Acta Cybernetica **18** (2008) 681–693.

Investigating the Behaviour of a Discrete Retrial System

Péter Kárász*

Abstract

Certain technological applications use queuing systems where the service time of entering entities cannot take any value, it can only be a multiple of a certain cycle-time. As examples of this, one can mention the landing of aeroplanes and the optical buffers of internet networks. Servicing an entering customer can be started immediately, or, if the server is busy, or there are waiting customers, the new customer joins a queue, moving along a closed path which can be completed within a fixed cycle-time of T units. Applications in digital technology induce the investigation of discrete systems. We give the mathematical description of systems serving two types of customers, where inter-arrival times follow a geometric distribution, and service times are distributed uniformly. A Markov-chain is defined and the generating functions of transition probabilities are calculated. The condition of ergodicity is established and the equilibrium distribution is given.

Keywords: discrete retrial systems, Lakatos-type queuing system

1 Introduction

The system investigated in the paper was originally based on a real problem connected with the landing of aeroplanes, but later many other applications emerged which are strongly related to information technology. We use the original problem to provide an initial description.

Consider an airport where aeroplanes come to land. The airport can serve only one plane at a time. Hence, if the runway is used or there are other planes waiting to land, the incoming plane has to wait. Unlike in classic queuing systems, special conditions prevail here, which result in significant differences from an ordinary system. We assume that a plane planning to land approaches the runway in the optimum position, and, if it is possible, it starts landing immediately. If the plane is forced to wait, then it starts a circular manoeuvre and can issue further requests to land only when reaching the original starting point of its trajectory. We assume

*Budapest Tech, John von Neumann Faculty of Informatics, Bécsi út 96/b, H–1034 Budapest, Hungary, E-mail: karasz.peter@nik.bmf.hu

that completing each full cycle takes equal time, T, thus the possible instants of starting to land can differ from the moment of arrival by integer multiples of T. Because of possible fuel shortage it is natural to use the first-come-first-served (FCFS) discipline.

We can model the above problem by investigating a retrial system, where the service of an incoming customer can be started upon arrival if the system is in free state, otherwise – if the server is busy or there are other entities waiting – the incoming customer joins a queue and its service is started at the nearest possible instant differing from the arrival by a multiple of the cycle-time T. The FCFS rule is obeyed. Different service time distributions lead to various problems, these were broadly investigated by Lakatos. In [9], service time distribution is exponential, whereas in [10] it is uniform. In the light of technical applications, it is important to consider discrete models. In this case the cycle-time is divided into n equal time-slices, which form the basis of the discrete distributions. A typical application in digital technology is the use of an optical buffer, which is a device capable of temporarily storing light (or rather, data in the form of light). As light cannot be frozen, a typical optical buffer is realised by a single loop, in which data circulate a variable number of times, and thus n can be the measure of the cycle-time in clock cycles. The model was investigated by Rogiest, Laevens et al. in [8, 14]. The involved optical buffers are implemented as a set of $N + 1$ Fiber Delay Lines (FDLs), with lengths that are typically multiples of a basic value D, called the granularity. This results in a degenerate waiting room with waiting times 0, D, $2D$, …, ND. The problem is investigated from the point of view of the customers, i. e. their waiting time. Lakatos and myself chose a different approach, the problem is described from the aspect of the server, i. e. the number of waiting customers, which is more significant for determining the number of necessary FDLs. The time elapsed between two arrivals was geometrically distributed and service times of customers were geometric and uniform in [11] and [12], respectively. The two different approaches to describe these systems coincide in the condition of ergodicity and the probability of free state; this was shown in [13]. A numerical investigation was carried out in [2].

It was Kovalenko's suggestion to generalise the problem for two different types of customers. Only one customer of the first type can be present in the system. Such a customer can be accepted for service only in the case of a free system, in all other cases its request is turned down. There is no such restriction for the customers of the second type; they are serviced immediately or join a queue, when the server is busy. This type of system was examined with different continuous distributions in [4, 5, 6]; simulation results were also included in [6]. In [7] this type of system using relative priorities was investigated with geometric inter-arrival and service time distributions. In the present paper the same system is considered with discrete uniform service time distributions. The endpoints of the interval of the uniform distributions are presumed to be multiples of the cycle-time T. This assumption does not restrict the generality of the theory, but without it formulae are much more complicated.

