$((M^*)_{12},w) = \sum_{z_1z_2=w}((a+bd^*c)^*,z_1)\varphi^{|z_1|}(bd^*,z_2) =$$
$$(a+bd^*c,\varepsilon)^*(bd^*,w) + \sum_{z_1z_2=w}(a+bd^*c,\varepsilon)^*\sum_{uv=z_1,\ u\neq\varepsilon}$$
$$(a+bd^*c,u)\varphi^{|u|}((a+bd^*c)^*,v)\varphi^{|z_1|}(bd^*,z_2) =$$
$$((a+bd^*c)^*,\varepsilon)(bd^*,w) +$$
$$\sum_{uvz_2=w,\ u\neq\varepsilon}((a+bd^*c)^*,\varepsilon)(a,u)\varphi^{|u|}((a+bd^*c)^*,v)\varphi^{|uv|}(bd^*,z_2) +$$
$$\sum_{uvz_2=w,\ u\neq\varepsilon}((a+bd^*c)^*,\varepsilon)(bd^*,\varepsilon)(c,u)\varphi^{|u|}((a+bd^*c)^*,v)\varphi^{|uv|}(bd^*,z_2) +$$
$$\sum_{z_1z_2=w}((a+bd^*c)^*,\varepsilon)\sum_{uv=z_1}\sum_{z_3z_4=u,\ z_3\neq\varepsilon}$$
$$(bd^*,z_3)\varphi^{|z_3|}(c,z_4)\varphi^{|u|}((a+bd^*c)^*,v)\varphi^{|z_1|}(bd^*,z_2)\,.$$

We call the first, second, third and fourth of these terms L_0, L_1, L_2 and L_3, respectively.

Moreover, we obtain

$$(M'^*,w)_{12} = \sum_{1\leq i,j\leq n}(M'^*,\varepsilon)_{1i}\sum_{uv=w,\ u\neq\varepsilon}(M',u)_{ij}\varphi^{|u|}((M'^*,v)_{j2}) =$$
$$\sum_{1\leq i,j\leq n}((M^*)_{1i},\varepsilon)\sum_{uv=w,\ u\neq\varepsilon}(M_{ij},u)\varphi^{|u|}((M^*)_{j2},v) =$$
$$((a+bd^*c)^*,\varepsilon)\sum_{uv=w,\ u\neq\varepsilon}(a,u)\varphi^{|u|}((a+bd^*c)^*bd^*,v) +$$
$$((a+bd^*c)^*,\varepsilon)\sum_{uv=w,\ u\neq\varepsilon}(b,u)\varphi^{|u|}((d+ca^*b)^*,v) +$$
$$((a+bd^*c)^*bd^*,\varepsilon)\sum_{uv=w,\ u\neq\varepsilon}(c,u)\varphi^{|u|}((a+bd^*c)^*bd^*,v) +$$
$$((a+bd^*c)^*bd^*,\varepsilon)\sum_{uv=w,\ u\neq\varepsilon}(d,u)\varphi^{|u|}((d+ca^*b)^*,v)\,.$$

We call the first, second, third and fourth of these terms R_1, R_2, R_3 and R_4, respectively.

It is clear that $L_1 = R_1$ and $L_2 = R_3$. Hence, we have only to prove that $L_0 + L_3 = R_2 + R_4$. We obtain

$$L_0 + L_3 =$$
$$((a+bd^*c)^*,\varepsilon)\sum_{z_1z_2=w}(b,z_1)\varphi^{|z_1|}(d^*,z_2) +$$
$$((a+bd^*c)^*,\varepsilon)\sum_{z_3z_5=w,\ z_3\neq\varepsilon}(bd^*,z_3)\varphi^{|z_3|}(c(a+bd^*c)^*bd^*,z_5) =$$
$$((a+bd^*c)^*,\varepsilon)(b,\varepsilon)(d^*w) +$$
$$((a+bd^*c)^*,\varepsilon)\sum_{z_1z_2=w,\ z_1\neq\varepsilon}(b,z_1)\varphi^{|z_1|}(d^*,z_2) +$$
$$((a+bd^*c)^*,\varepsilon)\sum_{z_3z_5=w,\ z_3\neq\varepsilon}(b,\varepsilon)(d^*,z_3)\varphi^{|z_3|}(c(a+bd^*c)^*bd^*,z_5) +$$
$$((a+bd^*c)^*,\varepsilon)\sum_{z_3z_5=w}\sum_{z_6z_7=z_3,\ z_6\neq\varepsilon}$$
$$(b,z_6)\varphi^{|z_6|}(d^*,z_7)\varphi^{|z_3|}(c(a+bd^*c)^*bd^*,z_5) =$$

$$((a+bd^*c)^*,\varepsilon)(b,\varepsilon)(d^*,\varepsilon)\sum_{uv=w,\ u\neq\varepsilon}(d,u)\varphi^{|u|}(d^*,v)+$$
$$((a+bd^*c)^*,\varepsilon)\sum_{z_1z_2=w,\ z_1\neq\varepsilon}(b,z_1)\varphi^{|z_1|}(d^*,z_2)+$$
$$((a+bd^*c)^*,\varepsilon)(b,\varepsilon)\sum_{z_3z_5=w}(d^*,\varepsilon)\sum_{uv=z_3,u\neq\varepsilon}$$
$$(d,u)\varphi^{|u|}(d^*,v)\varphi^{|z_3|}(c(a+bd^*c)^*bd^*,z_5)+$$
$$((a+bd^*c)^*,\varepsilon)\sum_{z_6z_8=w,z_6\neq\varepsilon}(b,z_6)\varphi^{|z_6|}(d^*c(a+bd^*c)^*bd^*,z_8)=$$
$$((a+bd^*c)^*bd^*,\varepsilon)\sum_{uv=w,u\neq\varepsilon}(d,u)\varphi^{|u|}(d^*,v)+$$
$$((a+bd^*c)^*bd^*,\varepsilon)\sum_{uz_6=w,u\neq\varepsilon}(d,u)\varphi^{|u|}(d^*c(a+bd^*c)^*bd^*,z_6)+$$
$$((a+bd^*c)^*,\varepsilon)\sum_{z_1z_2=w,\ z_1\neq\varepsilon}(b,z_1)\varphi^{|z_1|}(d^*,z_2)+$$
$$((a+bd^*c)^*,\varepsilon)\sum_{z_6z_8=w,\ z_6\neq\varepsilon}(b,z_6)\varphi^{|z_6|}(d^*c(a+bd^*c)^*bd^*,z_8)=R_4+R_2\,.$$

Here we have used in the last equality the equation $(d+ca^*b)^*=d^*+d^*c(a+bd^*c)^*bd^*$.

The equality of the (2,1)- and (2,2)-blocks is proved by symmetry: interchange 1 and 2, a and d, b and c. □

Corollary 9. *Let A be a Conway semiring and Σ be an alphabet. Then $(A\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$ and $A^{n\times n}\langle\langle\Sigma^*\rangle\rangle$ are isomorphic starsemirings.*

Let $\varphi,\varphi':A\to A$ be endomorphisms. Then we define the mapping $\varphi'_\Sigma: A_\varphi\langle\langle\Sigma^*\rangle\rangle\to A_\varphi\langle\langle\Sigma^*\rangle\rangle$ by $(\varphi'_\Sigma(r),w)=\varphi'(r,w)$, $r\in A_\varphi\langle\langle\Sigma^*\rangle\rangle$, for all $w\in\Sigma^*$. Moreover, φ and φ' are *commuting* if, for all $a\in A$, $\varphi(\varphi'(a))=\varphi'(\varphi(a))$.

The next theorem is a special case of Theorem 4.3 of Droste, Kuske [5].

Theorem 10. *Let $\varphi,\varphi':A\to A$ be commuting endomorphisms. Then $\varphi'_\Sigma: A_\varphi\langle\langle\Sigma^*\rangle\rangle\to A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is an endomorphism.*

Proof. Clearly, $\varphi'_\Sigma(0)=0$ and $\varphi'_\Sigma(\varepsilon)=\varepsilon$. Let now $r_1,r_2\in A_\varphi\langle\langle\Sigma^*\rangle\rangle$. Then, for all $w\in\Sigma^*$, $(\varphi'_\Sigma(r_1+r_2),w)=\varphi'(r_1+r_2,w)=\varphi'(r_1,w)+\varphi'(r_2,w)=(\varphi'_\Sigma(r_1),w)+(\varphi'_\Sigma(r_2),w)$, i. e.,

$$\varphi'_\Sigma(r_1+r_2)=\varphi'_\Sigma(r_1)+\varphi'_\Sigma(r_2)\,,$$

and $(\varphi'_\Sigma(r_1\odot_\varphi r_2),w)=\varphi'(r_1\odot_\varphi r_2,w)=\varphi'(\sum_{w_1w_2=w}(r_1,w_1)\varphi^{|w_1|}(r_2,w_2))=\sum_{w_1w_2=w}\varphi'(r_1,w_1)\varphi'(\varphi^{|w_1|}(r_2,w_2))=\sum_{w_1w_2=w}\varphi'(r_1,w_1)\varphi^{|w_1|}(\varphi'(r_2,w_2))=\sum_{w_1w_2=w}(\varphi'_\Sigma(r_1),w_1)\varphi^{|w_1|}(\varphi'_\Sigma(r_2),w_2))=(\varphi'_\Sigma(r_1)\odot_\varphi\varphi'_\Sigma(r_2),w)$, i. e.,

$$\varphi'_\Sigma(r_1\odot_\varphi r_2)=\varphi'_\Sigma(r_1)\odot_\varphi\varphi'_\Sigma(r_2)\,.$$

□

Corollary 11. *Let $\varphi:A\to A$ be an endomorphism. Then $\varphi_\Sigma: A_\varphi\langle\langle\Sigma^*\rangle\rangle\to A_\varphi\langle\langle\Sigma^*\rangle\rangle$ and $\varphi_\Sigma: A\langle\langle\Sigma^*\rangle\rangle\to A\langle\langle\Sigma^*\rangle\rangle$ are endomorphisms.*

Corollary 12. *Let A be a Conway semiring, $\varphi:A\to A$ be an endomorphism, and Σ_1,Σ_2 be alphabets. Then $(A_\varphi\langle\langle\Sigma_1^*\rangle\rangle)_{\varphi_{\Sigma_1}}\langle\langle\Sigma_2^*\rangle\rangle$, $(A_\varphi\langle\langle\Sigma_1^*\rangle\rangle)\langle\langle\Sigma_2^*\rangle\rangle$, $(A\langle\langle\Sigma_1^*\rangle\rangle)_{\varphi_{\Sigma_1}}\langle\langle\Sigma_2^*\rangle\rangle$ and $(A\langle\langle\Sigma_1^*\rangle\rangle)\langle\langle\Sigma_2^*\rangle\rangle$ are again Conway semirings.*

Theorem 13. *Let A be a Conway semiring, $\varphi,\psi:A\to A$ be commuting endomorphisms and Σ_1,Σ_2 be alphabets. Then $(A_\varphi\langle\langle\Sigma_1^*\rangle\rangle)_{\psi_{\Sigma_1}}\langle\langle\Sigma_2^*\rangle\rangle$ and $(A_\psi\langle\langle\Sigma_2^*\rangle\rangle)_{\varphi_{\Sigma_2}}\langle\langle\Sigma_1^*\rangle\rangle$ are isomorphic starsemirings.*

Proof. We will prove that $(A_\varphi\langle\langle\Sigma_1^*\rangle\rangle)_{\psi_{\Sigma_1}}\langle\langle\Sigma_2^*\rangle\rangle$ and $(A_\psi\langle\langle\Sigma_2^*\rangle\rangle)_{\varphi_{\Sigma_2}}\langle\langle\Sigma_1^*\rangle\rangle$ are isomorphic by the correspondence of $r \in (A_\varphi\langle\langle\Sigma_1^*\rangle\rangle)_{\psi_{\Sigma_1}}\langle\langle\Sigma_2^*\rangle\rangle$ and $r' \in (A_\psi\langle\langle\Sigma_2^*\rangle\rangle)_{\varphi_{\Sigma_2}}\langle\langle\Sigma_1^*\rangle\rangle$ given by $((r,w_2),w_1) = ((r',w_1),w_2)$, $w_1 \in \Sigma_1^*$, $w_2 \in \Sigma_2^*$.

