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Small Conjunctive Varieties of Regular Languages*

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Abstract

The author's modification of Eilenberg theorem relates the so-called conjunctive varieties of regular languages with pseudovarieties of idempotent semirings. Recent results by Pastijn and his co-authors lead to the description of the lattice of all (pseudo)varieties of idempotent semirings with idempotent multiplication. We describe here the corresponding 78 varieties of languages.

Keywords: varieties of languages, pseudovarieties of idempotent semirings

1 Introduction

Certain significant classes of regular languages can by characterized by properties of syntactic semigroups/monoids of their members. The underlining framework is the so-called Eilenberg correspondence. The books by Pin [9] (see also [10]) and Almeida [1] present the background and numerous both simple and sophisticated examples. Varieties of languages corresponding to pseudovarieties of idempotent monoids are described by Neto and Sezinando in [6] and in their previous papers.

The author introduced syntactic semirings and proved an Eilenberg-type theorem in [11]. In Section 2 we reformulate the main result of Pastijn and his collaborators [7, 3, 8] giving a description of the lattice of all varieties of idempotent semirings with idempotent multiplication. We solve the identity problems in all those varieties, we show that all of them have a finite basis of identities. Further we recall one of the main results by Kuřil and author [4] relating the above varieties with certain operators on relatively free semigroups. In Section 3 we recall the author's modification of Eilenberg theorem, we find which classes of languages correspond to pseudovarieties of idempotent semirings in term of the closure operators mentioned above. Then we formulate it concretely for all 78 varieties. We complete this section by a relationship with the so-called shuffle closed languages. The last part of our contributions shows which of our varieties of languages are positive ones; we generate by each of our variety a positive one. We end with a simple example giving a language with idempotent syntactic semigroup having a syntactic semiring with a non-idempotent multiplication.

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2 Varieties of idempotent semirings

A *semigroup* is a non-empty set equipped with an associative operation. Let A^+ and $A^* = A^+ \cup \{1\}$ be the free semigroup and the free monoid, respectively, over a non-empty set A. An *ordered semigroup* is a triple $(S, \cdot\ \leq)$ where $(S, \cdot)$ is a semigroup and $\leq$ is a (partial) order on S such that

$$(\ \forall\ a, b, c \in S\ )(\ a \leq b \text{ implies both } ac \leq bc \text{ and } ca \leq cb\ )\ .$$

Homomorphisms of ordered semigroups are isotone semigroup homomorphisms. An *idempotent semiring* is a structure $(S, \cdot, \vee)$ where $(S, \cdot)$ is a semigroup, $(S, \vee)$ is a semilattice, and

$$(\ \forall\ a, b, c \in S\ )(\ a(b \vee c) = ab \vee ac \text{ and } (a \vee b)c = ac \vee bc\ )$$

(we do not postulate here the existence of the neutral element for the operation $\cdot$ nor for the operation $\vee$). Such a structure becomes an ordered semigroup with respect to the relation $\leq$ defined by

$$a \leq b \iff a \vee b = b,\ a, b \in S\ .$$

Let $A^\square$ denote the set of all non-empty finite subsets of A^+. Note that this set with the operations $U \cdot V = \{\, uv \mid u \in U,\, v \in V\,\}$ and the usual union forms a free idempotent semiring over the set A.

A class of semigroups is a *variety* if it is closed with respect to the forming of homomorphic images, substructures and products. A class of finite semigroups is a *pseudovariety* if it is closed with respect to the forming of homomorphic images, substructures and finite products. Similarly for ordered semigroups and idempotent semirings.

Let $X = \{x_1, x_2, \dots\}$ be the set of *variables* and let $X_n = \{x_1, \dots, x_n\}$ for $n \in \mathbb{N}$. For a variety $\mathcal{V}$ of semigroups we put

$$\rho_{\mathcal{V}} = \{\ (u, v) \in X^+ \times X^+ \mid \text{ all members of } \mathcal{V} \text{ satisfy the identity } u = v\ \}\ .$$

As well-known, the assignment $\mathcal{V} \mapsto \rho_{\mathcal{V}}$ is an isomorphism of the lattice of all varieties of semigroups onto the set $\mathsf{Fic}\, X^+$ of all the so-called fully invariant congruences on the semigroup X^+ ordered by the opposite inclusion. We put $\rho_{\mathcal{V},n} = \rho_{\mathcal{V}} \cap (X_n^+ \times X_n^+)$. Then $X^+/\rho_{\mathcal{V}}$ is a free semigroup in $\mathcal{V}$ over X and $X_n^+/\rho_{\mathcal{V},n}$ is a free semigroup in $\mathcal{V}$ over X_n, $n \in \mathbb{N}$. Similarly, for a variety of idempotent semirings $\mathcal{X}$, we put

$$\sigma_{\mathcal{X}} = \{\ (\{u_1, \dots, u_k\}, \{v_1, \dots, v_l\}) \in X^\square \times X^\square \mid$$

$$\text{all members of } \mathcal{X} \text{ satisfy the identity } u_1 \vee \dots \vee u_k = v_1 \vee \dots \vee v_l\ \}\ .$$

Again, $\mathcal{X} \mapsto \sigma_{\mathcal{X}}$ is an isomorphism of the lattice of all varieties of idempotent semirings onto the set $\mathsf{Fic}\, X^\square$ of all fully invariant congruences on the semiring $X^\square$ ordered by the opposite inclusion. We put $\sigma_{\mathcal{X},n} = \sigma_{\mathcal{X}} \cap (X_n^\square \times X_n^\square)$. Then $X^\square/\sigma_{\mathcal{X}}$ is a free idempotent semiring in $\mathcal{X}$ over X and $X_n^{\square}/\sigma_{\mathcal{X},n}$ is a free idempotent semiring in $\mathcal{X}$ over X_n, $n \in \mathbb{N}$.

First of all we have to recall basics on varieties of idempotent semigroups. The class of all semigroups satisfying a set Σ of identities is denoted by $\mathsf{Mod}^{\cdot}\Sigma$. We denote :

- $\mathcal{T} = \mathsf{Mod}^{\cdot}(x = y)$ – the class of all trivial semigroups,
- $\mathcal{LZ} = \mathsf{Mod}^{\cdot}(xy = x)$ – the class of all semigroups of left zeros,
- $\mathcal{RZ} = \mathsf{Mod}^{\cdot}(xy = y)$ – the class of all semigroups of right zeros,
- $\mathcal{S}l = \mathsf{Mod}^{\cdot}(x^2 = x,\, xy = yx)$ – the class of all semilattices,
- $\mathcal{LNB} = \mathsf{Mod}^{\cdot}(x^2 = x,\, xyz = xzy)$ – the class of all left normal bands,
- $\mathcal{RNB} = \mathsf{Mod}^{\cdot}(x^2 = x,\, xyz = yxz)$ – the class of all right normal bands,
- $\mathcal{R}e\mathcal{B} = \mathsf{Mod}^{\cdot}(x^2 = x,\, xyx = x)$ – the class of all rectangular bands,
- $\mathcal{LRB} = \mathsf{Mod}^{\cdot}(x^2 = x,\, xy = xyx)$ – the class of all left regular bands,
- $\mathcal{RRB} = \mathsf{Mod}^{\cdot}(x^2 = x,\, xy = yxy)$ – the class of all right regular bands,
- $\mathcal{NB} = \mathsf{Mod}^{\cdot}(x^2 = x,\, xyzx = xzyx)$ – the class of all normal bands,
- $\mathcal{LQNB} = \mathsf{Mod}^{\cdot}(x^2 = x,\, xyz = xyxz)$ – the class of all left quasinormal bands,
- $\mathcal{RQNB} = \mathsf{Mod}^{\cdot}(x^2 = x,\, xyz = xzyz)$ – the class of all right quasinormal bands,
- $\mathcal{RB} = \mathsf{Mod}^{\cdot}(x^2 = x,\, xyzx = xyxzx)$ – the class of all regular bands.