There are several aspects which can be treated just exactly in the same way as in the case of continuous distributions. However, there are some differences between discrete and continuous systems. One such significant phenomenon is collision. With continuously distributed service times the probability of the appearance of two different types of customers at the very same instant is zero, but in discrete systems different types of customers do appear during the same time-slice with non-zero probability. There are several ways to deal with collisions, we suggest three methods for treating these, numbered I., II., and III.

The aim is to determine generating functions of transition probabilities, as well as to establish the condition of ergodicity.

2 Results

Consider a Lakatos-type queuing system serving two types of customers. The cycle-time T is divided into n equal time-slices. The probability of the appearance of a customer of type j during a certain time-slice is r_j, and there is no entry with probability $1 - r_j$, i. e. inter-arrival times are geometrically distributed with parameters r_j $(j = 1, 2)$. Service times are uniformly distributed in the interval $[\gamma_j, \delta_j]$, (where γ_j and δ_j are multiples of T), i. e. the probability that the service time of a customer of type j is in this interval is $q_j = \frac{T}{n(\delta_j - \gamma_j)}$.

For the description of the system we are going to use the *embedded Markov-chain* technique. Let us consider the number of customers in the system at the moment just before the service of a new customer begins. In other words, if t_k denotes the moment when the service of the k-th entity starts, we consider the sequence, whose states correspond to the number of customers at $t_k - 0$. For the sake of definiteness, at $t = 0$ let the system be free. To see that the process is Markovian we refer to the same argument as in [6], and the memoryless property of the geometric distribution.

For this chain we introduce the following transition probabilities:

a_{ji}: the probability of the appearance of i customers of the second type at the service of a type j customer $(j = 1, 2)$, if at the beginning there is only one customer in the system;

b_i: the probability of the appearance of i customers of the second type at the service of a second type customer, if at the beginning of the service there are at least two customers in the system;

c_i: the probability of the appearance of i customers of the second type in a free state.

As the process runs, the busy period can start with a customer of either type. During the service of this customer only second type customers are accepted for service, they join the queue. Requests of first type customers are refused. This explains the need for introducing c_i, the value of which will be determined using a_{ji}, depending on the type of customer being serviced. If there are no other requests present when the service of the next customer begins (which is obviously of the second type), the system turns into state 1, and the probabilities of turning into other states from this one are given by a_{2i}. The probabilities of all other transitions are b_i. The corresponding generating functions of all these probabilities are

$$A_j(z) = \sum_{i=0}^{\infty} a_{ji} z^i, \quad B(z) = \sum_{i=0}^{\infty} b_i z^i, \quad \text{and} \quad C(z) = \sum_{i=0}^{\infty} c_i z^i.$$

Collision disciplines. As far as collisions are considered, three different methods will be applied:

Method I. In the case of a collision, both types of customers are refused.

Method II. In the case of a collision, the first type customers are accepted for service, but the second type ones are refused.

Method III. In the case of a collision, customers of both types are accepted for service, but the ones of the first type are served first. When applying this method, in addition to previously defined transition probabilities, new ones have to be introduced. Let a_{12i} denote the probability of the appearance of i customers of the second type at the service of a first type customer, if the service process started with the simultaneous appearance of customers of both types; the generating function of these probabilities is $A_{12}(z) = \sum_{i=0}^{\infty} a_{12i} z^i$.

Let us now summarise the properties of the system and introduce some notation. Consider a discrete cyclic-waiting system serving two types of customers in which inter-arrival time distributions are geometric with parameters r_j, whereas service times are uniformly distributed in the intervals $[\gamma_j, \delta_j]$ $(j = 1, 2)$, respectively. The service of an entering customer may start immediately on arrival if the server is free, but in case of a busy server or waiting customers, first type customers are refused, and second type customers join the queue. The service of queued customers may start at times differing from their arrival times by multiples of the cycle-time T, which is divided into n equal time slices; these form the units of the geometric and uniform distributions. The states of the corresponding embedded Markov-chain are identified with the number of customers in the system at moments just before starting the service of a customer.