We prove only the compatibility of multiplication and star. Let $r_1, r_2 \in (A_\varphi\langle\langle\Sigma_1^*\rangle\rangle)_{\psi_{\Sigma_1}}\langle\langle\Sigma_2^*\rangle\rangle$ with corresponding $r_1', r_2' \in (A_\psi\langle\langle\Sigma_2^*\rangle\rangle)_{\varphi_{\Sigma_2}}\langle\langle\Sigma_1^*\rangle\rangle$, respectively. Then, for all $w_1 \in \Sigma_1^*$ and $w_2 \in \Sigma_2^*$, we obtain

$$\begin{aligned}
&((r_1'r_2',w_1),w_2) = (\textstyle\sum_{v_1v_2=w_1}(r_1',v_1)\odot_{\varphi_{\Sigma_2}}(r_2',v_2),w_2) = \\
&\textstyle\sum_{v_1v_2=w_1}((r_1',v_1)\varphi_{\Sigma_2}^{|v_1|}(r_2',v_2),w_2) = \\
&\textstyle\sum_{v_1v_2=w_1}\sum_{u_1u_2=w_2}((r_1',v_1),u_1)\psi^{|u_1|}(\varphi_{\Sigma_2}^{|v_1|}(r_2',v_2),u_2) = \\
&\textstyle\sum_{v_1v_2=w_1}\sum_{u_1u_2=w_2}((r_1',v_1),u_1)\psi^{|u_1|}(\varphi^{|v_1|}((r_2',v_2),u_2)) = \\
&\textstyle\sum_{u_1u_2=w_2}\sum_{v_1v_2=w_1}((r_1,u_1),v_1)\varphi^{|v_1|}(\psi^{|u_1|}((r_2,u_2),v_2)) = \\
&\textstyle\sum_{u_1u_2=w_2}\sum_{v_1v_2=w_1}((r_1,u_1),v_1)\varphi^{|v_1|}(\psi_{\Sigma_1}^{|u_1|}(r_2,u_2),v_2) = \\
&\textstyle\sum_{u_1u_2=w_2}((r_1,u_1)\psi_{\Sigma_1}^{|u_1|}(r_2,u_2),w_1) = ((r_1r_2,w_2),w_1)\,.
\end{aligned}$$

Let now $r \in (A_\varphi\langle\langle\Sigma_1^*\rangle\rangle)_{\psi_{\Sigma_1}}\langle\langle\Sigma_2^*\rangle\rangle$ correspond to $r' \in (A_\psi\langle\langle\Sigma_2^*\rangle\rangle)_{\varphi_{\Sigma_2}}\langle\langle\Sigma_1^*\rangle\rangle$. We show by induction on $|w_1|+|w_2|$, $w_1 \in \Sigma_1^*$, $w_2 \in \Sigma_2^*$, that $((r^*,w_2),w_1) = ((r'^*,w_1),w_2)$. The case $|w_1|+|w_2| = 0$, i.e., $w_1 = \varepsilon$, $w_2 = \varepsilon$, is clear. Assume now $|w_1|+|w_2| > 0$. Then we consider the three cases (i) $w_1 = \varepsilon$, $w_2 \neq \varepsilon$, (ii) $w_1 \neq \varepsilon$, $w_2 = \varepsilon$ and (iii) $w_1 \neq \varepsilon$, $w_2 \neq \varepsilon$.

(i) We obtain

$$\begin{aligned}
&((r^*,w_2),\varepsilon) = (\textstyle\sum_{u_1u_2=w_2,\ u_1\neq\varepsilon}(r^*,\varepsilon)(r,u_1)\psi_{\Sigma_1}^{|u_1|}(r^*,u_2),\varepsilon) = \\
&\textstyle\sum_{u_1u_2=w_2,\ u_1\neq\varepsilon}((r^*,\varepsilon),\varepsilon)((r,u_1),\varepsilon)\psi^{|u_1|}((r^*,u_2),\varepsilon)
\end{aligned}$$

and

$$\begin{aligned}
&((r'^*,\varepsilon),w_2) = ((r',\varepsilon)^*,w_2) = \\
&\textstyle\sum_{u_1u_2=w_2,\ u_1\neq\varepsilon}((r',\varepsilon),\varepsilon)^*((r',\varepsilon),u_1)\psi^{|u_1|}((r',\varepsilon)^*,u_2) = \\
&\textstyle\sum_{u_1u_2=w_2,\ u_1\neq\varepsilon}((r^*,\varepsilon),\varepsilon)((r,u_1),\varepsilon)\psi^{|u_1|}((r^*,u_2),\varepsilon)\,.
\end{aligned}$$

(ii) By the substitution $w_1 \leftrightarrow w_2$, $\varphi \leftrightarrow \psi$, $r \leftrightarrow r'$, $\Sigma_1 \leftrightarrow \Sigma_2$, the proof of the equality $((r^*,\varepsilon),w_1) = ((r'^*,w_1),\varepsilon)$ is symmetric to the proof of (i).

(iii) We obtain

$$\begin{aligned}
&((r^*,w_2),w_1) = (\textstyle\sum_{u_1u_2=w_2,\ u_1\neq\varepsilon}(r,\varepsilon)^*(r,u_1)\psi_{\Sigma_1}^{|u_1|}(r^*,u_2),w_1) = \\
&\textstyle\sum_{u_1u_2=w_2,\ u_1\neq\varepsilon}\sum_{v_1v_2v_3=w_1} \\
&\qquad((r^*,\varepsilon),v_1)\varphi^{|v_1|}((r,u_1),v_2)\varphi^{|v_1v_2|}(\psi^{|u_1|}((r^*,u_2),v_3)) = \\
&\textstyle\sum_{u_1u_2=w_2,\ u_1\neq\varepsilon}\sum_{v_1v_2v_3=w_1,\ v_1\neq\varepsilon} \\
&\qquad((r^*,\varepsilon),v_1)\varphi^{|v_1|}((r,u_1),v_2)\varphi^{|v_1v_2|}(\psi^{|u_1|}((r^*,u_2),v_3)) + \\
&\textstyle\sum_{u_1u_2=w_2,\ u_1\neq\varepsilon}\sum_{v_2v_3=w_1,\ v_2\neq\varepsilon}((r^*,\varepsilon),\varepsilon)((r,u_1),v_2)\varphi^{|v_2|}(\psi^{|u_1|}((r^*,u_2),v_3)) + \\
&\textstyle\sum_{u_1u_2=w_2,\ u_1\neq\varepsilon}((r^*,\varepsilon),\varepsilon)((r,u_1),\varepsilon)\psi^{|u_1|}((r^*,u_2),w_1)\,.
\end{aligned}$$

We call the first, second and third of these terms L_1, L_2 and L_3, respectively.

Moreover, we obtain

$$
\begin{aligned}
&((r'^*, w_1), w_2) = \textstyle\sum_{x_1x_2=w_1,\ x_1\neq\varepsilon}((r'^*,\varepsilon)(r',x_1)\varphi_{\Sigma_2}^{|x_1|}(r'^*,x_2),w_2) = \\
&\textstyle\sum_{x_1x_2=w_1,\ x_1\neq\varepsilon}\sum_{y_1y_2y_3=w_2} \\
&\quad ((r^*,y_1),\varepsilon)\psi^{|y_1|}((r,y_2),x_1)\varphi^{|x_1|}(\psi^{|y_1y_2|}((r^*,y_3),x_2)) = \\
&\textstyle\sum_{x_1x_2=w_1,\ x_1\neq\varepsilon}\sum_{y_1y_2y_3=w_2,\ y_1\neq\varepsilon} \\
&\quad ((r^*,y_1),\varepsilon)\psi^{|y_1|}((r,y_2),x_1)\varphi^{|x_1|}(\psi^{|y_1y_2|}((r^*,y_3),x_2)) + \\
&\textstyle\sum_{x_1x_2=w_1,\ x_1\neq\varepsilon}\sum_{y_2y_3=w_2,\ y_2\neq\varepsilon}((r^*,\varepsilon),\varepsilon)((r,y_2),x_1)\varphi^{|x_1|}(\psi^{|y_2|}((r^*,y_3),x_2)) + \\
&\textstyle\sum_{x_1x_2=w_1,\ x_1\neq\varepsilon}((r^*,\varepsilon),\varepsilon)((r,\varepsilon),x_1)\varphi^{|x_1|}((r^*,w_2),x_2)\,.
\end{aligned}
$$

We call the first, second and third of these terms R_1, R_2 and R_3, respectively. It is clear that $L_2 = R_2$. We will prove that $L_3 = R_1$ and $L_1 = R_3$. We obtain

$$
\begin{aligned}
&L_3 = \textstyle\sum_{u_1u_2=w_2,\ u_1\neq\varepsilon,\ u_2\neq\varepsilon}\sum_{t_1t_2=w_1,\ t_1\neq\varepsilon} \\
&((r^*,\varepsilon),\varepsilon)((r,u_1),\varepsilon)\psi^{|u_1|}((r'^*,\varepsilon)(r',t_1)\varphi_{\Sigma_2}^{|t_1|}(r'^*,t_2),u_2) + \\
&((r^*,\varepsilon),\varepsilon)((r,w_2),\varepsilon)\psi^{|w_2|}((r^*,\varepsilon),w_1) = \\
&\textstyle\sum_{u_1u_2=w_2,\ u_1\neq\varepsilon,\ u_2\neq\varepsilon}\sum_{t_1t_2=w_1,\ t_1\neq\varepsilon}\sum_{s_1s_2s_3=u_2} \\
&((r^*,\varepsilon),\varepsilon)((r,u_1),\varepsilon)\psi^{|u_1|}(((r^*,s_1),\varepsilon)\psi^{|s_1|}((r,s_2),t_1)\varphi^{|t_1|}(\psi^{|s_1s_2|}((r^*,s_3),t_2))) + \\
&\textstyle\sum_{t_1t_2=w_1,\ t_1\neq\varepsilon}((r^*,\varepsilon),\varepsilon)((r,w_2),\varepsilon)\psi^{|w_2|}(((r^*,\varepsilon),\varepsilon)((r,\varepsilon),t_1)\varphi^{|t_1|}((r^*,\varepsilon),t_2)) = \\
&\textstyle\sum_{u_1s_1s_2s_3=w_2,\ u_1\neq\varepsilon}\sum_{t_1t_2=w_1,\ t_1\neq\varepsilon} \\
&((r^*,\varepsilon),\varepsilon)((r,u_1),\varepsilon)\psi^{|u_1|}((r^*,s_1),\varepsilon)\psi^{|u_1s_1|}((r,s_2),t_1)\varphi^{|t_1|}(\psi^{|u_1s_1s_2|}((r^*,s_3),t_2))
\end{aligned}
$$

and

$$
\begin{aligned}
&R_1 = \textstyle\sum_{x_1x_2=w_1,\ x_1\neq\varepsilon}\sum_{p_1p_2y_2y_3=w_2,\ p_1\neq\varepsilon} \\
&((r^*\!,\varepsilon),\varepsilon)((r,p_1),\varepsilon)\psi^{|p_1|}((r^*\!,p_2),\varepsilon)\psi^{|p_1p_2|}((r,y_2),x_1)\varphi^{|x_1|}(\psi^{|p_1p_2y_2|}((r^*\!,y_3),x_2))\,.
\end{aligned}
$$

Hence, $L_3 = R_1$.