Note that the pairs $\mathcal{LZ}$ and $\mathcal{RZ}$, $\mathcal{LNB}$ and $\mathcal{RNB}$, $\mathcal{LRB}$ and $\mathcal{RRB}$, $\mathcal{LQNB}$ and $\mathcal{RQNB}$ consist of pairwise dual semigroups.

We need to introduce several operators on words from X^* :

- $\mathsf{c}(u)$ is the set of all variables in u,
- $\mathsf{h}(u)$ is the first variable of $u \in X^+$, $\mathsf{h}(1) = 1$,
- $\mathsf{t}(u)$ is the last variable of $u \in X^+$, $\mathsf{t}(1) = 1$ (it is dual to h),
- $\mathsf{l}(u)$ is the word resulting from $u \in X^+$ leaving only the first occurrence of each variable from the left, $\mathsf{l}(1) = 1$,
- $\mathsf{r}(u)$ is the word resulting from $u \in X^+$ leaving only the first occurrence of each variable from the right, $\mathsf{r}(1) = 1$ (it is dual to l),

- u_Y, for $Y \subseteq X$, is the word resulting from u by substituting 1 for each occurrence of each variable from Y.

Next we formulate how to solve the identity problem (i.e., to describe the congruences $\rho_{\mathcal{X}}$) in varieties mentioned above.

Result 1 (see for instance [13]). *The lattice of all varieties of regular bands consists of 13 varieties introduced above; the order by the inclusion is given by the diagram below.*

Further, for $u, v \in X^+$ *we have*

(i) $u \; \rho_{\mathcal{T}} \; v$ *for all* u, v*; thus* $X^+/\rho_{\mathcal{T}} \cong (\{x_1\}, \circ)$,

(ii) $u \; \rho_{\mathcal{LZ}} \; v$ *iff* $\mathsf{h}(u) = \mathsf{h}(v)$*; thus* $X^+/\rho_{\mathcal{LZ}} \cong (X, \circ)$ *where* $x \circ y = x$,

(iii) $u \; \rho_{\mathcal{S}l} \; v$ *iff* $\mathsf{c}(u) = \mathsf{c}(v)$*; thus* $X^+/\rho_{\mathcal{S}l} \cong$

$$(\; \{ \, Y \mid Y \text{ is a non-empty finite subset of } X \, \}, \circ \;) \text{ where } Y \circ Z = Y \cup Z \; ,$$

(iv) $u \; \rho_{\mathcal{LNB}} \; v$ *iff* $\mathsf{c}(u) = \mathsf{c}(v)$, $\mathsf{h}(u) = \mathsf{h}(v)$*; thus* $X^+/\rho_{\mathcal{LNB}} \cong$

$$(\; \{ \, (y, Y) \mid Y \text{ is a non-empty finite subset of } X, \; y \in Y \, \}, \circ \;)$$

$$\text{where } (y, Y) \circ (z, Z) = (y, Y \cup Z),$$

(v) $u \; \rho_{\mathcal{R}e\mathcal{B}} \; v$ *iff* $\mathsf{h}(u) = \mathsf{h}(v)$, $\mathsf{t}(u) = \mathsf{t}(v)$*; thus* $X^+/\rho_{\mathcal{R}e\mathcal{B}} \cong$

$$(X \times X, \circ) \text{ where } (x, y) \circ (z, t) = (x, t) \; ,$$

(vi) $u \; \rho_{\mathcal{LRB}} \; v$ *iff* $\mathsf{l}(u) = \mathsf{l}(v)$*; thus* $X^+/\rho_{\mathcal{LRB}} \cong$

$$(\; \{ \, u \in X^+ \mid u \text{ has pairwise different variables } \}, \circ \;) \text{ where } u \circ v = \mathsf{l}(uv) \; ,$$

(vii) $u \; \rho_{\mathcal{NB}} \; v$ *iff* $\mathsf{c}(u) = \mathsf{c}(v)$, $\mathsf{h}(u) = \mathsf{h}(v)$, $\mathsf{t}(u) = \mathsf{t}(v)$*; thus* $X^+/\rho_{\mathcal{NB}} \cong$

$$(\; \{ \, (x, Y, y) \mid Y \text{ is a non-empty finite subset of } X, \; x, y \in Y \, \}, \circ \;)$$

$$\text{where } (x, Y, y) \circ (z, Z, t) = (x, Y \cup Z, t) \; ,$$

(viii) $u \; \rho_{\mathcal{LQNB}} \; v$ *iff* $\mathsf{l}(u) = \mathsf{l}(v)$, $\mathsf{t}(u) = \mathsf{t}(v)$*; thus* $X^+/\rho_{\mathcal{LQNB}} \cong$

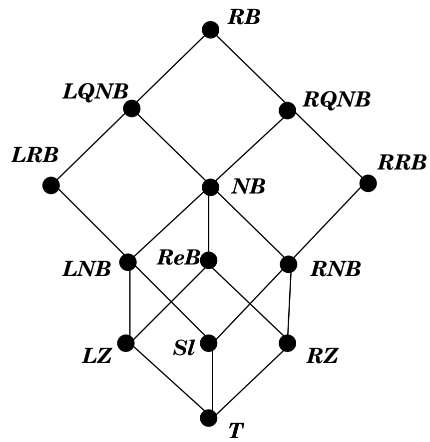
$$(\; \{ \, (u, y) \in X^+ \mid u \text{ has pairwise different variables, } y \in \mathsf{c}(u) \, \}, \circ \;)$$

$$\text{where } (u, y) \circ (v, z) = (\mathsf{l}(uv), z) \; ,$$

(ix) $u \; \rho_{\mathcal{RB}} \; v$ *iff* $\mathsf{l}(u) = \mathsf{l}(v)$, $\mathsf{r}(u) = \mathsf{r}(v)$*; thus* $X^+/\rho_{\mathcal{RB}} \cong$

$$(\; \{ \, (u, v) \in X^+ \times X^+ \mid \mathsf{c}(u) = \mathsf{c}(v), \text{each of } u, v \text{ has pairwise diff. variables } \}, \circ \;)$$

$$\text{where } (u, v) \circ (u', v') = (\mathsf{l}(uu'), \mathsf{r}(vv')) \; .$$



Now we introduce several important finite idempotent semirings :

- L is the left zero semigroup with elements a and b ordered by $a < b$,
- R is the right zero semigroup with elements a and b ordered by $a < b$,
- D is the distributive lattice with elements a and b ordered by $a < b$ (multiplication is the meet),
- M has the elements a, b and both operations equal to the join with respect to the order $a < b$,
- B is the left zero semigroup with elements a and b and with an extra neutral element 1 ordered by $a < 1 < b$,
- C is the right zero semigroup with elements a and b and with an extra neutral element 1 ordered by $a < 1 < b$.