Theorem 1. *The matrix of the transition probabilities of the defined chain has the form:*

$$\begin{pmatrix} c_0 & c_1 & c_2 & c_3 & \cdots \\ a_{20} & a_{21} & a_{22} & a_{23} & \cdots \\ 0 & b_0 & b_1 & b_2 & \cdots \\ 0 & 0 & b_0 & b_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1}$$

The elements of the matrix are determined by their generating functions below.

$$
\begin{aligned}
A_j(z) = \frac{q_j}{r_2}\left[(1-r_2)^{\frac{\gamma_j}{T}n+1} - (1-r_2)^{\frac{\delta_j}{T}n+1}\right] + \\
+ zq_j\left[(1-r_2)^{\frac{\gamma_j}{T}n} - (1-r_2)^{\frac{\delta_j}{T}n}\right] + zq_j\left[(1-r_2)^{\frac{\gamma_j}{T}n} - (1-r_2)^{\frac{\delta_j}{T}n}\right] \times \\
\times \frac{1-(1-r_2)^n}{r_2(1-r_2)^n}(1-r_2+r_2z)^n \frac{1-\left(\frac{1-r_2+r_2z}{1-r_2}\right)^{\frac{\gamma_j}{T}n}}{1-\left(\frac{1-r_2+r_2z}{1-r_2}\right)^n} + \\
+ zq_j(1-r_2+r_2z)^n\left[n\frac{(1-r_2+r_2z)^{\frac{\gamma_j}{T}n} - (1-r_2+r_2z)^{\frac{\delta_j}{T}n}}{1-(1-r_2+r_2z)^n} - \right. \\
\left. - (1-r_2)^{\frac{\delta_j}{T}n}\frac{1-(1-r_2)^n}{r_2(1-r_2)^n}\frac{\left(\frac{1-r_2+r_2z}{1-r_2}\right)^{\frac{\gamma_j}{T}n} - \left(\frac{1-r_2+r_2z}{1-r_2}\right)^{\frac{\delta_j}{T}n}}{1-\left(\frac{1-r_2+r_2z}{1-r_2}\right)^n}\right],
\end{aligned}
\tag{2}
$$

$$
A_{12}(z) = znq_1\frac{(1-r_2+r_2z)^{\left(\frac{\gamma_1}{T}+1\right)n} - (1-r_2+r_2z)^{\left(\frac{\delta_1}{T}+1\right)n}}{1-(1-r_2+r_2z)^n},
\tag{3}
$$

$$
\begin{aligned}
B(z) = \frac{r_2q_2}{1-(1-r_2^n)}\frac{(1-r_2+r_2z)^{\frac{\gamma_2}{T}n} - (1-r_2+r_2z)^{\frac{\delta_2}{T}n}}{1-(1-r_2+r_2z)^n} \times \\
\times \left[\left((1-r_2+r_2z) - (1-r_2+r_2z)^{n+1}\right) \times \right. \\
\times \left(\frac{1-(1-r_2)^n(1-r_2+r_2z)^n}{(1-(1-r_2)(1-r_2+r_2z))^2} - \frac{n(1-r_2)^n(1-r_2+r_2z)^n}{1-(1-r_2)(1-r_2+r_2z)}\right) + \\
\left. + n(1-r_2+r_2z)^{n+1}\frac{1-(1-r_2)^n(1-r_2+r_2z)^n}{1-(1-r_2)(1-r_2+r_2z)}\right]
\end{aligned}
\tag{4}
$$

and $C(z)$ depends on collision policies:

I. $C(z) = \frac{r_1(1-r_2)}{r_1+r_2-r_1r_2}A_1(z) + \frac{r_2(1-r_1)}{r_1+r_2-r_1r_2}A_2(z) + \frac{r_1r_2}{r_1+r_2-r_1r_2},$ (5a)

II. $C(z) = \frac{r_1}{r_1+r_2-r_1r_2}A_1(z) + \frac{r_2(1-r_1)}{r_1+r_2-r_1r_2}A_2(z)\,,$ (5b)

III. $C(z) = \frac{r_1(1-r_2)}{r_1+r_2-r_1r_2}A_1(z) + \frac{r_2(1-r_1)}{r_1+r_2-r_1r_2}A_2(z) + \frac{r_1r_2}{r_1+r_2-r_1r_2}A_{12}(z)\,.$ (5c)

Proof. Because of the definitions, the construction of the matrix of transition probabilities is straightforward. However, we draw the attention of the reader to the fact that probabilities a_{1i} do not appear in it explicitly, as customers of the first type can only be accepted when the system is free. These probabilities are represented through probabilities c_i.