We now write L_1 and R_3 in an other form, using the isomorphism of the induction hypothesis. Then we obtain

$$
\begin{aligned}
&L_1 = \textstyle\sum_{u_1u_2=w_2,\ u_1\neq\varepsilon}\sum_{v_1v_2v_3=w_1,\ v_1\neq\varepsilon} \\
&((r'^*,v_1),\varepsilon)\varphi^{|v_1|}((r',v_2),u_1)\varphi^{|v_1v_2|}(\psi^{|u_1|}((r'^*,v_3),u_2))
\end{aligned}
$$

and

$$
R_3 = \textstyle\sum_{x_1x_2=w_1,\ x_1\neq\varepsilon}((r'^*,\varepsilon),\varepsilon)((r',x_1),\varepsilon)\varphi^{|x_1|}((r'^*,x_2),w_2)\,.
$$

By the substitution $x_1 \leftrightarrow u_1$, $x_2 \leftrightarrow u_2$, $v_1 \leftrightarrow y_1$, $v_2 \leftrightarrow y_2$, $v_3 \leftrightarrow y_3$, $w_1 \leftrightarrow w_2$, $\phi \leftrightarrow \psi$, $r \leftrightarrow r'$, $\Sigma_1 \leftrightarrow \Sigma_2$, $L \leftrightarrow R$, the proof of the equality $L_1 = R_3$ is symmetric to the proof of the equality $R_1 = L_3$. □

Corollary 14. *Let A be a Conway semiring, $\varphi : A \to A$ be an endomorphism and Σ_1, Σ_2 be alphabets. Then $(A_\varphi\langle\langle\Sigma_1^*\rangle\rangle)_{\varphi_{\Sigma_1}}\langle\langle\Sigma_2^*\rangle\rangle$, $(A\langle\langle\Sigma_1^*\rangle\rangle)_{\varphi_{\Sigma_1}}\langle\langle\Sigma_2^*\rangle\rangle$, $(A_\varphi\langle\langle\Sigma_1^*\rangle\rangle)\langle\langle\Sigma_2^*\rangle\rangle$ and $(A\langle\langle\Sigma_1^*\rangle\rangle)\langle\langle\Sigma_2^*\rangle\rangle$, and $(A_\varphi\langle\langle\Sigma_2^*\rangle\rangle)_{\varphi_{\Sigma_2}}\langle\langle\Sigma_1^*\rangle\rangle$, $(A_\varphi\langle\langle\Sigma_2^*\rangle\rangle)\langle\langle\Sigma_1^*\rangle\rangle$, $(A\langle\langle\Sigma_2^*\rangle\rangle)_{\varphi_{\Sigma_2}}\langle\langle\Sigma_1^*\rangle\rangle$ and $(A\langle\langle\Sigma_2^*\rangle\rangle)\langle\langle\Sigma_1^*\rangle\rangle$ are isomorphic starsemirings, respectively.*

3 Finite automata and Kleene Theorems over Conway semiring-semimodule pairs

In this section we consider finite automata over semirings and quemirings and prove some Kleene Theorems.

By $\langle A_\varphi\langle\langle\Sigma^\omega\rangle\rangle, +, 0\rangle$ we denote the set of skew power series $\sum_{v\in\Sigma^\omega}(s,v)v$, $(s,v)\in A$, with pointwise addition. We define a (left) action $\otimes_\varphi : A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle \to A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$, $(r,s)\mapsto r\otimes_\varphi s$, by

$$(r\otimes_\varphi s, v) = \sum_{w\in\Sigma^*,\ u\in\Sigma^\omega,\ wu=v} (r,w)\varphi^{|w|}(s,u), \qquad v\in\Sigma^\omega\,.$$

Theorem 15. *Let A be a complete semiring, $\varphi : A\to A$ be an endomorphism of complete semirings and Σ be an alphabet. Then $A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$ is a (left) $A_\varphi\langle\langle\Sigma^*\rangle\rangle$-semimodule.*

Throughout this section, A is a *Conway semiring*, such that $(A_\varphi\langle\langle\Sigma^*\rangle\rangle, A_\varphi\langle\langle\Sigma^\omega\rangle\rangle)$ is a starsemiring-omegasemimodule pair (see Elgot [8], Ésik, Kuich [9]). Moreover, we assume $0^\omega = 0$. Furthermore, we use the notation $A_\varphi\langle\Sigma\cup\varepsilon\rangle = \{a\varepsilon + \sum_{x\in\Sigma} a_x x \mid a, a_x\in A\}$, $A_\varphi\langle\Sigma\rangle = \{\sum_{x\in\Sigma} a_x x \mid a_x\in A\}$, $A_\varphi\langle\varepsilon\rangle = \{a\varepsilon \mid a\in A\}$.

A *finite automaton over the semiring* $A_\varphi\langle\langle\Sigma^*\rangle\rangle$

$$\mathfrak{A} = (n, I, M, P)$$

is given by

(i) a finite *set of states* $\{1,\ldots,n\}$, $n\ge 1$,

(ii) a *transition matrix* $M\in (A_\varphi\langle\Sigma\cup\varepsilon\rangle)^{n\times n}$,

(iii) an *initial state vector* $I\in (A_\varphi\langle\varepsilon\rangle)^{1\times n}$,

(iv) a *final state vector* $P\in (A_\varphi\langle\varepsilon\rangle)^{n\times 1}$.

The *behavior* of $\mathfrak{A}$ is a skew power series in $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ and is defined by

$$||\mathfrak{A}|| = IM^*P\,.$$

(See Conway [4], Bloom, Ésik [2], Kuich, Salomaa [14].)

A finite automaton over the quemiring $A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$

$$\mathfrak{A} = (n, I, M, P, k)$$

is given by

(i) a *finite automaton* (n, I, M, P) *over* $A_\varphi\langle\langle\Sigma^*\rangle\rangle$,

(ii) a *set of repeated states* $\{1,\ldots,k\}$, $0\le k\le n$.

The behavior of $\mathfrak{A}$ is a pair of skew power series in $A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$ and is defined by

$$||\mathfrak{A}|| = IM^*P + IM^{\omega_k}\,.$$

(See Bloom, Ésik [2], Ésik, Kuich [11].)

Observe that, if $\mathfrak{A} = (n, I, M, P)$ is a finite automaton over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ and $\mathfrak{A}' = (n, I, M, P, 0)$ is a finite automaton over $A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$ without repeated states, then $||\mathfrak{A}'|| = ||\mathfrak{A}||$.

A finite automaton $\mathfrak{A} = (n, I, M, P)$ over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ or $\mathfrak{A}' = (n, I, M, P, k)$ over $A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$ is called *ε-free* if the entries of M are in $A_\varphi\langle\Sigma\rangle$.

A subsemiring of $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is *rationally closed* if it is closed under the operations $+, \cdot, {}^*$. A subquemiring of the generalized starquemiring $A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$ is *ω-rationally closed* if it is closed under the operations $+, \cdot, \P, {}^\otimes$. By definition, $A_\varphi^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$ (resp. ω-$\mathfrak{Rat}(A_\varphi\langle\Sigma\cup\varepsilon\rangle)$) is the *smallest* rationally (resp. ω-rationally) closed semiring (resp. quemiring) that contains $A_\varphi\langle\Sigma\cup\varepsilon\rangle$.

Since A is a Conway semiring, we can specialize the Kleene Theorem (Theorem 3.10) of Ésik, Kuich [11].

Theorem 16. *Let $(A_\varphi\langle\langle\Sigma^*\rangle\rangle, A_\varphi\langle\langle\Sigma^\omega\rangle\rangle)$ be a starsemiring-omegasemimodule pair, where A is a Conway semiring and $0^\omega = 0$. Then the following statements are equivalent for $(r, s) \in A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$:*

(i) $(r, s) = ||\mathfrak{A}||$, *where $\mathfrak{A}$ is a finite automaton over $A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$,*

(ii) $(r, s) \in \omega\text{-Rat}(A_\varphi\langle\Sigma\cup\varepsilon\rangle)$,

(iii) $r \in A_\varphi^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$, $s = \sum_{1\le j\le m} u_j v_j^\omega$ *with* $u_j, v_j \in A_\varphi^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$.

Proof. By Theorem 3.10 of Ésik, Kuich [11] and by Corollary 5. □

Moreover, Conway [4], Bloom, Ésik [2], or Aleshnikov, Boltnev, Ésik, Ishanov, Kuich, Malachowskij [1] imply at once the following generalization of the Kleene-Schützenberger Theorem.

Theorem 17. *Let A be a Conway semiring. Then the following statements are equivalent for $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$:*

(i) $r = ||\mathfrak{A}||$, *where $\mathfrak{A}$ is a finite automaton over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$,*

(ii) $r = ||\mathfrak{A}||$, *where $\mathfrak{A}$ is an ε-free finite automaton over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$,*

(iii) $r \in A_\varphi^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$.

Proof. Theorems 3.2 and 3.3 of Aleshnikov, Boltnev, Ésik, Ishanov, Kuich, Malachowskij [1]. □

This theorem can also be seen to be a specialization of Theorem 16 for finite automata over $A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$ with empty repeated states set.

4 Cycle-free finite automata and a Kleene Theorem over complete semiring-semimodule pairs

We first prove that, for a complete star-omega semiring A and an endomorphism $\varphi : A \to A$ compatible with infinite sums and products, $(A_\varphi\langle\langle\Sigma^*\rangle\rangle, A_\varphi\langle\langle\Sigma^\omega\rangle\rangle)$ is a complete semiring-semimodule pair.