For any idempotent semiring S, we denote by S^0 the semiring obtained from S by adding an extra element 0 and where $0 \cdot a = a \cdot 0 = 0$, $0 \vee a = a \vee 0 = a$, for every $a \in S$.

Result 2 ([4] Thm.2.9, [7] Thm.2.3). *Each idempotent semiring with an idempotent multiplication satisfies the identity $xyxzx = xyzx$; that is, its multiplicative reduct is a regular band.*

The following varieties play here a crucial role :

- $\mathcal{TS}$ – the class of all trivial (i.e., one element) semirings,
- $\mathcal{L} = < L >$ – the class of all idempotent semirings whose multiplicative reducts are left zero semigroups,
- $\mathcal{R} = < R >$ – the class of all idempotent semirings whose multiplicative reducts are right zero semigroups,

- $\mathcal{D} = < D >$ – the class of all distributive lattices,
- $\mathcal{M} = < M >$ – the class of all monobisemilattices,
- $\mathcal{S} = < M^0 >$ – the class of all bisemilattices,
- $\mathcal{B} = < B >,\ \mathcal{C} = < C >,\ \mathcal{L}^0 = < L^0 >,\ \mathcal{R}^0 = < R^0 >,\ \mathcal{B}^0 = < B^0 >,$ $\mathcal{C}^0 = < C^0 >,$
- $\mathcal{I}$ – the class of all idempotent semirings whose multiplicative reducts are idempotent.

Notice that the pairs $\mathcal{L}$ and $\mathcal{R}$, $\mathcal{B}$ and $\mathcal{C}$, $\mathcal{B}^0$ and $\mathcal{C}^0$ consist of pairwise dual semirings. Next we will solve the identity problems (i.e., to describe the congruences $\sigma_{\mathcal{X}}$) for the varieties $\mathcal{L}$, $\mathcal{D}$, $\mathcal{M}$, $\mathcal{B}$, $\mathcal{L}^0$, $\mathcal{S}$ and $\mathcal{B}^0$. These results can be extracted from [5, 7, 3, 8]. We present here simple and transparent proofs. Notice that each set of identities is equivalent to the inequalities of the form

$$u \leq u_1 \vee \cdots \vee u_k \ . \tag{$*$}$$

Result 3.

(i) *L satisfies $(*)$ iff* $\mathsf{h}(u) \in \{\, \mathsf{h}(u_1), \ldots, \mathsf{h}(u_k)\,\}$,

(ii) *D satisfies $(*)$ iff there exists $i \in \{1, \ldots, k\}$ such that* $\mathsf{c}(u) \supseteq \mathsf{c}(u_i)$,

(iii) *M satisfies $(*)$ iff* $\mathsf{c}(u) \subseteq \mathsf{c}(u_1) \cup \cdots \cup \mathsf{c}(u_k)$,

(iv) *B satisfies $(*)$ iff for each $Y \subseteq X$* $\mathsf{h}(u_Y) \in \{\, \mathsf{h}((u_1)_Y), \ldots, \mathsf{h}((u_k)_Y)\,\}$,

(v) *S^0 satisfies $(*)$ iff S satisfies* $u \leq \bigvee\{\ u_i \mid i \in \{1, \ldots, k\},\ \mathsf{c}(u) \supseteq \mathsf{c}(u_i)\ \}$.

Proof. (i) L does not satisfy $(*)$ iff we can find a substitution $\xi : X \to L$ such that $\xi(\mathsf{h}(u))$ is b and of $\xi(\mathsf{h}(u_1)) = \cdots = \xi(\mathsf{h}(u_k)) = a$.

(ii) D does not satisfy $(*)$ iff we can find a substitution $\xi : X \to D$ such that $\xi(u) = b$ and $\xi(u_1) = \cdots = \xi(u_k) = a$. This is equivalent to

$$\forall\ i \in \{1, \ldots, k\}\ \exists\ x \in \mathsf{c}(u_i) \setminus \mathsf{c}(u)\ .$$

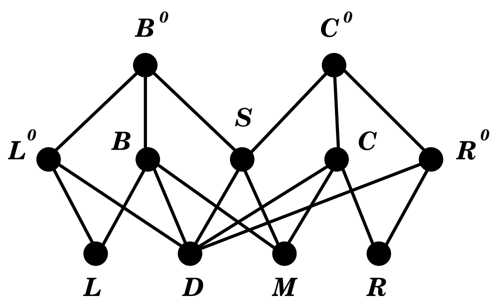
(iii) M does not satisfy $(*)$ iff we can find a substitution $\xi : X \to M$ such that $\xi(u) = b$ and $\xi(u_1) = \cdots = \xi(u_k) = a$. This is equivalent to the existence of $x \in \mathsf{c}(u) \setminus \{\, \mathsf{c}(u_i) \cup \cdots \cup \mathsf{c}(u_k)\,\}$.

(iv) $(*)$ is valid in B for all substitutions $X \to B$ where exactly all variables from Y go to 1 iff L satisfies $u_Y \leq (u_1)_Y \vee \cdots \vee (u_k)_Y$.

(v) If we substitute 0 for a variable from $\mathsf{c}(u)$ the inequality $(*)$ holds trivially. So substitute for all of them elements from S. The worst case is to substitute for all other variables the element 0. □

We denote by $\mathscr{V}(\mathcal{X})$ the lattice of all subvarieties of a variety $\mathcal{X}$. By McKenzie and Romanowska [5], all non-trivial varieties of idempotent semirings with commutative and idempotent multiplication are exactly : $\mathcal{D}$, $\mathcal{M}$, $\mathcal{D} \vee \mathcal{M}$ and $\mathcal{S}$. Later Ghosh, Pastijn and Zhao in [3] found a description of the lattice of all varieties of idempotent semirings whose multiplicative reducts are normal bands (35 varieties). They use combination of semantical methods (congruences, Green relations,...) with syntactical ones (calculating with identities,...). The result was previously announced by the authors of [4] : they used purely syntactical approach (operators on relatively free semigroups) - see Result 5. In [8] Pastijn accomplished the task of the description of the lattice $\mathscr{V}(\mathcal{I})$. Up to now we are not able to get it purely syntactically. We formulate this deep result next in a modified form. Recall that a subset B of an ordered set $(A, \leq)$ is *hereditary* if $b \in B,\ a \in A,\ a \leq b$ implies $a \in B$.

Result 4 (extracted from [7, 3, 8]). *The lattice of all varieties of idempotent semirings with an idempotent multiplication is distributive. Its non-trivial join-irreducible elements are exactly the eleven varieties mentioned above. They form the partially ordered set* $(\mathfrak{O}, \leq)$ *depicted below. Consequently, the varieties of idempotent semirings with an idempotent multiplication correspond to the 78 hereditary subsets of* $(\mathfrak{O}, \leq)$*; more precisely, they are exactly joins of hereditary sets and joins of different hereditary sets are different.*