First we determine a_{ji}. In this case only one customer is present at the beginning of a service (the one whose service is about to start). Time units are of length $\frac{T}{n}$, and all time intervals are measured using this unit. The service time of the actual customer is denoted by u, and the next one appears v time units after the servicing of the first customer started. In order to get a_{ji}, the distribution of $u - v$ must be known. Two separate calculations have to be carried out.

If $0 < l \leq \frac{\gamma_j}{T} n$:

$$P(u-v=l) = \sum_{k=\frac{\gamma_j}{T}n+1}^{\frac{\delta_j}{T}} q_j \left(1-r_2\right)^{k-l-1} r_2 = q_j \left[\left(1-r\right)^{\frac{\gamma_j}{T}n-l} - \left(1-r\right)^{\frac{\delta_j}{T}n-l}\right];$$

and if $\frac{\gamma_j}{T} n < l \leq \frac{\delta_j}{T} n$:

$$P(u-v=l) = \sum_{k=l+1}^{\frac{\delta_j}{T}} q_j \left(1-r_2\right)^{k-l-1} r_2 = q_j \left[1 - \left(1-r\right)^{\frac{\delta_j}{T}n-l}\right].$$

The waiting time can be determined on the basis of these probabilities. If $u - v = 0$ (the next customer appears in the time-slice in which the service of the present customer is completed), then the service of the next customer can be started immediately, the waiting time is 0. If $u-v$ is in $\overline{1,n}$[1], then the waiting time is T, i. e. n units; if it is in $\overline{n+1,2n}$, then the waiting time is $2n$; and in general if $u-v$ takes some value from $\overline{(i-1)\,n+1, in}$, then the waiting time of the next customer is in.

The probability that the waiting time of the second customer is in (i. e. $u-v$ is in $\overline{(i-1)\,n+1, in}$) is the following.

If $0 < i \leq \frac{\gamma_j}{T}$:

$$P\big(\,(i-1)\,n+1 \leq u-v \leq in\big) = \sum_{l=(i-1)n+1}^{in} q_j \left[\left(1-r_2\right)^{\frac{\gamma_j}{T}n-l} - \left(1-r_2\right)^{\frac{\delta_j}{T}n-l}\right] =$$
$$= q_j \left[\left(1-r_2\right)^{\frac{\gamma_j}{T}n} - \left(1-r_2\right)^{\frac{\delta_j}{T}n}\right] \frac{1-\left(1-r_2\right)^n}{r_2\left(1-r_2\right)^n} \frac{1}{\left(1-r_2\right)^{(i-1)n}}.$$

The generating function of the number of customers appearing in a time slice is $1 - r_2 + r_2 z$, hence, the generating function of the number of customers entering

[1]Notation $\overline{a,b}$ is used for integer intervals, i. e. $\overline{a,b} = [a,b] \cap \mathbb{Z}$.

during the waiting time is:

$$\sum_{i=1}^{\frac{\gamma_j}{T}} q_j \left[(1-r_2)^{\frac{\gamma_j}{T}n} - (1-r_2)^{\frac{\delta_j}{T}n}\right] \frac{1-(1-r_2)^n}{r_2\,(1-r_2)^n} \frac{(1-r_2+r_2z)^{in}}{(1-r_2)^{(i-1)n}} =$$
$$= q_j \left[(1-r_2)^{\frac{\gamma_j}{T}n} - (1-r_2)^{\frac{\delta_j}{T}n}\right] \times$$
$$\times \frac{1-(1-r_2)^n}{r_2\,(1-r_2)^n} (1-r_2+r_2z)^n \frac{1-\left(\frac{1-r_2+r_2z}{1-r_2}\right)^{\frac{\gamma_j}{T}n}}{1-\left(\frac{1-r_2+r_2z}{1-r_2}\right)^n}. \tag{6}$$