Then, for a subsemiring A' of A, such that, for any cycle-free $q \in A'\langle\Sigma \cup \varepsilon\rangle$, q^ω is in $A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$, we consider cycle-free finite automata over the quemiring $A'_\varphi\langle\langle\Sigma^*\rangle\rangle \times A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$ and prove a Kleene Theorem.

We then show that the star-omega semiring $\mathbb{R}^\infty_{\max}$ is complete. This implies then the Kleene Theorem of Droste, Kuske [5].

Assume that A is a complete star-omega semiring, i. e., there exists an infinite product subject to three conditions appearing in the definition of a complete semiring-semimodule pair. Then we define an infinite product for skew power series in the following way:

$$(r_1, r_2, \dots) \mapsto \prod_{j\geq 1}^{\varphi} r_j \in A_\varphi\langle\langle\Sigma^\omega\rangle\rangle, \qquad r_j \in A_\varphi\langle\langle\Sigma^*\rangle\rangle,\ j \geq 1\,,$$

where, for all $v \in \Sigma^\omega$,

$$(\prod_{j\geq 1}^{\varphi} r_j, v) = \sum_{v=v_1v_2\dots} \prod_{j\geq 1} \varphi^{|v_1\dots v_{j-1}|}(r_j, v_j)\,.$$

Observe that now, for $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$,

$$r^\omega = \prod_{j\geq 1}^{\varphi} r\,.$$

Theorem 18. *Let A be a complete star-omega semiring, $\varphi : A \to A$ be an endomorphism compatible with infinite sums and products and Σ be an alphabet. Then $(A_\varphi\langle\langle\Sigma^*\rangle\rangle, A_\varphi\langle\langle\Sigma^\omega\rangle\rangle)$ is a complete semiring-semimodule pair satisfying $(a\varepsilon)^\omega = 0$ for $a \in A$.*

Proof. We only prove the equation

$$\prod_{j\geq 1}^{\varphi} (\sum_{i_j\in I_j} r_j) = \sum_{(i_1,i_2,\dots)\in I_1\times I_2\times\dots} \prod_{j\geq 1}^{\varphi} r_j, \qquad r_j \in A_\varphi\langle\langle\Sigma^*\rangle\rangle,\ j \geq 1\,.$$

We obtain, for $v \in \Sigma^\omega$,

$$\begin{aligned}
&(\textstyle\prod_{j\geq 1}^{\varphi}(\sum_{i_j\in I_j} r_j), v) = \\
&\textstyle\sum_{v=v_1v_2\dots} \prod_{j\geq 1} \varphi^{|v_1\dots v_{j-1}|}(\sum_{i_j\in I_j}(r_j, v_j)) = \\
&\textstyle\sum_{v=v_1v_2\dots} \sum_{(i_1,i_2,\dots)\in I_1\times I_2\times\dots} \prod_{j\geq 1} \varphi^{|v_1\dots v_{j-1}|}(r_j, v_j) = \\
&\textstyle\sum_{(i_1,i_2,\dots)\in I_1\times I_2\times\dots} \sum_{v=v_1v_2\dots} \prod_{j\geq 1} \varphi^{|v_1\dots v_{j-1}|}(r_j, v_j) = \\
&\textstyle\sum_{(i_1,i_2,\dots)\in I_1\times I_2\times\dots} (\prod_{j\geq 1}^{\varphi} r_j, v) = \\
&\textstyle(\sum_{(i_1,i_2,\dots)\in I_1\times I_2\times\dots} \prod_{j\geq 1}^{\varphi} r_j, v)\,.
\end{aligned}$$

Consider now, for $a \in A$, $v \in \Sigma^\omega$, $(\prod_{j\geq 1}^{\varphi} a\varepsilon, v) = \sum_{v=v_1v_2\ldots} \prod_{j\geq 1} \varphi^{|v_1\ldots v_{j-1}|}(a\varepsilon, v_j)$. Then infinitely many of the v_j are unequal to ε. Hence, $(a\varepsilon, v_j) = 0$ for infinitely many j and $(\prod_{j\geq 1}^{\varphi} a\varepsilon, v) = 0$. ☐

In the sequel, we often denote $\otimes_\varphi$ simply by $\cdot$ or concatenation.

Corollary 19. *Let A be a complete star-omega semiring, $\varphi : A \to A$ be an endomorphism compatible with infinite sums and products and Σ be an alphabet. Then $(A_\varphi\langle\langle\Sigma^*\rangle\rangle, A_\varphi\langle\langle\Sigma^\omega\rangle\rangle)$ is a Conway semiring-semimodule pair satisfying $(a\varepsilon)^\omega = 0$ for $a \in A$.*

Proof. By Theorem 3.1 of Ésik, Kuich [9]. ☐

Corollary 20. *Let A be a complete star-omega semiring, $\varphi : A \to A$ be an endomorphism compatible with infinite sums and products and Σ be an alphabet. Then, for $n \geq 1$, $((A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n\times n}, (A_\varphi\langle\langle\Sigma^\omega\rangle\rangle)^n)$ is a complete semiring-semimodule pair satisfying $(M\varepsilon)^\omega = 0$ for $M \in A^{n\times n}$.*

Proof. By Ésik, Kuich [9] and an easy proof by induction on n. ☐

Corollary 21. *Let A be a complete star-omega semiring, $\varphi : A \to A$ be an endomorphism compatible with infinite sums and products and Σ be an alphabet. Then the following statements are equivalent for $(r, s) \in A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$:*

(i) $(r, s) = ||\mathfrak{A}||$, *where $\mathfrak{A}$ is a finite automaton over $A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$,*

(ii) $(r, s) \in \omega\text{-Rat}(A_\varphi\langle\Sigma \cup \varepsilon\rangle)$,

(iii) $r \in A_\varphi^{\text{rat}}\langle\langle\Sigma^*\rangle\rangle$, $s = \sum_{1\leq j\leq m} u_j v_j^\omega$ *with* $u_j, v_j \in A_\varphi^{\text{rat}}\langle\langle\Sigma^*\rangle\rangle$.

(iv) $(r, s) = ||\mathfrak{A}||$, *where $\mathfrak{A}$ is an ε-free finite automaton over $A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$.*

(v) $(r, s) \in \hat{\omega}\text{-Rat}(A_\varphi\langle\Sigma \cup \varepsilon\rangle)$,

(vi) $r \in A_\varphi^{\text{rat}}\langle\langle\Sigma^*\rangle\rangle$, $s = \sum_{1\leq j\leq m} u_j v_j^\omega$ *with* $u_j, v_j \in A_\varphi^{\text{rat}}\langle\langle\Sigma^*\rangle\rangle$ *where* $(u_j, \varepsilon) = 0$, $(v_j, \varepsilon) = 0$.

Proof. Since $(A_\varphi\langle\langle\Sigma^*\rangle\rangle, A_\varphi\langle\langle\Sigma^\omega\rangle\rangle)$ is a complete semiring-semimodule pair, it is also a Conway semiring-semimodule pair by Corollary 19. Moreover, $(a\varepsilon)^\omega = 0$ for $a \in A$. Hence, the corollary is implied by Theorems 16 and 17. ☐

A semiring A is called *zerosumfree* if, for all $a_1, a_2 \in A$, $a_1 + a_2 = 0$ implies $a_1 = 0$ and $a_2 = 0$. A semiring A is called *positive* if A is zerosumfree and if, for all $a_1, a_2 \in A$, whenever $s_1 \cdot s_2 = 0$ then $s_1 = 0$ or $s_2 = 0$ (see Eilenberg [7]). An element $a \in A$ is called *nilpotent* if there exists a $k \geq 1$ such that $a^k = 0$. The following lemma is from Ésik, Kuich [10].

Lemma 22. *(i) Let A be a complete positive semiring. Assume that*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A^{n\times n}, \text{ where } a \in A^{1\times 1},\ d \in A^{(n-1)\times(n-1)}.$$

If M is nilpotent then $a + bd^*c = 0$.

(ii) Let A be a zerosumfree semiring. Assume that

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A^{n\times n}, \text{ where } a \in A^{n_1\times n_1},\ d \in A^{n_2\times n_2},\ n_1 + n_2 = n.$$

If M is nilpotent then a, d, bc and cb are nilpotent.

A skew power series $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is called *cycle-free* if there exists a $k \geq 1$ such that $(r,\varepsilon)^k = 0$, i. e., if (r,ε) is nilpotent. A finite automaton $\mathfrak{A} = (n, I, M, P)$ (resp. $\mathfrak{A} = (n, I, M, P, k)$) over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ (resp. $A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$) is called *cycle-free* if M is cycle-free.

For the rest of this section, A is a complete star-omega semiring and $\varphi : A \to A$ is an endomorphism compatible with infinite sums and products.

Theorem 23. *Let A be a positive complete star-omega semiring, $\varphi : A \to A$ be an endomorphism compatible with infinite sums and products and Σ be an alphabet. Let A' be a subsemiring of A such that, for any cycle-free $q \in A'_\varphi\langle\Sigma\cup\varepsilon\rangle$, $q^\omega \in A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$. Assume that $M \in (A'_\varphi\langle\Sigma\cup\varepsilon\rangle)^{n\times n}$ is cycle-free. Then $M^\omega \in (A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle)^n$.*

Proof. The proof is by induction on n. The case $n = 1$ is clear. Assume now that $n > 1$ and partition M as usual into blocks a, b, c, d, where $a \in A'_\varphi\langle\Sigma\cup\varepsilon\rangle$ and $d \in (A'_\varphi\langle\Sigma\cup\varepsilon\rangle)^{(n-1)\times(n-1)}$. Consider $(M^\omega)_1 = (a + bd^*c)^\omega + (a + bd^*c)^*bd^\omega$. By Lemma 22, $(a + bd^*c, \varepsilon) = 0$ and d is cycle-free. Hence, $(a + bd^*c)^\omega \in A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$ and $d^\omega \in (A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle)^{n-1}$. Moreover, $(a + bd^*c)^* \in A'_\varphi\langle\langle\Sigma^*\rangle\rangle$. This implies that $(M^\omega)_1 \in A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$. By application of the omega-permutation-equation (see Bloom, Ésik [2]) we obtain that $M^\omega \in (A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle)^n$. □

By definition, $\mathfrak{R\hat{a}t}(A_\varphi\langle\Sigma\cup\varepsilon\rangle) \subseteq A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is the smallest semiring containing $A_\varphi\langle\Sigma\cup\varepsilon\rangle$ such that, for $q \in \mathfrak{R\hat{a}t}(A_\varphi\langle\Sigma\cup\varepsilon\rangle)$ where $(q,\varepsilon) = 0$, q^* is again in $\mathfrak{R\hat{a}t}(A_\varphi\langle\Sigma\cup\varepsilon\rangle)$.