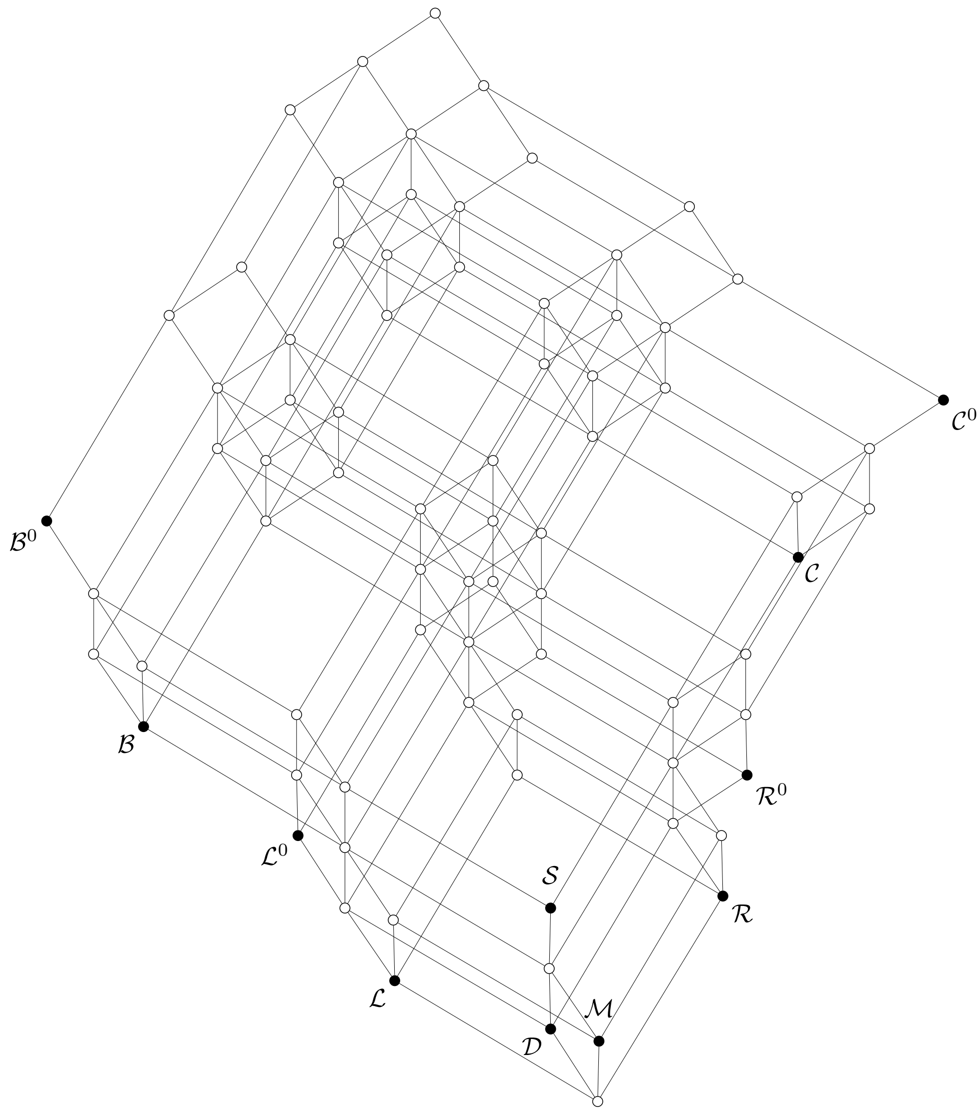


Proof. By Theorem 3.4. of [3], the kernel of the mapping

$$\phi : \mathscr{V}(\mathcal{I}) \to \mathscr{V}(\mathcal{S}),\ \mathcal{X} \mapsto \mathcal{X} \cap \mathcal{S}$$

decomposes $\mathscr{V}(\mathcal{I})$ into five intervals with the lower ends $\mathcal{TS}$, $\mathcal{D}$, $\mathcal{M}$, $\mathcal{D} \vee \mathcal{M}$, $\mathcal{S}$, respectively. By Result 4.1, the first interval is $[\mathcal{TS}, \mathcal{L} \vee \mathcal{R}]$ and it consists exactly of $\mathcal{TS}$, $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{L} \vee \mathcal{R}$. Since $\mathcal{R} \not\subseteq \mathcal{B}^0$ by Result 3, this interval intersects $\mathscr{L}(\mathcal{B}^0)$ in $\{\mathcal{TS}, \mathcal{L}\}$. Similarly, by Theorems 4.5. and 4.7., the second and the third interval intersect $\mathscr{L}(\mathcal{B}^0)$ in $\{\mathcal{D}, \mathcal{D} \vee \mathcal{L}, \mathcal{L}^0\}$ and $\{\mathcal{M}, \mathcal{M} \vee \mathcal{L}\}$, respectively.

Further by Corollary 2.5 and 3.4. of [8], the fourth and the fifth interval intersect $\mathscr{L}(\mathcal{B}^0)$ in $\{\, \mathcal{D} \vee \mathcal{M},\ \mathcal{D} \vee \mathcal{M} \vee \mathcal{L},\ \mathcal{D} \vee \mathcal{M} \vee \mathcal{L}^0,\ \mathcal{B},\ \mathcal{B} \vee \mathcal{L}^0 \,\}$ and $\{\, \mathcal{S},\ \mathcal{S} \vee \mathcal{L},\ \mathcal{S} \vee \mathcal{L}^0,\ \mathcal{S} \vee \mathcal{B},\ \mathcal{B}^0 \,\}$, respectively.



Finally, by Theorem 4.1, the mapping

$$\psi : \{\ (\mathcal{X},\mathcal{Y}) \in [\mathcal{TS},\mathcal{B}^0] \times [\mathcal{TS},\mathcal{C}^0] \mid \mathcal{X} \cap \mathcal{S} = \mathcal{Y} \cap \mathcal{S}\ \} \to \mathscr{L}(\mathcal{I}),\ (\mathcal{X},\mathcal{Y}) \mapsto \mathcal{X} \vee \mathcal{Y}$$

is a bijection. □

One of the main results of [4] is recalled below. For our purposes it is not necessary to put here the 10 axioms defining the so-called $\mathcal{V}$-admissible closure operators from subsets of $X^+/\rho_{\mathcal{V}}$ to subsets of $X^+/\rho_{\mathcal{V}}$. Notice only that one of the axioms is to be of finite character (all closures are determined by closures of finite sets).

Result 5 (**[4], Theorem 4.7**). *Varieties of idempotent semirings corres-pond to the pairs* $(\mathcal{V}, [\ ])$ *where* $\mathcal{V}$ *is a variety of semigroups and* $[\ ]$ *is a* $\mathcal{V}$*-admissible closure operator on* $X^+/\rho_{\mathcal{V}}$*; we write* $\mathcal{X} \mapsto (\underline{\mathcal{X}}, [\ ]_{\mathcal{X}})$*. Moreover,*

$$\{u_1, \ldots, u_k\}\ \sigma_{\mathcal{X}}\ \{v_1, \ldots, v_l\}\ \textit{iff}$$

$$[\,\{u_1\rho_{\underline{\mathcal{X}}}, \ldots, u_k\rho_{\underline{\mathcal{X}}}\}\,]_{\mathcal{X}} = [\,\{v_1\rho_{\underline{\mathcal{X}}}, \ldots, v_l\rho_{\underline{\mathcal{X}}}\}\,]_{\mathcal{X}}\ .$$

Conversely, given a variety $\mathcal{X}$*, we get* $\underline{\mathcal{X}}$ *and* $[\ ]_{\mathcal{X}}$ *by*

$$u\ \rho_{\underline{\mathcal{X}}}\ v\ \textit{if and only if}\ \{u\}\ \sigma_{\mathcal{X}}\ \{v\},\ \textit{and}$$

$$u\rho_{\underline{\mathcal{X}}} \in [\,(u_1)\rho_{\underline{\mathcal{X}}}, \ldots, (u_k)\rho_{\underline{\mathcal{X}}}\,]\ \textit{iff}\ \{u_1, \ldots, u_k, u\}\ \sigma_{\mathcal{X}}\ \{u_1, \ldots, u_k\}\ ,$$

which is also equivalent to the fact that $\mathcal{X}$ *satisfies* $u \le u_1 \vee \cdots \vee u_k$.