If $\frac{\gamma_j}{T} < i \leq \frac{\delta_j}{T}$:

$$P\big((i-1)\,n+1 \leq u-v \leq in\big) = \sum_{l=(i-1)n+1}^{in} q_j \left[1-(1-r_2)^{\frac{\delta_j}{T}n-l}\right] =$$
$$= q_j n - q_j\,(1-r_2)^{\frac{\delta_j}{T}n} \frac{1-(1-r_2)^n}{r_2\,(1-r_2)^n} \frac{1}{(1-r_2)^{(i-1)n}},$$

and the generating function of entering customers is:

$$\sum_{i=\frac{\gamma_j}{T}+1}^{\frac{\delta_j}{T}} \left[q_j n - q_j\,(1-r_2)^{\frac{\delta_j}{T}n} \frac{1-(1-r_2)^n}{r_2\,(1-r_2)^n} \frac{1}{(1-r_2)^{(i-1)n}}\right] (1-r_2+r_2z)^{in} =$$
$$= q_j n\,(1-r_2+r_2z)^n \frac{(1-r_2+r_2z)^{\frac{\gamma_j}{T}n} - (1-r_2+r_2z)^{\frac{\delta_j}{T}n}}{1-(1-r_2+r_2z)^n} -$$
$$- q_j\,(1-r_2)^{\frac{\delta_j}{T}n} \frac{1-(1-r_2)^n}{r_2\,(1-r_2)^n} (1-r_2+r_2z)^n \frac{\left(\frac{1-r_2+r_2z}{1-r_2}\right)^{\frac{\gamma_j}{T}n} - \left(\frac{1-r_2+r_2z}{1-r_2}\right)^{\frac{\delta_j}{T}n}}{1-\left(\frac{1-r_2+r_2z}{1-r_2}\right)^n}. \tag{7}$$

The probability that the waiting time is zero (which happens when the next customer enters during the last time-slice of the service of the previous one) is:

$$\sum_{k=\frac{\gamma_j}{T}n+1}^{\frac{\delta_j}{T}n} q_j\,(1-r_2)^{k-1}\,r_2 = q_j \left[(1-r_2)^{\frac{\gamma_j}{T}n} - (1-r_2)^{\frac{\delta_j}{T}n}\right], \tag{8}$$

while the probability that there is no entry at all is:

$$\sum_{k=\frac{\gamma_j}{T}n+1}^{\frac{\delta_j}{T}n} q_j\,(1-r_2)^k = \frac{q_j}{r_2} \left[(1-r_2)^{\frac{\gamma_j}{T}n+1} - (1-r_2)^{\frac{\delta_j}{T}n+1}\right]. \tag{9}$$

Bearing in mind that we examined those possibilities when a customer enters obligatorily, generating functions (6), (7), and (8) have to be multiplied by z, then added to (9), which yields (2).

If the third method of collision treatment is applied, then $A_{12}(z)$ has to be determined. In this case the service starts with the simultaneous arrival of two customers of different types, and the one of the first type is served first. The duration of its service can take any value between $\frac{\gamma_1}{T}n+1$ and $\frac{\delta_1}{T}n$ with equal probability $q_1 = \frac{T}{n(\delta_1-\gamma_1)}$, so

$$P\big((i-1)\,n+1 \le u \le in\big) = \sum_{l=(i-1)n+1}^{in} q_1 = q_1 n$$

for all $\frac{\gamma_1}{T} < i \le \frac{\delta_1}{T}$.

Taking into account that one customer of the second type is already present at the start of the service of the first type customer, the generating function is:

$$A_{12}(z) = z \sum_{i=\frac{\gamma_1}{T}+1}^{\frac{\delta_1}{T}} nq_1\,(1-r_2+r_2z)^{in} =$$

$$= znq_1 \frac{(1-r_2+r_2z)^{\left(\frac{\gamma_1}{T}+1\right)n} - (1-r_2+r_2z)^{\left(\frac{\delta_1}{T}+1\right)n}}{1-(1-r_2+r_2z)^n},$$

which is identical to (3).

The probability of the appearance of at least one customer of any type in a time-slice is

$$1-(1-r_1)\,(1-r_2) = r_1+r_2-r_1r_2.$$

The busy period can start with the arrival of a first type customer alone with the probability $\frac{r_1(1-r_2)}{r_1+r_2-r_1r_2}$, with the arrival of a single second type customer with the probability $\frac{r_2(1-r_1)}{r_1+r_2-r_1r_2}$, and with the arrival both customers, with the probability $\frac{r_1r_2}{r_1+r_2-r_1r_2}$. These easily explain (5c) (collision discipline III.). In the case of collision treatment method I., customers of both types are lost if they arrive during the same time-slice, and this may be interpreted as a service of zero length (the system stays in the free state with this probability), which results in the generating function (5a). If collision discipline II. is applied, then the busy period can start with the service of a first type customer with the probability $\frac{r_1}{r_1+r_2-r_1r_2}$ (no matter whether there was a refused customer of the second type at the same time), and starts with the service of a second type customer with the probability $\frac{r_2(1-r_1)}{r_1+r_2-r_1r_2}$, which explains (5b).