Theorem 24. *Let A be a positive complete star-omega semiring, $\varphi : A \to A$ be an endomorphism compatible with infinite sums and products and Σ be an alphabet. Let A' be a subsemiring of A such that, for any cycle-free $q \in A'_\varphi\langle\Sigma\cup\varepsilon\rangle$, $q^\omega \in A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$. Assume that $M \in (A'_\varphi\langle\Sigma\cup\varepsilon\rangle)^{n\times n}$ is cycle-free. Then, for $1 \leq i \leq n$, $1 \leq j \leq m$, there exist $u_{ij}, v_{ij} \in \mathfrak{R\hat{a}t}(A_\varphi\langle\Sigma\cup\varepsilon\rangle)$, where $(u_{ij},\varepsilon) = 0$, $(v_{ij},\varepsilon) = 0$, such that $(M^\omega)_i = \sum_{1\leq j\leq m} u_{ij}v_{ij}^\omega$.*

Proof. The proof is by induction on n. The case $n = 1$ is clear. Assume now that $n > 1$ and partition M as usual into blocks a, b, c, d, where $a \in A'_\varphi\langle\Sigma\cup\varepsilon\rangle$ and $d \in (A'_\varphi\langle\Sigma\cup\varepsilon\rangle)^{(n-1)\times(n-1)}$. The entries of $a + bd^*c$, $(a + bd^*c)^*b$ and d

are in $\mathfrak{R}\hat{\mathfrak{a}}\mathfrak{t}(A_\varphi\langle\Sigma \cup \varepsilon\rangle)$. Hence, by Lemma 22 , there exist $t \in \mathfrak{R}\hat{\mathfrak{a}}\mathfrak{t}(A_\varphi\langle\Sigma \cup \varepsilon\rangle)$, $u \in (\mathfrak{R}\hat{\mathfrak{a}}\mathfrak{t}(A_\varphi\langle\Sigma \cup \varepsilon\rangle))^{1\times(n-1)}$, where $(t, \varepsilon) = 0$, such that $(M^\omega)_1 = t^\omega + ud^\omega = t^\omega + u(d^k)^\omega = t^\omega + ud^k(d^k)^\omega$ for all $k \geq 1$. Here the second equality follows by Corollaries 4.3 and 4.2. Since d is cycle-free there exists a $k \geq 1$ such that $(d^k, \varepsilon) = 0$. Let now $(ud^k)_i = u_i$, $(d^k)^\omega_i = v_i$. By induction hypothesis, $v_i = \sum_{1\leq j\leq m} u'_{ij}{v'}^\omega_{ij}$, where $(u'_{ij}, \varepsilon) = 0$, $(v'_{ij}, \varepsilon) = 0$. Then $(M^\omega)_1 = t^\omega + \sum_{1\leq i\leq n}\sum_{1\leq j\leq m} u_i u'_{ij}{v'}^\omega_{ij}$, where $(t, \varepsilon) = 0$, $(u_i, \varepsilon) = 0$, $(u'_{ij}, \varepsilon) = 0$, $(v'_{ij}, \varepsilon) = 0$. The omega-permutation-equation proves the theorem for $(M^\omega)_i$, $2 \leq i \leq n$. □

Theorem 25. *Let A be a complete semiring and A' be a subsemiring of A. Let $\mathfrak{A} = (n, I, M, P)$ be a cycle-free finite automaton over the semiring $A'_\varphi\langle\langle\Sigma^*\rangle\rangle$. Then $||\mathfrak{A}|| \in A'_\varphi\langle\langle\Sigma^*\rangle\rangle$.*

Proof. Since $\mathfrak{A}$ is cycle-free, $(M, \varepsilon)^* \in {A'}^{n\times n}$. Let $M_1 = \sum_{x\in\Sigma}(M, x)x$. Then, since $((M, \varepsilon)^* M_1, \varepsilon) = 0$,

$$M^* = ((M, \varepsilon)^* M_1)^* (M, \varepsilon)^* \in (A'_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n\times n} .$$

(Here we have applied already the forthcoming Theorem 38.) Hence, $||\mathfrak{A}|| \in A'_\varphi\langle\langle\Sigma^*\rangle\rangle$. □

Theorem 26. *Let A be a positive complete star-omega semiring, $\varphi : A \to A$ be an endomorphism compatible with infinite sums and products and Σ be an alphabet. Let A' be a subsemiring of A such that, for any cycle-free $q \in A'_\varphi\langle\Sigma \cup \varepsilon\rangle$, $q^\omega \in A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$. Let $\mathfrak{A} = (n, I, M, P, k)$ be a cycle-free finite automaton over the quemiring $A'_\varphi\langle\langle\Sigma^*\rangle\rangle \times A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$. Then $||\mathfrak{A}|| \in A'_\varphi\langle\langle\Sigma^*\rangle\rangle \times A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$.*

Proof. By the proof of Theorem 25, $M^* \in (A'_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$. By Theorem 23, $M^\omega \in (A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle)^n$. Hence, $||\mathfrak{A}|| \in A'_\varphi\langle\langle\Sigma^*\rangle\rangle \times A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$. □

Theorem 27. *Let A be a positive complete star-omega semiring, $\varphi : A \to A$ be an endomorphism compatible with infinite sums and products and Σ be an alphabet. Let A' be a subsemiring of A such that, for any cycle-free $q \in A'_\varphi\langle\Sigma\cup\varepsilon\rangle$, $q^\omega \in A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$. Then the behaviors of cycle-free finite automata over $A'_\varphi\langle\langle\Sigma^*\rangle\rangle \times A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$ form a subquemiring $\hat{T}_\varphi$ of $A'_\varphi\langle\langle\Sigma^*\rangle\rangle \times A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$ containing $A'_\varphi\langle\Sigma\cup\varepsilon\rangle$, such that for $r \in \hat{T}_\varphi$, where $(r\P, \varepsilon) = 0$, $r^\otimes$ is again in $\hat{T}_\varphi$.*

Proof. Inspection of the proofs of Theorems 3.3–3.8 of Ésik, Kuich [11] shows that all constructed finite automata are again cycle-free. This is seen by the proofs of Lemmas 3.15–3.17 of Ésik, Kuich [10]. Hence, Theorem 26 proves our theorem. □

Theorem 28. *Let A be a positive complete star-omega semiring, $\varphi : A \to A$ be an endomorphism compatible with infinite sums and products and Σ be an alphabet. Let A' be a subsemiring of A such that, for any cycle-free $q \in A'_\varphi\langle\Sigma\cup\varepsilon\rangle$, $q^\omega \in A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$. Then the following statements are equivalent for $(r, s) \in A'_\varphi\langle\langle\Sigma^*\rangle\rangle \times A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$:*

(i) *$(r, s) = ||\mathfrak{A}||$, where $\mathfrak{A}$ is a cycle-free finite automaton over $A'_\varphi\langle\langle\Sigma^*\rangle\rangle \times A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$,*

(ii) $(r,s) \in \hat{\omega}\text{-}\mathfrak{Rat}(A'_\varphi\langle\Sigma \cup \varepsilon\rangle)$,

(iii) $r \in \mathfrak{R\hat{a}t}(A_\varphi\langle\Sigma \cup \varepsilon\rangle)$ *and* $s = \sum_{1 \le i \le m} u_i v_i^\omega$ *with* $u_i, v_i \in \mathfrak{R\hat{a}t}(A_\varphi\langle\Sigma \cup \varepsilon\rangle)$ *and* $(u_i, \varepsilon) = 0$, $(v_i, \varepsilon) = 0$.

Proof. (i) ⇒ (iii): By Theorems 24 and 25.

(iii) ⇒ (ii): Since $r \in \mathfrak{R\hat{a}t}(A_\varphi\langle\Sigma \cup \varepsilon\rangle)$ and $s \in \hat{\omega}\text{-}\mathfrak{Rat}(A'_\varphi\langle\Sigma \cup \varepsilon\rangle.0$, we obtain $(r,s) \in \hat{\omega}\text{-}\mathfrak{Rat}(A'_\varphi\langle\Sigma \cup \varepsilon\rangle$.

(ii) ⇒ (i): By Theorem 27. □

We now want to prove the Kleene Theorem of Droste, Kuske [5]. We first consider the complete semiring

$$\mathbb{R}^\infty_{\max} = \langle \{a \ge 0 \mid a \in \mathbb{R}\} \cup \{-\infty, \infty\}, \max, +, -\infty, 0\rangle\,.$$

Here the operations are as usual, with $-\infty + \infty = -\infty$, *infinite sums* are defined by $\sum'_{i\in I} a_i = \sup\{a_i \mid i \in I\}$ and *infinite products* are defined by $\prod'_{i\ge 1} a_i = \sum_{i\ge 1} a_i$. Here $\sum_{i \ge 1} a_i$ denotes $\sup\{\sum_{1\le i\le n} a_i \mid n \ge 1\}$. We now show that this infinite product satisfies the three laws of a complete star-omega semiring.

(i) Let $a_i \ge 0$ and $0 = n_0 \le n_1 \le n_2 \le \ldots$ and define $b_i = a_{n_{i-1}+1} \ldots a_{n_i} = \sum_{n_{i-1}+1 \le j \le n_i} a_j$, $i \ge 1$. We have to show that $\prod'_{i\ge 1} a_i = \prod'_{i\ge 1} b_i$. We obtain $\prod'_{i\ge 1} b_i = \sum_{i\ge 1} b_i = \sum_{i \ge 1}\sum_{n_{i-1}+1\le j\le n_i} a_j = \sum_{i\ge 1} a_i = \prod'_{i\ge 1} a_i$.

(ii) Let $a_i \ge 0$, $i \ge 1$. Then we obtain $a_1 + \prod'_{i\ge 1} a_{i+1} = a_1 + \sum_{i\ge 1} a_{i+1} = \sum_{i\ge 1} a_i = \prod'_{i \ge 1} a_i$.

(iii) Let $a_{i_j} \ge 0$, $i_j \in I_j$, $j \ge 1$. Then we have to show that $\prod'_{j\ge 1}\sum'_{i_j\in I_j} a_{i_j} = \sum'_{(i_1,i_2,\ldots)\in I_1\times I_2\times\ldots}\prod'_{j\ge 1} a_{i_j}$. We obtain $\prod'_{j\ge 1}\sum'_{i_j\in I_j} a_{i_j} = \sum_{j\ge 1}\sup\{a_{i_j} \mid i_j \in I_j\} = \sup\{\sum_{j\ge 1} a_{i_j} \mid (i_1, i_2, \ldots) \in I_1 \times I_2 \times \ldots\} = \sum'_{(i_1,i_2,\ldots)\in I_1\times I_2\times\ldots}\prod'_{j\ge 1} a_{i_j}$.

Hence, we have proved the next theorem.

Theorem 29. $\mathbb{R}^\infty_{\max}$ *is a complete star-omega semiring.*

The only endomorphisms of $\mathbb{R}^\infty_{\max}$ are of the form $\varphi(a) = q \cdot a$ for some $q \in \mathbb{R}$, $q \ge 0$. (See Droste, Kuske [5], Lemma 5.1.) Denote $(\mathbb{R}^\infty_{\max})_\varphi\langle\langle\Sigma^*\rangle\rangle$ by $\mathbb{R}^\infty_{\max,q}\langle\langle\Sigma^*\rangle\rangle$ and $(\mathbb{R}^\infty_{\max})_\varphi\langle\langle\Sigma^\omega\rangle\rangle$ by $\mathbb{R}^\infty_{\max,q}\langle\langle\Sigma^\omega\rangle\rangle$ if φ is defined as above, and observe that the multiplication $+_q$ in $\mathbb{R}^\infty_{\max,q}\langle\langle\Sigma^*\rangle\rangle$ is defined by

$$(r_1 +_q r_2, w) = \max\{(r_1, w_1) + q^{|w_1|}(r_2, w_2) \mid w_1 w_2 = w\}\,,$$

where $r_1, r_2 \in \mathbb{R}^\infty_{\max,q}\langle\langle\Sigma^*\rangle\rangle$, $w \in \Sigma^*$.