The situation above leads also to a closure operator on subsets of $X_n^+/\rho_{\mathcal{V},n}$ defined by

$$[\,u_1\rho_{\mathcal{V},n}, \ldots, u_k\rho_{\mathcal{V},n}\,]_{\mathcal{X},n} = [\,u_1\rho_{\mathcal{V}}, \ldots, u_k\rho_{\mathcal{V}}\,]_{\mathcal{X}} \cap (X_n^+/\rho_{\mathcal{V},n} \times X_n^+/\rho_{\mathcal{V},n})\ .$$

It follows how to get $(\underline{\mathcal{X}}, [\ ]_{\mathcal{X}})$ for all varieties knowing the data for the join-irreducible ones.

Result 6 (**[4], Theorem 4.9**). *Let* $\mathcal{X}$, $\mathcal{Y}$ *be varieties of idempotent semirings with idempotent multiplication. Then* $\underline{\mathcal{X} \vee \mathcal{Y}} = \underline{\mathcal{X}} \vee \underline{\mathcal{Y}}$ *and*

$$u(\rho_{\underline{\mathcal{X}}} \cap \rho_{\underline{\mathcal{Y}}}) \in [\,\{u_1(\rho_{\underline{\mathcal{X}}} \cap \rho_{\underline{\mathcal{Y}}}), \ldots, u_k(\rho_{\underline{\mathcal{X}}} \cap \rho_{\underline{\mathcal{Y}}})\}\,]_{\mathcal{X}\vee\mathcal{Y}}\ \textit{if and only if}$$

$$u\rho_{\underline{\mathcal{X}}} \in [\,u_1\rho_{\underline{\mathcal{X}}}, \ldots, u_k\rho_{\underline{\mathcal{X}}}\,]_{\mathcal{X}}\ \textit{and}\ u\rho_{\underline{\mathcal{Y}}} \in [\,u_1\rho_{\underline{\mathcal{Y}}}, \ldots, u_k\rho_{\underline{\mathcal{Y}}}\,]_{\mathcal{Y}}\ .$$

We can extract from [3, 8] that each variety from $\mathscr{V}(\mathcal{I})$ is finitely based. We formulate and prove it now in a transparent way.

Theorem 7. *For each variety* $\mathcal{X} \in \mathfrak{O}$ *there exists a variety* $\mathcal{X}^{\mathsf{c}} \in \mathscr{V}(\mathcal{I})$ *such that*

$$(\ \forall\ \mathcal{Y} \in \mathscr{V}(\mathcal{I})\ )\ \mathcal{X} \not\subseteq \mathcal{Y}\ \textit{if and only if}\ \mathcal{Y} \subseteq \mathcal{X}^{\mathsf{c}}\ .$$

More precisely,

- $\mathcal{L}^{\mathsf{c}} = \mathsf{Mod}\,(xy \le y \vee yx)$,
- $\mathcal{D}^{\mathsf{c}} = \mathsf{Mod}\,(x \le xyx)$,
- $\mathcal{M}^{\mathsf{c}} = \mathsf{Mod}\,(xyz \le x \vee z)$,
- $\mathcal{S}^{\mathsf{c}} = \mathsf{Mod}\,(xyx \le x \vee yzy)$,
- $\mathcal{B}^{\mathsf{c}} = \mathsf{Mod}\,(xyz \le x \vee zyz)$,

- $(\mathcal{L}^0)^{\mathsf{c}} = \mathsf{Mod}\,(xy \le xz \vee yxy)$,
- $(\mathcal{B}^0)^{\mathsf{c}} = \mathsf{Mod}\,(xyz \le xz \vee zyz \vee xyt)$.

Consequently,

$$(\forall\ \mathcal{Y} \in \mathscr{V}(\mathcal{I})\ )\ \mathcal{Y} = \bigcap \{\ \mathcal{X}^{\mathsf{c}} \mid \mathcal{X} \in \mathfrak{O},\ \mathcal{X} \not\subseteq \mathcal{Y}\ \}\ .$$

It follows that each $\mathcal{Y} \in \mathscr{V}(\mathcal{I})$ is finitely based.

Proof. We see that

- L does not satisfy $xy \le y \vee yx$ but C^0 does,
- D does not satisfy $x \le xyx$ but $L,\ M,\ R$ do,
- M does not satisfy $xyz \le x \vee z$ but $L^0,\ R^0$ do,
- S does not satisfy $xyx \le x \vee yzy$ but $L^0,\ B,\ C,\ R^0$ do,
- B does not satisfy $xyz \le x \vee zyz$ but $L^0,\ C^0$ does,
- L^0 does not satisfy $xy \le xz \vee yxy$ but $B,\ C^0$ do,
- B^0 does not satisfy $xyz \le xz \vee zyz \vee xyt$ but $L^0,\ B,\ C^0$ do.

Let $\mathcal{Y} \in \mathscr{V}(\mathcal{I})$, $\{\mathcal{X}_1, \dots, \mathcal{X}_k\} = \{\ \mathcal{X} \in \mathfrak{O} \mid \mathcal{X} \not\subseteq \mathcal{Y}\ \}$ and $\{\mathcal{X}_{k+1}, \dots, \mathcal{X}_{11}\} = \{\ \mathcal{X} \in \mathfrak{O} \mid \mathcal{X} \subseteq \mathcal{Y}\ \}$. Then $\mathcal{X}_{k+1} \vee \cdots \vee \mathcal{X}_{11} = \mathcal{Y} \subseteq \mathcal{X}_1^{\mathsf{c}} \cap \cdots \cap \mathcal{X}_k^{\mathsf{c}}$. The last inclusion is, in fact, an equality since there is no $\mathcal{X}_i,\ i \in \{1, \dots, k\}$, such that $\mathcal{X}_i \subseteq \mathcal{X}_1^{\mathsf{c}} \cap \cdots \cap \mathcal{X}_k^{\mathsf{c}}$. □

3 Varieties of languages

A language $L \subseteq A^+$ defines the *syntactic congruence* $\sim_L$ on $(A^\Box, \cdot, \cup)$ by

$$\{u_1, \dots, u_k\} \sim_L \{v_1, \dots, v_l\} \text{ if and only if}$$

$$(\forall\ p, q \in A^*\ )\ (\ pu_1q, ..., pu_kq \in L \iff pv_1q, ..., pv_lq \in L\ )\ .$$

The factor-structure $(A^\Box, \cdot, \cup)/{\sim_L}$ is called the *syntactic semiring* of L; we denote it by $(\mathsf{S}\,(L), \cdot, \vee)$. This structure is finite if and only if the language L is regular.