Finally, we are going to determine the transition probabilities b_i. In this case, when the service of the actual customer begins, the next one is already present. Let $x = u - \left\lfloor \frac{u-1}{n} \right\rfloor n$, i. e. x is the service time mod n ($1 \le x \le n$), and let y denote the inter-arrival time mod n ($1 \le y \le n$). The time elapsed between the starting moments of services of these two consecutive customers will be:

$$t_0 = \begin{cases} \left\lfloor \frac{u-1}{n} \right\rfloor n + y, & \text{if } x \le y; \\ \left(\left\lfloor \frac{u-1}{n} \right\rfloor + 1 \right) n + y, & \text{if } x > y. \end{cases}$$

Now, let us fix y, and consider the usual set of integers $\overline{in+1, (i+1)\,n}$. If the service is completed until $in + y$ (inclusive), then the time in question is $in + y$. The probability of this event is yq_2. If the service finishes later than $in+y$, then the time difference between starting services is $(i+1)\,n+y$ with probability $(n-y)\,q_2$. Summation has to be extended over all possible values of service times, therefore, the generating function of the number of entering customers on condition that inter-arrival time mod n is equal to y is:

$$\sum_{i=\frac{\gamma_2}{T}}^{\frac{\delta_2}{T}-1} \left[q_2 y \left(1 - r_2 + r_2 z\right)^{in+y} + q_2 \left(n-y\right) \left(1 - r_2 + r_2 z\right)^{(i+1)n+y} \right] =$$

$$= q_2 \left(1 - r_2 + r_2 z\right)^y \left[y + (n-y) \left(1 - r_2 + r_2 z\right)^n \right] \times$$

$$\times \frac{\left(1 - r_2 + r_2 z\right)^{\frac{\gamma_2}{T} n} - \left(1 - r_2 + r_2 z\right)^{\frac{\delta_2}{T} n}}{1 - \left(1 - r_2 + r_2 z\right)^n}.$$

The random variable of y has truncated geometric distribution with probabilities $\frac{(1-r_2)^{k-1} r_2}{1-(1-r_2)^n}$ $(k = 1, 2, \ldots, n)$. The previously calculated sum has to be multiplied by $\frac{(1-r_2)^{y-1} r_2}{1-(1-r_2)^n}$, and summed up for y, from 1 to n. Expanding this sum we finally receive (4).

□

Theorem 2. *The generating function of ergodic distribution of this chain is:*

$$P(z) = \sum_{i=0}^{\infty} p_i z^i = \frac{p_0 \big(zC(z) - B(z) \big) + p_1 z \big(A_2(z) - B(z) \big)}{z - B(z)}, \tag{10}$$

where p_0 and p_1 are the first two probabilities of the equilibrium distribution. They are connected with the relation $p_1 = \frac{1-c_0}{a_{20}} p_0$, and

$$p_0 = \frac{1 - B'(1)}{1 - B'(1) + C'(1) + \frac{1-c_0}{a_{20}} \big(A_2'(1) - B'(1) \big)}, \tag{11}$$

where

$$A_j'(1) = -a_{j0} + \frac{T}{\delta_j - \gamma_j} \left[\left(1 - r_2\right)^{\frac{\gamma_j}{T} n} - \left(1 - r_2\right)^{\frac{\delta_j}{T} n} \right] \frac{\left(1 - r_2\right)^n}{1 - \left(1 - r_2\right)^n} +$$

$$+ \frac{n r_2}{T} \frac{\gamma_j + \delta_j + T}{2},$$

$$A'_{12}(1) = 1 + \frac{nr_2}{T}\frac{\gamma_1 + \delta_1 + T}{2},$$

$$B'(1) = \frac{nr_2}{T}\frac{\gamma_2 + \delta_2 + T}{2}, \tag{12}$$

and $C'(1)$ *depends on collision policies:*

$$\text{I.}\quad C'(1) = \tfrac{r_1(1-r_2)}{r_1+r_2-r_1r_2}A'_1(1) + \tfrac{r_2(1-r_1)}{r_1+r_2-r_1r_2}A'_2(1)\,,$$

$$\text{II.}\quad C'(1) = \tfrac{r_1}{r_1+r_2-r_1r_2}A'_1(1) + \tfrac{r_2(1-r_1)}{r_1+r_2-r_1r_2}A'_2(1)\,,$$

$$\text{III.}\quad C'(1) = \tfrac{r_1(1-r_2)}{r_1+r_2-r_1r_2}A'_1(1) + \tfrac{r_2(1-r_1)}{r_1+r_2-r_1r_2}A'_2(1) + \tfrac{r_1r_2}{r_1+r_2-r_1r_2}A'_{12}(1)\,.$$