Corollary 30 (Ésik, Kuich [10])**.** $(\mathbb{R}^\infty_{\max,q}\langle\langle\Sigma^*\rangle\rangle, \mathbb{R}^\infty_{\max,q}\langle\langle\Sigma^\omega\rangle\rangle)$ *is a complete semiring-semimodule pair.*

Let $\mathbb{R}_{\max}$ be the following subsemiring of $\mathbb{R}^\infty_{\max}$:

$$\mathbb{R}_{\max} = \langle \{a \ge 0 \mid a \in \mathbb{R}\} \cup \{-\infty\}, \max, +, -\infty, 0\rangle\,.$$

Denote $(\mathbb{R}_{\max})_\varphi\langle\langle\Sigma^*\rangle\rangle$ by $\mathbb{R}_{\max,q}\langle\langle\Sigma^*\rangle\rangle$ and $(\mathbb{R}_{\max})_\varphi\langle\langle\Sigma^\omega\rangle\rangle$ by $\mathbb{R}_{\max,q}\langle\langle\Sigma^\omega\rangle\rangle$.

Theorem 31 (Droste, Kuske [5]). *The following statements are equivalent for* $(r, s) \in \mathbb{R}_{\max,q}\langle\langle\Sigma^*\rangle\rangle \times \mathbb{R}_{\max,q}\langle\langle\Sigma^\omega\rangle\rangle$, $0 \le q < 1$*:*

(i) $(r, s) = ||\mathfrak{A}||$*, where* $\mathfrak{A}$ *is a cycle-free finite automaton over* $\mathbb{R}_{\max,q}\langle\langle\Sigma^*\rangle\rangle \times \mathbb{R}_{\max,q}\langle\langle\Sigma^\omega\rangle\rangle$,

(ii) $(r, s) \in \hat{\omega}\text{-}\mathfrak{Rat}(\mathbb{R}_{\max,q}\langle\Sigma \cup \varepsilon\rangle)$,

(iii) $r \in \mathfrak{R\hat{a}t}(\mathbb{R}_{\max,q}\langle\Sigma \cup \varepsilon\rangle)$ *and* $s = \max\{u_i +_q v_i \mid 1 \le i \le m\}$ *with* $u_i, v_i \in \mathfrak{R\hat{a}t}(\mathbb{R}_{\max,q}\langle\Sigma \cup \varepsilon\rangle)$ *and* $(u_i, \varepsilon) = -\infty$, $(v_i, \varepsilon) = -\infty$.

Proof. By Theorem 28. □

5 Skew power series over arbitrary semirings

We assume that the reader is familiar with the axiomatic theory of convergence considered in Section 2 of Kuich, Salomaa [14]. We also use the notations and isomorphisms used there.

In this section we define a convergence in the semiring $A_\varphi\langle\langle\Sigma^*\rangle\rangle$. This is done mainly for the purpose to define the star of a cycle-free power series in $A_\varphi\langle\langle\Sigma^*\rangle\rangle$. If A is a starsemiring, these considerations on a convergence are not necessary. Hence, we assume that A is *not* a starsemiring. (Or, if A is a starsemiring, we do not consider explicitly the star operation in A.) We then show variants of the sum-star-equation, the product-star-equation and the matrix-star-equation. Eventually, we prove a Kleene Theorem due to Droste, Kuske [5] by application of these equations.

By $(A_\varphi\langle\langle\Sigma^*\rangle\rangle)^\mathbb{N}$ we denote the set of sequences in $A_\varphi\langle\langle\Sigma^*\rangle\rangle$. We denote by o and η the sequences defined by $o(n) = 0$ and $\eta(n) = \varepsilon$, $n \ge 0$. For $\alpha_1, \alpha_2 \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^\mathbb{N}$ we define $\alpha_1 + \alpha_2$ and $\alpha_1 \odot_\varphi \alpha_2$ in $(A_\varphi\langle\langle\Sigma^*\rangle\rangle)^\mathbb{N}$ by $(\alpha_1 + \alpha_2)(n) = \alpha_1(n) + \alpha_2(n)$ and $(\alpha_1 \odot_\varphi \alpha_2)(n) = \alpha_1(n) \odot_\varphi \alpha_2(n)$, $n \ge 0$. For $\alpha \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^\mathbb{N}$, $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$, we define $r \odot_\varphi \alpha$ and $\alpha \odot_\varphi r$ in $(A_\varphi\langle\langle\Sigma^*\rangle\rangle)^\mathbb{N}$ by $(r \odot_\varphi \alpha)(n) = r \odot_\varphi \alpha(n)$ and $(\alpha \odot_\varphi r)(n) = \alpha(n) \odot_\varphi r$, $n \ge 0$. Observe that $\langle (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^\mathbb{N}, +, \odot_\varphi, o, \eta\rangle$ is a semiring, the full Cartesian product of ω copies of the semiring $A_\varphi\langle\langle\Sigma^*\rangle\rangle$. In the sequel, we often denote $\odot_\varphi$ by $\cdot$ or by concatenation.

Consider $\alpha \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^\mathbb{N}$ and $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$. Then $\alpha_r \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^\mathbb{N}$ denotes the sequence defined by $\alpha_r(0) = r$, $\alpha_r(n+1) = \alpha(n)$, $n \ge 0$. Moreover, for a sequence $\beta \in A^\mathbb{N}$, $\varphi(\beta)$ is the sequence in A defined by $\varphi(\beta)(n) = \varphi(\beta(n))$, $n \ge 0$.

By $D_\varphi\langle\langle\Sigma^*\rangle\rangle \subseteq (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^\mathbb{N}$ we denote the set of sequences $\alpha : \mathbb{N} \to A_\varphi\langle\langle\Sigma^*\rangle\rangle$ such that for all $w \in \Sigma^*$ there exists an $n_{\alpha,w} \ge 0$ with $(\alpha(n_{\alpha,w} + k), w) = (\alpha(n_{\alpha,w}), w)$ for all $k \ge 0$. Let D_d be the set of convergent sequences of the discrete convergence in A. Then $\alpha \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$ iff $(\alpha, w) \in D_d$ for all $w \in \Sigma^*$.

We now will show that $D_\varphi\langle\langle\Sigma^*\rangle\rangle$ is a set of convergent sequences. Hence, we have to prove that the following conditions are satisfied:

(D1) $\eta \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$,

(D2) (i) if $\alpha_1, \alpha_2 \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$ then $\alpha_1 + \alpha_2 \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$,

(ii) if $\alpha \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$ and $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ then $r \odot_\varphi \alpha, \alpha \odot_\varphi r \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$,

(D3) if $\alpha \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$ and $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ then $\alpha_r \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$.

Lemma 32. *$D_\varphi\langle\langle\Sigma^*\rangle\rangle$ is a set of convergent sequences in $(A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{\mathbb{N}}$.*

Proof. We only prove (D2)(ii), i. e., we prove that for $\alpha \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$, $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$, the sequences $r \odot_\varphi \alpha$ and $\alpha \odot_\varphi r$ are again in $D_\varphi\langle\langle\Sigma^*\rangle\rangle$. We obtain, for all $w \in \Sigma^*$,

$$(r \odot_\varphi \alpha, w) = \sum_{w_1 w_2 = w} (r, w_1)\varphi^{|w_1|}(\alpha, w_2)$$

and

$$(\alpha \odot_\varphi r, w) = \sum_{w_1 w_2 = w} (\alpha, w_1)\varphi^{|w_1|}(r, w_2)\,.$$

Since $\varphi^{|w_1|}(\alpha, w_2)$ and (α, w_1) are in D_d, these sequences $r \odot_\varphi \alpha$ and $\alpha \odot_\varphi r$ are in $D_\varphi\langle\langle\Sigma^*\rangle\rangle$.

The rest of the proof is analogous to the proof of Lemma 2.10 of Kuich, Salomaa [14]. □

We now will show that the mapping $\lim : D_\varphi\langle\langle\Sigma^*\rangle\rangle \to A_\varphi\langle\langle\Sigma^*\rangle\rangle$ defined by $\lim \alpha = \sum_{w\in\Sigma^*} \lim_d(\alpha, w)w$, $\alpha \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$, is a limit function on $D_\varphi\langle\langle\Sigma^*\rangle\rangle$. Here $\lim_d : D_d \to A$ is the limit function of the discrete convergence in A defined by $\lim_d \beta = \beta(n_\beta)$ if $\beta \in D_d$ with $\beta(n_\beta + k) = \beta(n_\beta)$ for all $k \geq 0$. Hence, we have to prove that the following conditions are satisfied:

(lim1) $\lim \eta = 1$,

(lim2) (i) if $\alpha_1, \alpha_2 \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$ then $\lim(\alpha_1 + \alpha_2) = \lim \alpha_1 + \lim \alpha_2$,

(ii) if $\alpha \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$ and $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ then $\lim(r\alpha) = r \lim \alpha$ and $\lim(\alpha r) = (\lim \alpha)r$,

(lim3) if $\alpha \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$ and $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ then $\lim \alpha_r = \lim \alpha$.

Theorem 33. *The mapping $\lim : D_\varphi\langle\langle\Sigma^*\rangle\rangle \to A_\varphi\langle\langle\Sigma^*\rangle\rangle$ defined by $\lim \alpha = \sum_{w\in\Sigma^*} \lim_d(\alpha, w)w$, $\alpha \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$, is a limit function on $D_\varphi\langle\langle\Sigma^*\rangle\rangle$.*

Proof. We only prove (lim2)(ii). Let $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$, $\alpha \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$ and $w \in \Sigma^*$. Then

$$\begin{aligned}&(\lim r\alpha, w) = \lim_d(r\alpha, w) = \lim_d(\textstyle\sum_{w_1 w_2 = w}(r, w_1)\varphi^{|w_1|}(\alpha, w_2)) =\\ &\textstyle\sum_{w_1 w_2 = w}(r, w_1)\lim_d \varphi^{|w_1|}(\alpha, w_2) = \sum_{w_1 w_2 = w}(r, w_1)\varphi^{|w_1|}(\lim_d(\alpha, w_2)) =\\ &\textstyle\sum_{w_1 w_2 = w}(r, w_1)\varphi^{|w_1|}(\lim \alpha, w_2) = (r \lim \alpha, w)\end{aligned}$$

and

$$\begin{aligned}&(\lim \alpha r, w) = \lim_d(\alpha r, w) = \lim_d(\textstyle\sum_{w_1 w_2 = w}(\alpha, w_1)\varphi^{|w_1|}(r, w_2)) =\\ &\textstyle\sum_{w_1 w_2 = w}\lim_d(\alpha, w_1)\varphi^{|w_1|}(r, w_2) =\\ &\textstyle\sum_{w_1 w_2 = w}(\lim \alpha, w_1)\varphi^{|w_1|}(r, w_2) = ((\lim \alpha)r, w)\,.\end{aligned}$$

We now obtain

$$\lim(r\alpha) = \sum_{w\in\Sigma^*} \lim_d(r\alpha, w)w = \sum_{w\in\Sigma^*}(r \lim \alpha, w)w = r \lim \alpha$$

and

$$\lim(\alpha r) = \sum_{w \in \Sigma^*} \lim_d(\alpha r, w)w = \sum_{w \in \Sigma^*} ((\lim \alpha)r, w)w = (\lim \alpha)r\,.$$

The rest of the proof is analogous to the proof of Lemma 2.11 of Kuich, Salomaa [14]. □

We make now the following conventions throughout the rest of this paper: In A we use always the discrete convergence; in $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ we use always the convergence defined in Theorem 33; in $A^{n\times n}$ we use always the discrete convergence; and in $A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$ (and isomorphically in $(A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$) we use always the convergence defined in Theorem 33.