For non-empty finite sets A and B, a semiring homomorphism

$$f : (B^\Box, \cdot, \cup) \to (A^\Box, \cdot, \cup)$$

and $K \subseteq A^+$, we define

$$f^{[-1]}(K) = \{\ v \in B^+ \mid f(\{v\}) \subseteq K\ \}\ .$$

Similarly, for a semigroup homomorphism $g : (B^+, \cdot) \to (A^+, \cdot)$ and $K \subseteq A^+$ we put

$$g^{(-1)}(K) = \{ v \in B^+ \mid g(v) \in K \} .$$

A *class* of (regular) languages is an operator $\mathscr{L}$ assigning to every non-empty finite set A a set $\mathscr{L}(A)$ of regular languages over the alphabet A. Such a class is a *conjunctive variety* if

(i) each $\mathscr{L}(A)$ contains both $\emptyset$ and A^+,

(ii) each $\mathscr{L}(A)$ is closed with respect to finite intersections and quotients, and

(iii) for each finite sets A and B and a semiring homomorphism $f : B^\square \to A^\square$, $K \in \mathscr{L}(A)$ implies $f^{[-1]}(K) \in \mathscr{L}(B)$.

Similarly, it is a *positive* variety if (i) holds,
(ii') each $\mathscr{L}(A)$ is closed with respect to finite intersections, finite unions and quotients, and
(iii') for each finite sets A and B and a semigroup homomorphism $g : B^+ \to A^+$, $K \in \mathscr{L}(A)$ implies $f^{(-1)}(K) \in \mathscr{L}(B)$.

Adding to (ii') the closeness with respect to complements, we get the notion of a *boolean* variety.

We can assign to any class of languages $\mathscr{L}$ the pseudovariety

$$\mathsf{S}\,(\mathscr{L}) = [\ \{\ (\mathsf{S}\,(L), \cdot, \vee) \mid A \text{ is a non-empty finite set, } L \in \mathscr{L}(A)\ \}\ ]$$

of idempotent semirings generated by all syntactic semirings of members of all $\mathscr{L}(A)$'s. Conversely, for a class $\mathcal{X}$ of idempotent semirings and a non-empty finite set A, we put

$$(\mathsf{L}\,(\mathcal{X}))(A) = \{ L \subseteq A^* \mid (\mathsf{S}\,(L), \cdot, \vee) \in \mathcal{X} \} .$$

Result 8 ([11], Theorem 14). *The assignments $\mathscr{L} \mapsto \mathsf{S}\,(\mathscr{L})$ and $\mathcal{X} \mapsto \mathsf{L}\,(\mathcal{X})$ are mutually inverse bijections between the conjunctive varieties of languages and pseudovarieties of finite idempotent semirings.*

Similarly, by the classical Eilenberg theorem, the boolean varieties of languages correspond to pseudovarieties of semigroups, and the positive varieties of languages correspond to pseudovarieties of ordered semigroups (see Pin [10]).

Theorem 9. *The pseudovarieties of finite idempotent semirings with idempotent multiplication are exactly the classes $\mathsf{Fin}\,\mathcal{X}$ consisting of finite members of a variety $\mathcal{X}$ of idempotent semirings with idempotent multiplication. Finite members of different varieties form different pseudovarieties.*

Proof. Since the lattice $\mathscr{V}(\mathcal{I})$ is finite, the pseudovarieties are exactly of the form $\mathsf{Fin}\,\mathcal{X}$, $\mathcal{X} \in \mathscr{V}(\mathcal{I})$ (see [1], Proposition 3.2.4). Since all varieties are generated by finite members, the mapping Fin is injective. □

Our key result follows.

Theorem 10. *Let $\mathcal{X}$ be a variety of idempotent semirings with idempotent multiplication. For each $n \in \mathbb{N}$, the set $(\mathsf{L}\,(\mathsf{Fin}\,\mathcal{X}))(X_n)$ consists exactly of unions of $[\;]_{\mathcal{X},n}$-closed subsets of $X_n^+/\rho_{\underline{\mathcal{X}},n}$.*

Proof. Let $L \subseteq X_n^+,\ u, u_1, \dots, u_k \in X_m^+$. We show that

$$(X_n^{\Box}, \cdot, \cup)/\sim_L \ \text{ satisfies } u \le u_1 \vee \dots \vee u_k \qquad (\dagger)$$

if and only if $(\ \forall\ w_1, \dots, w_m \in X_n^+,\ p, q \in X_n^*\ )$

$$(\ pu_1(w_1, \dots, w_m)q, \dots, pu_k(w_1, \dots, w_m)q \in L \ \Rightarrow\ pu(w_1, \dots, w_m)q \in L\ )\ . \qquad (\ddagger)$$

Indeed, $(\dagger)$ means that $(\ \forall\ w_1, \dots, w_m \in X_n^+\ )$

$$(\ u(w_1 \sim_L, \dots, w_k \sim_L) \le u_1(w_1 \sim_L, \dots, w_k \sim_L) \vee \dots \vee u_k(w_1 \sim_L, \dots, w_k \sim_L)\ ,$$

which is equivalent to $(\ \forall\ w_1, \dots, w_m \in X_n^+\ )$

$$u(w_1, \dots, w_k) \sim_L \ \le (\ u_1(w_1, \dots, w_k) \vee \dots \vee u_k(w_1, \dots, w_k)\ ) \sim_L\ ,$$

and this is equivalent to $(\ddagger)$.

Now let $u_1, \dots, u_k \in L,\ u \in X_n^+$ and let $\mathcal{X}$ satisfy $u \le u_1 \vee \dots u_k$. Put $p = q = 1$ and $w_1 = x_1, \dots, w_n = x_n$ (we have $m = n$) in $(\ddagger)$.

Conversely, let L be a union of $[\;]_{\mathcal{X}}$-closed set of classes. Let $u, u_1, \dots, u_k \in X_m^+,\ u \le u_1 \vee \dots \vee u_k$ in $\mathcal{X}$. Let $w_1, \dots, w_m \in X_n^+, p, q \in X_n^*$. Then also $pu(w_1, \dots, w_m)q \le pu_1(w_1, \dots, w_m)q \vee \dots \vee pu_k(w_1, \dots, w_m)q$ in $\mathcal{X}$, which gives $(\ddagger)$ and therefore also $(\dagger)$. Moreover, L is a regular language since it is recognized by a finite semigroup $X_n^+/\rho_{\underline{\mathcal{X}},n}$. □

Similar results for semigroups and ordered semigroups are almost obvious. Recall that the operator L there uses syntactic semigroups and ordered semigroups instead of semirings; we write $\mathsf{L}^{\cdot}$ and $\mathsf{L}^{\le}$. In the ordered case $\rho_{\mathcal{V}}$ consists of all pairs (u, v) such that $u \le v$ in $\mathcal{V}$. Let $\tau_{\mathcal{V}} = \rho_{\mathcal{V}} \cap (\rho_{\mathcal{V}})^{-1}$. Then $X_n^+/\tau_{\mathcal{V}}$ is ordered by $u\tau_{\mathcal{V}} \le v\tau_{\mathcal{V}}$ iff $u\ \rho_{\mathcal{V}}\ v$.

Result 11.

(i) Let $\mathcal{V}$ be a variety of semigroups. For each $n \in \mathbb{N}$, the set $(\mathsf{L}^{\cdot}(\mathsf{Fin}\,\mathcal{V}))(X_n)$ consists exactly of unions of classes of $X_n^+/\rho_{\mathcal{V},n}$ which are regular languages.

(ii) Let $\mathcal{V}$ be a variety of ordered semigroups. For each $n \in \mathbb{N}$, the set $(\mathsf{L}^{\le}(\mathsf{Fin}\,\mathcal{V}))(X_n)$ consists exactly of unions of hereditary sets of classes of $(X_n^+/\tau_{\mathcal{V},n}, \le)$ which are regular languages.

Now we describe conjunctive varieties of languages corresponding to our pseudovarieties.