Proof. The matrix of transition probabilities has the form (1). Using this we can determine the probabilities of the equilibrium distribution denoted by p_l. They satisfy the equations

$$\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & \dots \\ a_{20} & a_{21} & a_{22} & a_{23} & \dots \\ 0 & b_0 & b_1 & b_2 & \dots \\ 0 & 0 & b_0 & b_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^T \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ \vdots \end{pmatrix},$$

i. e.

$$p_0 = p_0c_0 + p_1a_{20}, \tag{13a}$$

$$p_l = \sum_{k=2}^{l+1} p_k b_{l-k+1} + p_0c_l + p_1a_{2l} \qquad (l \geq 1)\,, \tag{13b}$$

from which we receive the following expression for the generating function:

$$\begin{aligned} P(z) = \sum_{l=0}^{\infty} p_l z^l &= p_0C(z) + p_1A_2(z) + \sum_{l=1}^{\infty}\sum_{k=2}^{l+1} p_k b_{l-k+1} z^l = \\ &= p_0C(z) + p_1A_2(z) + \sum_{k=2}^{\infty}\sum_{l=k-1}^{\infty} p_k b_{l-k+1} z^{l-k+1} z^{k-1} = \\ &= p_0C(z) + p_1A_2(z) + B(z)\left(\frac{P(z)}{z} - \frac{p_0}{z} - p_1\right), \end{aligned}$$

which yields (10). From (13a)

$$p_1 = \frac{1-c_0}{a_{20}}p_0.$$

To determine p_0, the condition $P(1) = 1$ is used, from which (11) is obtained. $\square$

Lemma 1. *The expression* $C'(1)+\frac{1-c_0}{a_{20}}\big(A_2'(1)-B'(1)\big)$ *is always positive, for any values of the parameters, and for any collision discipline.*

Proof.

$$\frac{1-c_0}{a_{20}}\big(A_2'(1)-B'(1)\big)=$$
$$=(1-c_0)\left[-1+\frac{T}{a_{20}\left(\delta_2-\gamma_2\right)}\left((1-r_2)^{\frac{\gamma_2}{T}n}-(1-r_2)^{\frac{\delta_2}{T}n}\right)\frac{(1-r_2)^n}{1-(1-r_2)^n}\right].$$

Substituting $a_{20}=\frac{T}{n(\delta_2-\gamma_2)}\frac{1-r_2}{r_2}\left((1-r_2)^{\frac{\gamma_2}{T}n}-(1-r_2)^{\frac{\delta_2}{T}n}\right)$ in the formula, we get:

$$\frac{1-c_0}{a_{20}}\big(A_2'(1)-B'(1)\big)=(1-c_0)\left(-1+\frac{nr_2}{1-r_2}\cdot\frac{(1-r_2)^n}{1-(1-r_2)^n}\right).$$

Next, $C'(1)$ is transformed in the following way:

$$C'(1)=\sum_{i=0}^{\infty}ic_i=\sum_{i=0}^{\infty}c_i-c_0+\sum_{i=2}^{\infty}(i-1)\,c_i=1-c_0+\sum_{i=2}^{\infty}(i-1)\,c_i.$$

Substituting all in, we obtain:

$$C'(1)+\frac{1-c_0}{a_{20}}\big(A_2'(1)-B'(1)\big)=\sum_{i=2}^{\infty}(i-1)\,c_i+(1-c_0)\,\frac{nr_2}{1-r_2}\frac{(1-r_2)^n}{1-(1-r_2)^n}.$$