If, for $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ the sequence $(\sum_{j=0}^n r^j)$ is in $D_\varphi\langle\langle\Sigma^*\rangle\rangle$ then we write $\lim_{n\to\infty} \sum_{j=0}^n r^j = r^*$ and call r^* the *star* of r.

Clearly, a skew power series $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is cycle-free iff $\lim_{n\to\infty}((r,\varepsilon),\varepsilon)^n = 0$. A proof analogous to the proof of Theorem 3.8 of Kuich, Salomaa [14] yields the next theorem.

Theorem 34. *If $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is cycle-free then there exists a $k \geq 1$ such that*

$$(r^{(n+1)k+j}, w) = 0$$

for all $w \in \Sigma^*$, $|w| = n$, and $j \geq 0$. Furthermore, r^* exists and

$$(r^*, w) = \sum_{j=0}^{(n+1)k-1} (r^j, w), \quad w \in \Sigma^*\,.$$

Corollary 35. *If $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is cycle-free then $\lim_{n\to\infty} r^n = 0$ and r^* exists. Moreover,*

$$r^* = \varepsilon + rr^* = \varepsilon + r^*r\,.$$

Proof. The second statement follows from Kuich, Salomaa [14], Theorem 2.3. □

Theorem 36. *Let $r, s \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$. Then rs is cycle-free iff sr is cycle-free and, in this case,*

$$s(rs)^* = (sr)^*s\,.$$

Proof. If rs is cycle-free there exists a $k \geq 1$ such that $((rs)^k, \varepsilon) = 0$. This implies that $((sr)^{k+1}, \varepsilon) = (s(rs)^k r, \varepsilon) = 0$. Hence, rs is cycle-free iff sr is cycle-free. Now apply Theorem 2.7 of Kuich, Salomaa [14]. □

Recall that, in case of a Conway semiring A, for $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$, r^* is defined by a formula given in Section 1. In case of a cycle-free skew power series we can prove the validity of that formula in arbitrary semirings.

Theorem 37. *If $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is cycle-free then*

$$(r^*, \varepsilon) = (r, \varepsilon)^*$$

and, for all $w \in \Sigma^*$, $w \neq \varepsilon$,

$$(r^*, w) = \sum_{uv=w,\ u\neq\varepsilon} (r^*, \varepsilon)(r, u)(r^*, v)\,.$$

Proof. Analogous to the proofs of Lemmas 3.3, 3.4 and Theorem 3.5 of Kuich, Salomaa [14]. □

The next theorem shows that the sum-star-equation and the product-star-equation are valid for certain skew power series.

Theorem 38. *Let $r, s \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$. If r is cycle-free and $(s, \varepsilon) = 0$, or $(r, \varepsilon) = 0$ and s is cycle-free then*

$$(r + s)^* = (r^* s)^* r^*\,.$$

If rs or sr is cycle-free then

$$(rs)^* = \varepsilon + r(sr)^* s\,.$$

Proof. If r is cycle-free (resp. $(r, \varepsilon) = 0$) and $(s, \varepsilon) = 0$ (resp. s is cycle-free) then $r + s$ is cycle-free. Hence, $\lim_{n\to\infty}(r + s)^n = 0$ and $(r + s)^*$ exists by Corollary 35. Moreover, $(r^* s, \varepsilon) = 0$ (resp. $(r^* s, \varepsilon) = (s, \varepsilon)$). Hence, $r^* s$ is cycle-free and $(r^* s)^*$ exists by Theorem 34. Eventually, r^* exists, again by Theorem 34. Now, Theorems 2.8 and 2.7 of Kuich, Salomaa [14] prove the first statement of our theorem.

By Corollary 36, $s(rs)^* = (sr)^* s$. Hence, $\varepsilon + rs(rs)^* = \varepsilon + r(sr)^* s$. By Corollary 35, we obtain the equality $(rs)^* = \varepsilon + rs(rs)^*$. □

Corollary 39. *Let $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ be cycle-free and $r_0 = (r, \varepsilon)\varepsilon$, $r_1 = \sum_{w\in\Sigma^*,\ w\neq\varepsilon}(r, w)w$. Then*

$$r^* = (r_0 + r_1)^* = (r_0^* r_1)^* r_0^*\,.$$

We now turn to matrices $M \in A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$. In Theorem 40 and Corollary 41, we partition M and M^* into blocks

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \quad \text{and} \quad M^* = \begin{pmatrix} M^*(n_1, n_1) & M^*(n_1, n_2) \\ M^*(n_2, n_1) & M^*(n_2, n_2) \end{pmatrix},$$

where $n_1 + n_2 = n$, $M_{11}, M^*(n_1, n_1) \in A_\varphi^{n_1\times n_1}\langle\langle\Sigma^*\rangle\rangle$ and $M_{22}, M^*(n_2, n_2) \in A_\varphi^{n_2\times n_2}\langle\langle\Sigma^*\rangle\rangle$. The next theorem shows that, under certain conditions, the matrix-star-equation is valid.

Theorem 40. *Let $M \in A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$ and assume that M_{11} and M_{22} are cycle-free and $(M_{21},\varepsilon)=0$. Then M is cycle-free and*

$$\begin{aligned} M^*(n_1,n_1) &= (M_{11}+M_{12}M_{22}^*M_{21})^*\,,\\ M^*(n_1,n_2) &= (M_{11}+M_{12}M_{22}^*M_{21})^*M_{12}M_{22}^*\,,\\ M^*(n_2,n_1) &= (M_{22}+M_{21}M_{11}^*M_{12})^*M_{21}M_{11}^*\,,\\ M^*(n_2,n_2) &= (M_{22}+M_{21}M_{11}^*M_{12})^*\,. \end{aligned}$$

Proof. In the proof of Theorem 4.22 of Kuich, Salomaa [14] it is shown that, for $j\geq 1$,

$$(M,\varepsilon)^j = \begin{pmatrix} (M_{11},\varepsilon)^j & \sum_{j_1+j_2=j-1}(M_{11},\varepsilon)^{j_1}(M_{12},\varepsilon)(M_{22},\varepsilon)^{j_2} \\ 0 & (M_{22},\varepsilon)^j \end{pmatrix}.$$

Since M_{11} and M_{22} are cycle-free there exist $k_1,k_2\geq 1$ such that $(M_{11},\varepsilon)^{k_1}=0$ and $(M_{22},\varepsilon)^{k_2}=0$. Hence, $(M,\varepsilon)^{k_1+k_2+1}=0$ and M is cycle-free.

Let now

$$a_1 = \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 0 & M_{12} \\ M_{21} & 0 \end{pmatrix}$$

and consider the matrix

$$(a_1+a_2a_1^*a_2,\varepsilon) = \begin{pmatrix} (M_{11},\varepsilon) & 0 \\ 0 & (M_{22},\varepsilon) \end{pmatrix} + \begin{pmatrix} 0 & (M_{12},\varepsilon) \\ (M_{21},\varepsilon) & 0 \end{pmatrix}\begin{pmatrix} (M_{11}^*,\varepsilon) & 0 \\ 0 & (M_{22}^*,\varepsilon) \end{pmatrix}\begin{pmatrix} 0 & (M_{12},\varepsilon) \\ (M_{21},\varepsilon) & 0 \end{pmatrix}.$$

Since $(M_{21},\varepsilon)=0$ this matrix equals (a_1,ε). Since $a_1+a_2=M$, and a_1 and $a_1+a_2a_1^*a_2$ are cycle-free, we can apply Theorem 2.9 of Kuich, Salomaa [14]:

$$(a_1+a_2)^* = (a_1+a_2a_1^*a_2)^*(1+a_2a_1^*)\,.$$

Computation of the right side of this equality yields the equations of our theorem. □

Corollary 41. *Let $M \in A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$ and assume that M_{11} and M_{22} are cycle-free and $M_{21}=0$. Then M is cycle-fee and*

$$M^* = \begin{pmatrix} M_{11}^* & M_{11}^*M_{12}M_{22}^* \\ 0 & M_{22}^* \end{pmatrix}.$$

Corollary 42. *Let $M \in A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$ be of the form*

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ 0 & M_{22} & M_{23} \\ 0 & 0 & M_{33} \end{pmatrix},$$

where M_{11}, M_{22} and M_{33} are square blocks and assume that these blocks are cycle-free matrices. Then M is cycle-free and

$$M^* = \begin{pmatrix} M_{11}^* & M_{11}^* M_{12} M_{22}^* & M_{11}^* M_{12} M_{22}^* M_{23} M_{33}^* + M_{11}^* M_{13} M_{33}^* \\ 0 & M_{22}^* & M_{22}^* M_{23} M_{33}^* \\ 0 & 0 & M_{33}^* \end{pmatrix} .$$

Theorem 43. *Let $M \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n_1 \times n_2}$ and $M' \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n_2 \times n_1}$. Then MM' is cycle-free iff $M'M$ is cycle-free and, in this case,*

$$(MM')^* M = M(M'M)^* .$$

Proof. If MM' is cycle-free there exists a $k \geq 1$ such that $((MM')^k, \varepsilon) = 0$. This implies that $((M'M)^{k+1}, \varepsilon) = (M'(MM')^k M, \varepsilon) = 0$. Hence MM' is cycle-free iff $M'M$ is cycle-free.

We now distinguish three cases: $n_1 = n_2$, $n_1 > n_2$ and $n_1 < n_2$.

(i) If $n_1 = n_2$ then Theorem 36 proves our theorem.

(ii) If $n_1 > n_2$, write $M = \begin{pmatrix} a \\ b \end{pmatrix}$, $M' = (a' \ c')$, where $a, a' \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n_2 \times n_2}$. Denote $M_0 = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$, $M_0' = \begin{pmatrix} a' & c' \\ 0 & 0 \end{pmatrix}$ and observe that $M_0 M_0' = MM'$ and $M_0' M_0 = \begin{pmatrix} M'M & 0 \\ 0 & 0 \end{pmatrix}$. Moreover, by Corollary 41, $(M_0' M_0)^* = \begin{pmatrix} (M'M)^* & 0 \\ 0 & E \end{pmatrix}$. We now apply Theorem 36 and obtain, by $(M_0 M_0')^* M_0 = M_0 (M_0' M_0)^*$, the equation $(MM')^* M = M(M'M)^*$.