Theorem 12.

(i) $\underline{\mathcal{D}} = \mathcal{S}l$,

$$\text{and } Y \in [\{Y_1, \ldots, Y_k\}]_{\mathcal{D}} \text{ iff } (\ \exists\, i \in \{1, \ldots, k\}\ )\ Y \supseteq Y_i\ .$$

Consequently,

$$L \in (\mathsf{L}\,(\mathcal{D}))(X_n) \text{ iff } (\ \exists\, Y_1, \ldots, Y_k \subseteq X_n\ )$$

$$\text{such that } L = \{\ u \in X_n^+ \mid (\ \exists\, i \in \{1, \ldots, k\}\ )\ \mathsf{c}(u) \supseteq Y_i\ \}\ ,$$

or equivalently L is a union of languages of the form

$$\{\, u \in X_n^+ \mid \mathsf{c}(u) \supseteq Y\,\},\ Y \subseteq X\ .$$

(ii) $\underline{\mathcal{M}} = \mathcal{S}l$,

$$\text{and } Y \in [\{Y_1, \ldots, Y_k\}]_{\mathcal{D}} \text{ iff } Y \subseteq Y_1 \cup \cdots \cup Y_k\ .$$

Consequently,

$$L \in (\mathsf{L}\,(\mathcal{M}))(X_n) \text{ iff } (\ \exists\, Y \subseteq X_n\ )\ L = Y^+\ .$$

(iii) $\underline{\mathcal{S}} = \mathcal{S}l$,

$$\text{and } Y \in [\{Y_1, \ldots, Y_k\}]_{\mathcal{S}} \text{ iff } (\ \exists\, i_1, \ldots, i_l \in \{1, \ldots, k\}\ )\ Y = Y_{i_1} \cup \cdots \cup Y_{i_l}\ .$$

Consequently,

$$L \in (\mathsf{L}\,(\mathcal{S}))(X_n) \text{ iff } (\ \exists\, Y_1, \ldots, Y_k \subseteq X_n\ )$$

$$\text{with } L = \{\ u \in X_n^+ \mid (\ \exists\, i_1, \ldots, i_l \in \{1, \ldots, k\}\ )\ \mathsf{c}(u) = Y_{i_1} \cup \cdots \cup Y_{i_l}\ \}\ .$$

(iv) $\underline{\mathcal{L}} = \mathcal{LZ}$, *and* $[\{y_1, \ldots, y_k\}]_{\mathcal{L}} = \{y_1, \ldots, y_k\}$. *Consequently,*

$$L \in (\mathsf{L}\,(\mathcal{L}))(X_n) \text{ iff } (\ \exists\, Y \subseteq X_n\ )\ L = \{\, u \in X_n^+ \mid \mathsf{h}(u) \in Y\,\} = Y X_n^*.$$

(v) $\underline{\mathcal{L}}^0 = \mathcal{LNB}$, *and*

$$(y, Y) \in [\{\,(y_1, Y_1), \ldots, (y_k, Y_k)\,\}]_{\mathcal{L}^0}$$

$$\text{iff } (\ \exists\, i \in \{1, \ldots, k\}\ )\ (\ y = y_i,\ Y \supseteq Y_i\ )\ .$$

Consequently,

$$L \in (\mathsf{L}\,(\mathcal{L}^0))(X_n) \text{ iff } (\ \exists\, Y_1, \ldots, Y_k \subseteq X_n,\, y_1 \in Y_1, \ldots, y_k \in Y_k\ )$$

$$\text{such that } L = \{\ u \in X_n^+ \mid (\ \exists\, i \in \{1, \ldots, k\}\ )\ \mathsf{h}(u) = y_i \text{ and } \mathsf{c}(u) \supseteq Y_i\ \}\ ,$$

or equivalently L is a union of languages of the form

$$\{\, u \in X_n^+ \mid \mathsf{h}(u) = y,\ \mathsf{c}(u) \supseteq Y\,\},\ y \in Y \subseteq X\ .$$

(vi) $\underline{\mathcal{B}} = \mathcal{LRB}$, *and*

$$u \in [\{u_1,\dots,u_k\}]_{\mathcal{B}} \textit{ iff } (\ \forall\ Y \subseteq X\ )\ \mathsf{h}(u_Y) \in \{\, \mathsf{h}((u_1)_Y),\dots,\mathsf{h}((u_k)_Y)\,\}\ .$$

Consequently,

$$L \in (\mathsf{L}\,(\mathcal{B}))(X_n) \textit{ iff } (\ \exists\ u_1,\dots,u_k \in X_n^+\ ) \textit{ such that}$$

$$L = \{\ u \in X_n^+ \mid (\ \forall\ Y \subseteq X_n\ )\ \mathsf{h}(u_Y) \in \{\, \mathsf{h}((u_1)_Y),\dots,\mathsf{h}((u_k)_Y)\,\}\ \}\ .$$

(vii) $\underline{\mathcal{B}}^0 = \mathcal{LRB}$,

$$\textit{and } u \in [\{u_1,\dots,u_k\}]_{\mathcal{B}^0} \textit{ iff } u \in [\{\ u_i \mid i=1,\dots,k,\ \mathsf{c}(u_i) \subseteq \mathsf{c}(u)\ \}]_{\mathcal{B}}$$

Consequently,

$$L \in (\mathsf{L}\,(\mathcal{B}^0))(X_n) \textit{ iff } (\ \exists\ u_1,\dots,u_k \in X_n^+\ ) \textit{ such that } L =$$

$$\{\ u \in X_n^+ \mid (\ \forall\ Y \subseteq X_n\ )\ \mathsf{h}(u_Y) \in \{\, \mathsf{h}((u_i)_Y) \mid i=1,\dots,k,\ \mathsf{c}(u_i) \subseteq \mathsf{c}(u)\,\}\ .$$

Proof. We find the values of $\underline{\mathcal{X}}$ first. It would follow from the observations below. We will use Result 3.

- The identity $x^2 = x$ holds in B^0; by duality also in C^0 and therefore in all eleven varieties from $\mathfrak{O}$.
- The identity $xy = yx$ holds in M^0 but not in L.
- The identity $xy = x$ holds in L but not in $D,\ M,\ R$.
- The identity $xyz = xzy$ holds in L^0 but not in $M,\ R$.
- The identity $xy = xyx$ holds in B^0 but not in R.

The descriptions of the operators $[\ ]_{\mathcal{X}}$ follows immediately from Result 3. Use Theorem 10 for the formulas for the corresponding languages. □

The corresponding results for the varieties $\mathcal{R},\ \mathcal{R}^0,\ \mathcal{C}$ and $\mathcal{C}^0$ we get by duality. We can describe the joins of irreducible varieties of languages by Result 6 or we can use the following simple construction.

Theorem 13. *For conjunctive varieties of languages $\mathscr{K}$ and $\mathscr{L}$ and a non-empty finite set A, we have*

$$(\mathscr{K} \vee \mathscr{L})(A) = \{\, K \cap L \mid K \in \mathscr{K}(A) \textit{ and } L \in \mathscr{L}(A)\,\}\,.$$

Proof. Obvious. □

A language $L \subseteq A^+$ is *closed* if $u, v \in L$ implies $uv \in L$. Recall that the *shuffle* of words $u, v \in A^+$ is the set $u \sqcup v =$

$$\{ u_1 v_1 \dots u_k v_k \mid k \in \mathbb{N},\, u = u_1 \dots u_k,\ v = v_1 \dots v_k,\, u_1, \dots, u_k, v_1, \dots, v_k \in A^* \}.$$

Thus the following system of identities characterizes languages all quotients of which are shuffle-closed

$$x_1 y_1 \dots x_k y_k \le x_1 \dots x_k \vee y_1 \dots y_k,\ x_1 y_1 \dots x_k y_k x_{k+1} \le x_1 \dots x_{k+1} \vee y_1 \dots y_k \ ,$$

$k \in \mathbb{N}$. Now the following is straightforward.