From the formula rewritten in this form, it is obvious that

$$C'(1)+\frac{1-c_0}{a_{20}}\big(A_2'(1)-B'(1)\big)>0.$$

□

Theorem 3. *The condition of the existence of ergodic distribution is the fulfilment of the following inequality:*

$$\frac{nr_2}{T}\frac{\gamma_2+\delta_2+T}{2}<1. \tag{14}$$

Proof. As the embedded Markov-chain is irreducible and aperiodic, the condition of the existence of ergodic distribution is $0<p_0<1$. Applying Theorem 2 and Lemma 1, $p_0=\frac{1-B'(1)}{1-B'(1)+K}$, where K is a positive constant. Thus, the condition simplifies into

$$1-B'(1)>0,$$

which – together with (12) – gives (14). □

3 Conclusions

We investigated a special queuing system which serves two types of customers: customers of the first type are accepted for service only if the system is free; customers of the second type – if not serviced immediately – join a queue and can start their service after a multiple of the cycle-time has elapsed. Inter-arrival time distributions were geometric, service times were uniformly distributed; three collision treatment methods were considered.

By applying exact methods we gave formulae for transition probabilities and established the condition of ergodicity (14). One remarkable thing about (14) is that it does not depend on the customers of the first type, i. e. such customers have no effect on the ergodicity of the process. Moreover, the formula expressing the condition has a clear probabilistic interpretation. Considering that $\frac{T}{n}\frac{1}{r_2}$ is the average inter-arrival time, the condition rewritten in the form

$$\frac{\gamma_2 + \delta_2}{2} + \frac{T}{2} < \frac{T}{nr_2}$$

expresses the constraint that the sum of the average service and average idle times (on average $\frac{T}{2}$ time is needed for the next customer in the queue to reach the starting position) should be less than the average inter-arrival time.

Although the model was motivated by a real problem, it certainly is only a simplified version of it, which affects its applicability. For instance, it is assumed, without any statistical investigation of real data, that arriving entities form Poisson-processes. Presumably, even if they really do so, the Poisson-processes cannot be homogeneous, almost certainly. The FCFS rule is often broken in real life, too; normally the plane reaching the starting position first is to commence landing. Nevertheless, this simplified model provides exciting tasks to solve, and it can be modified later to fit the real case more precisely.

References

[1] Falin, G.I. and Templeton, J.G.C. *Retrial Queues.* Monographs on Statistics and Applied Probability **75** Chapman & Hall, 1997.

[2] Farkas, G. and Kárász, P. Investigation of a discrete cyclic-waiting problem by simulation. *Acta Acad. Paed. Agriensis, Sectio Mathematicae*, 27:57–62, 2000.

[3] Gnedenko, B.V. and Kovalenko, I.N. *Introduction to Queuing Theory.* Birkhäuser, Boston, 1989.

[4] Kárász, P. Special retrial systems with exponentially and uniformly distributed service times. In proc. of *International Conference in Memoriam John von Neumann*, pages 199–207, Budapest, 2003.

[5] Kárász, P. Special retrial systems with requests of two types. *Theory of Stochastic Processes*, 10(26)3-4:51–56, 2004.

[6] Kárász, P. and Farkas, G. Exact solution for a two-type customers retrial system. *Computers and Mathematics with Applications*, 49:95–102, 2005.

[7] Kárász, P. A special discrete cyclic-waiting queuing system. Submitted to *Central European Journal of Operations Research*, 2008.

[8] Laevens, K., Van Houdt, B., Blondia, C., and Bruneel, H. On the sustainable load of fiber delay line buffers. *Electronic Letters*, 40:137–138, 2004.

[9] Lakatos, L. On a simple continuous cyclic-waiting problem. *Annales Univ. Sci. Budapest., Sect. Comp.*, 14:105–113, 1994.

[10] Lakatos, L. On a cyclic-waiting queuing system. *Theory of Stochastic Processes*, 2(18)1-2:176–180, 1996.

[11] Lakatos, L. On a simple discrete cyclic-waiting queuing problem. *J. Math. Sci. (New York)*, 92(4):4031–4034, 1998.

[12] Lakatos, L. and Koltai, T. A discrete retrial system with uniformly distributed service time. *Annales Univ. Sci. Budapest., Sect. Comp.*, 22:225–234, 2003.

[13] Lakatos, L. and Zbăganu, G. Waiting time in cyclic-waiting systems. *Annales Univ. Sci. Budapest., Sect. Comp.*, 27:217–228, 2007.

[14] Rogiest, W., Laevens, K, Walraevens, J., and Bruneel, H. Analyzing a degenerate buffer with general inter-arrival and service times in discrete time. *Queueing Systems*, 56:203–212, 2007.