(iii) If $n_2 > n_1$, write $M = (a \ c)$, $M' = \begin{pmatrix} a' \\ b' \end{pmatrix}$, where $a, a' \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n_1 \times n_2}$. Denote $M_0 = \begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix}$, $M_0' = \begin{pmatrix} a' & 0 \\ b' & 0 \end{pmatrix}$ and observe that $M_0 M_0' = \begin{pmatrix} MM' & 0 \\ 0 & 0 \end{pmatrix}$ and $M_0' M_0 = M'M$. Moreover, by Corollary 41, $(M_0 M_0')^* = \begin{pmatrix} (MM')^* & 0 \\ 0 & E \end{pmatrix}$. We now apply Theorem 36 and obtain, by $(M_0 M_0')^* M_0 = M_0 (M_0' M_0)^*$, the equation $(MM')^* M = M(M'M)^*$. □

We now show part of the Kleene Theorem of Droste, Kuske [5], Theorem 3.6. Before, some auxiliary results are necessary.

A finite automaton $\mathfrak{A} = (n, I, M, P)$ over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is called *normalized* if $n \geq 2$ and

(i) $I_1 = \varepsilon$, $I_i = 0$, $2 \leq i \leq n$;

(ii) $P_n = \varepsilon$, $P_i = 0$, $1 \leq i \leq n-1$;

(iii) $M_{i1} = M_{ni} = 0$, $1 \leq i \leq n$.

Theorem 44. *Let $\mathfrak{A}$ be a cycle-free finite automaton over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$. Then there exists a normalized cycle-free finite automaton $\mathfrak{A}'$ over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ with $||\mathfrak{A}'|| = ||\mathfrak{A}||$.*

Proof. Let $\mathfrak{A} = (n, I, M, P)$. Define

$$\mathfrak{A}' = (1 + n + 1, \begin{pmatrix} 0 & I & 0 \\ 0 & M & P \\ 0 & 0 & 0 \end{pmatrix}, (\varepsilon\ 0\ 0), \begin{pmatrix} 0 \\ 0 \\ \varepsilon \end{pmatrix}).$$

Then $\mathfrak{A}'$ is normalized. Moreover, by Corollary 42, $\mathfrak{A}'$ is cycle-free. Applying Corollary 42 yields the proof that $||\mathfrak{A}'|| = ||\mathfrak{A}||$. □

Theorem 45. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be cycle-free finite automata over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$. Then there exist cycle-free finite automata $\mathfrak{A}_1 + \mathfrak{A}_2$ and $\mathfrak{A}_1\mathfrak{A}_2$ over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ with $||\mathfrak{A}_1 + \mathfrak{A}_2|| = ||\mathfrak{A}_1|| + ||\mathfrak{A}_2||$ and $||\mathfrak{A}_1\mathfrak{A}_2|| = ||\mathfrak{A}_1||\,||\mathfrak{A}_2||$.*

Proof. Let $\mathfrak{A}_i = (n_i, I_i, M_i, P_i)$, $i = 1, 2$. Define

$$\begin{aligned} \mathfrak{A}_1 + \mathfrak{A}_2 &= (n_1 + n_2, \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, (I_1\ I_2), \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}), \\ \mathfrak{A}_1\mathfrak{A}_2 &= (n_1 + n_2, \begin{pmatrix} M_1 & P_1 I_2 \\ 0 & M_2 \end{pmatrix}, (I_1\ 0), \begin{pmatrix} 0 \\ P_2 \end{pmatrix}). \end{aligned}$$

Then, by Corollary 41, $\mathfrak{A}_1 + \mathfrak{A}_2$ and $\mathfrak{A}_1\mathfrak{A}_2$ are cycle-free. Applying Corollary 41 yields the proof that $||\mathfrak{A}_1 + \mathfrak{A}_2|| = ||\mathfrak{A}_1|| + ||\mathfrak{A}_2||$ and $||\mathfrak{A}_1\mathfrak{A}_2|| = ||\mathfrak{A}_1||\,||\mathfrak{A}_2||$. □

A finite automaton $\mathfrak{A} = (n, I, M, P)$ over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is called *ε-free* if $(M, \varepsilon) = 0$.

Theorem 46. *Let $\mathfrak{A}$ be a cycle-free finite automaton over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$. Then there exists an ε-free finite automaton $\mathfrak{A}'$ over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ with $||\mathfrak{A}'|| = ||\mathfrak{A}||$.*

Proof. Let $\mathfrak{A} = (n, I, M, P)$. Define

$$\mathfrak{A}' = (n, I, M_0^* M_1, M_0^* P),$$

where $M_0 = (M, \varepsilon)$ and $M_1 = \sum_{x\in\Sigma}(M, x)x$. Then $\mathfrak{A}'$ is ε-free. We now apply the sum-star-equation of Corollary 39: $||\mathfrak{A}'|| = I(M_0^* M_1)^* M_0^* P = I(M_0 + M_1)^* P = IM^* P = ||\mathfrak{A}||$. □

Theorem 47. *Let $\mathfrak{A}$ be an ε-free finite automaton over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$. Then there exists a cycle-free finite automaton $\mathfrak{A}^*$ over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ with $||\mathfrak{A}^*|| = ||\mathfrak{A}||^*$.*

Proof. Let $\mathfrak{A} = (n, I, M, P)$. Define

$$\mathfrak{A}^+ = (n, I, M + PI, P).$$

Since $\mathfrak{A}$ is ε-free, we obtain $IP = 0$. Hence, $(PI)^2 = 0$ and $\mathfrak{A}^+$ is cycle-free. We now apply Theorems 38 and 43: $||\mathfrak{A}^+|| = I(M + PI)^* P = I(M^* PI)^* M^* P = IM^* P(IM^* P)^*$.

Consider now the ε-free finite automata $\mathfrak{A}_\varepsilon = (1, \varepsilon, 0, \varepsilon)$ and $\mathfrak{A}^* = \mathfrak{A}_\varepsilon + \mathfrak{A}^+$ over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ with $||\mathfrak{A}_\varepsilon|| = \varepsilon$ and $||\mathfrak{A}^*|| = ||\mathfrak{A}||^*$. Here the second equality is obtained by Theorem 45 and Corollary 35. □

Theorem 48. *Given $r \in A_\varphi\langle \Sigma \cup \varepsilon\rangle$, there exists a cycle-free finite automaton $\mathfrak{A}$ over $A_\varphi\langle\langle \Sigma^*\rangle\rangle$ with $||\mathfrak{A}|| = r$.*

Proof. For $a \in A$, the finite automaton $\mathfrak{A}_a = (1, a\varepsilon, 0, \varepsilon)$ has behavior $||\mathfrak{A}_a|| = a\varepsilon$. For $x \in \Sigma$, the finite automaton

$$\mathfrak{A}_x = (2, (\varepsilon\ 0), \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix})$$

has behavior $||\mathfrak{A}_x|| = x$.

Since each $r \in A_\varphi\langle \Sigma \cup \varepsilon\rangle$ is generated from $a\varepsilon$, $a \in A$, and x, $x \in \Sigma$, by addition and multiplication, Theorem 45 proves our theorem. □

Corollary 49. *If $r \in \mathfrak{R}\hat{\mathfrak{a}}\mathfrak{t}(A_\varphi\langle \Sigma \cup \varepsilon\rangle)$ then there exists a cycle-free finite automaton $\mathfrak{A}$ over $A_\varphi\langle\langle \Sigma^*\rangle\rangle$ such that $||\mathfrak{A}|| = r$.*

Theorem 50. *Let $M \in (A_\varphi\langle\langle \Sigma^*\rangle\rangle)^{n\times n}$ with $(M, \varepsilon) = 0$. Then $M^* \in (\mathfrak{R}\hat{\mathfrak{a}}\mathfrak{t}(A_\varphi\langle \Sigma \cup \varepsilon\rangle))^{n\times n}$.*

Proof. An easy proof by induction on n using the matrix-star-equation of Theorem 40 proves our theorem (see Theorem 8.1 of Kuich, Salomaa [14]). □

Theorem 51 (Droste, Kuske [5])**.** *Let A be a semiring, $\varphi : A \to A$ be an endomorphism and Σ be an alphabet. Then the following statements are equivalent for $r \in A_\varphi\langle\langle \Sigma^*\rangle\rangle$:*

(i) $r = ||\mathfrak{A}||$, *where $\mathfrak{A}$ is a cycle-free finite automaton over $A_\varphi\langle\langle \Sigma^*\rangle\rangle$,*

(ii) $r = ||\mathfrak{A}||$, *where $\mathfrak{A}$ is an ε-free finite automaton over $A_\varphi\langle\langle \Sigma^*\rangle\rangle$,*

(iii) $r \in \mathfrak{R}\hat{\mathfrak{a}}\mathfrak{t}(A_\varphi\langle \Sigma \cup \varepsilon\rangle)$.

Proof. (i) ⇒ (ii): By Theorem 46. (ii) ⇒ (iii): By Theorem 50. (iii) ⇒ (i): By Corollary 49. □

Droste, Kuske [5] introduce generalized weighted automata. This model of a finite automaton is captured by our next definition.

A *generalized finite automaton* $\mathfrak{A} = (n, I, M, P)$ *over* $A_\varphi\langle\langle \Sigma^*\rangle\rangle$ is defined as a finite automaton over $A_\varphi\langle\langle \Sigma^*\rangle\rangle$, except that $M \in (\mathfrak{R}\hat{\mathfrak{a}}\mathfrak{t}(A_\varphi\langle \Sigma \cup \varepsilon\rangle))^{n\times n}$. If $M \in (\mathfrak{R}\hat{\mathfrak{a}}\mathfrak{t}(A_\varphi\langle \Sigma \cup \varepsilon\rangle))^{n\times n}$ with $(M, \varepsilon) = 0$, then we obtain by an easy proof by induction on n using the matrix-star-equation of Theorem 40 that $M^* \in (\mathfrak{R}\hat{\mathfrak{a}}\mathfrak{t}(A_\varphi\langle \Sigma \cup \varepsilon\rangle))^{n\times n}$ (see Theorem 8.1 of Kuich, Salomaa [14]). This together with a generalized version of Theorem 46 yields the following result, due to Droste, Kuske [5].

Theorem 52 (Droste, Kuske [5])**.** *Let A be a semiring, $\varphi : A \to A$ be an endomorphism and Σ be an alphabet. Then the following statements on $r \in A_\varphi\langle\langle \Sigma^*\rangle\rangle$ are equivalent to the statements of Theorem 51:*

(iv) $r = ||\mathfrak{A}||$, *where $\mathfrak{A}$ is a cycle-free generalized finite automaton over $A_\varphi\langle\langle \Sigma^*\rangle\rangle$,*

(v) $r = ||\mathfrak{A}||$, *where $\mathfrak{A}$ is an ε-free generalized finite automaton over $A_\varphi\langle\langle \Sigma^*\rangle\rangle$.*

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