Theorem 14.

(i) All quotients of a language whose syntactic semiring has idempotent multiplication are shuffle closed.

(ii) Each language with idempotent syntactic semigroup with all quotients being closed has syntactic semiring with idempotent multiplication.

□

4 Conjunctive versus positive varieties of languages

For a class $\mathcal{V}$ of semigroups, we put

$$\mathcal{V}^{\le} = \{ (S, \cdot, \le) \text{ is an ordered semigroup } \mid (S, \cdot) \in \mathcal{V} \} ,$$

and for a class $\mathcal{W}$ of ordered semigroups, we set

$$\mathcal{W}_+ = \{ (S, \cdot, \le) \in \mathcal{W} \mid (S, \cdot, \le) \text{ satisfies } xyx \le x \} \text{ and}$$

$$\mathcal{W}_- = \{ (S, \cdot, \le) \in \mathcal{W} \mid (S, \cdot, \le) \text{ satisfies } x \le xyx \} .$$

Result 15 (**[2]**). *The lattice of all varieties of ordered normal bands consists of 8 varieties of the form $\mathcal{V}^{\le}$ where $\mathcal{V}$ is a variety of normal bands and 8 varieties of the forms $\mathcal{W}_+$, $\mathcal{W}_-$ where $\mathcal{W} \in \{ \mathcal{S}l^{\le}, \mathcal{LNB}^{\le}, \mathcal{RNB}^{\le}, \mathcal{NB}^{\le} \}$.*

The operator Fin *is a bijection of this lattice onto the lattice of all pseudovarieties of ordered normal bands.*

As announced in [2] and also proved by the authors of [4] (unpublished) any other variety of ordered bands is of the form $\mathcal{V}^{\le}$ for a variety $\mathcal{V}$ of bands. Therefore we have exactly 21 varieties of ordered regular bands and (since they are generated by their finite members) exactly 21 pseudovarieties of ordered regular bands.

Theorem 16. *Let $\mathscr{L}$ be a conjunctive variety of languages. Then the smallest positive variety of languages is of the form $\mathscr{L}^{\cup}$, where for each non-empty finite set A :*

$$\mathscr{L}^{\cup}(A) = \{ L_1 \cup \dots \cup L_k \mid k \in \mathbb{N},\ L_1, \dots, L_k \in \mathscr{L}(A) \} .$$

Proof. The result is obvious. □

Theorem 17.

(i) *For* $\mathcal{X} \in \mathscr{V}(\mathcal{I})$ *with* $\underline{\mathcal{X}} = \{\mathcal{T}, \mathcal{S}l\}$, *exactly* $\mathsf{L}(\mathcal{TS}) = \mathsf{L}^{\leq}(\mathcal{T})$ *and* $\mathsf{L}(\mathcal{D}) = \mathsf{L}^{\leq}(\mathcal{S}l^{\leq}_{+})$ *are positive varieties of languages. Moreover,*

$$(\mathsf{L}(\mathcal{M}))^{\cup} = \mathsf{L}^{\leq}(\mathcal{S}l^{\leq}_{-}) \text{ and } (\mathsf{L}(\mathcal{D} \vee \mathcal{M}))^{\cup} = (\mathsf{L}(\mathcal{S}))^{\cup} = \mathsf{L}^{\leq}(\mathcal{S}l^{\leq}) \ .$$

(ii) *For* $\mathcal{X} \in \mathscr{V}(\mathcal{I})$ *with* $\underline{\mathcal{X}} = \{\mathcal{LZ}, \mathcal{LNB}\}$, *exactly* $\mathsf{L}(\mathcal{L}) = \mathsf{L}^{\leq}(\mathcal{LZ})$ *and* $\mathsf{L}(\mathcal{L}^0) = \mathsf{L}^{\leq}(\mathcal{LNB}^{\leq}_{+})$ *are positive varieties of languages. Moreover,*

$$(\mathsf{L}(\mathcal{L} \vee \mathcal{M}))^{\cup} = \mathsf{L}^{\leq}(\mathcal{LNB}^{\leq}_{-}),\ (\mathsf{L}(\mathcal{L} \vee \mathcal{D}))^{\cup} = \mathsf{L}(\mathcal{L}^0), \quad \text{and}$$

$$(\mathsf{L}(\mathcal{L} \vee \mathcal{D} \vee \mathcal{M}))^{\cup} = \cdots = \mathsf{L}^{\leq}(\mathcal{LNB}^{\leq}) \ .$$

(iii) *For* $\mathcal{X} \in \mathscr{V}(\mathcal{I})$ *with* $\underline{\mathcal{X}} = \{\mathcal{R}e\mathcal{B}, \mathcal{NB}\}$, *exactly* $\mathsf{L}(\mathcal{L} \vee \mathcal{R}) = \mathsf{L}^{\leq}(\mathcal{R}e\mathcal{B})$ *and* $\mathsf{L}(\mathcal{L}^0 \vee \mathcal{R}^0) = \mathsf{L}^{\leq}(\mathcal{NB}^{\leq}_{+})$ *are positive varieties of languages. Moreover,*

$$(\mathsf{L}(\mathcal{L} \vee \mathcal{M} \vee \mathcal{R}))^{\cup} = \mathsf{L}^{\leq}(\mathcal{NB}^{\leq}_{-}),\ (\mathsf{L}(\mathcal{L} \vee \mathcal{D} \vee \mathcal{R}))^{\cup} = \mathsf{L}(\mathcal{L}^0 \vee \mathcal{R}^0), \quad \text{and}$$

$$(\mathsf{L}(\mathcal{L} \vee \mathcal{D} \vee \mathcal{M} \vee \mathcal{R}))^{\cup} = \cdots = \mathsf{L}^{\leq}(\mathcal{NB}^{\leq}) \ .$$

(iv) *For all other* $\mathcal{X} \in \mathscr{V}(\mathcal{I})$, $\mathsf{L}(\mathcal{X})$ *is not a positive variety and*

$$(\mathsf{L}(\mathcal{X}))^{\cup} = \mathsf{L}^{\leq}(\underline{\mathcal{X}}^{\leq}) \ .$$

Proof. All follows from simple calculations. □

Example. Let $A = \{a, b\}$, $L = a^+ \cup b^+$. Then the (ordered) syntactic semigroup is idempotent, but the syntactic semiring is not. Indeed, using the notation from [12] we have $D = \{\, a^+ \cup b^+, a^*, b^*, \emptyset \,\}$ and the transformation semigroup consists of transformations given by a, b, ab having the presentation $a^2 = a$, $b^2 = b$, $ab = ba = 0$. Further, $\overline{D} = D \cup \{a^+, b^+, 1\}$ and there is a new transformation given by $\{a, b\}$. This element is not an idempotent.

We can derive the result from Theorem 14 : $a^2, b \in L$ but $aba \in a^2 \sqcup b$, $aba \notin L$.

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