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M-Solid Varieties of Languages

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Abstract

In this paper, a characterization of the language varieties and congruence varieties corresponding to M-solid pseudovarieties is presented. Taking into account the isomorphisms of the Eilenberg-type correspondences, each complete sublattice of pseudovarieties corresponds to a complete sublattice of language varieties, as well as another one of congruence varieties. For the varieties of tree language, we present the complete sublattices of varieties of languages and the complete sublattice of varieties of congruences isomorphic to the complete sublattice of all M-solid pseudovarieties.

Keywords: tree languages, Eilenberg-type correspondences, M-solid pseudovarieties, M-solid varieties of languages

1 Introduction

Motivated by the connection between star-free languages and aperiodic monoids, and other important similar results, Eilenberg [6] establishes an isomorphism between the lattice of all monoid pseudovarieties and the lattice of all varieties of regular languages. At the beginning of the eighties, Thérien [14] proved that these two lattices are also isomorphic to the lattice of all varieties of congruences of the free monoids. These connections were independently extended to tree languages by Almeida [1] and Steinby [12]. Due to the original result achieved by Eilenberg these kind of connections have come to be known as Eilenberg-type correspondences. Some of the complete sublattices of the complete lattice $\mathcal{L}^{ps}(\tau)$ of all pseudovarieties of type τ were described by Denecke and Pibaljommee in [4]. They showed that for each monoid M of hypersubstitutions, the set $\mathcal{S}_M^{ps}(\tau)$ of all M-solid pseudovarieties of type τ is a complete sublattice of $\mathcal{L}^{ps}(\tau)$. So, it is a natural problem to find a characterization of the complete sublattices corresponding to $\mathcal{S}_M^{ps}(\tau)$, under the Eilenberg-type correspondences. This work is based on the final remarks of Ésik's in [7], where he points out a more wide framework to characterize varieties of tree languages. Following Ésik suggestions we show how monoids of hypersubstitutions and solid pseudovarieties can be used in the characterization of varieties of languages.

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We assume the reader is familiar with the basic notions and results of Universal Algebra [2]. Throughout this article we fix an algebraic type τ consisting of finitary operations. For technical reasons we will consider a type of algebras without nullary operations. Let $\{f_i : i \in I\}$ be a set of operational symbols of type τ, were f_i is an operational symbol of arity $n_i \geqslant 1$. We will denote by $Alg_f(\tau)$ the class of all finite algebras of type τ. Let $X_\omega = \{x_1, \ldots, x_n, \ldots\}$ be a countable infinite set of variables disjoint from the set of operational symbols, and $X_n = \{x_1, \ldots, x_n\}$ be the set of the first n variables. We will use X to represent any of the previous sets of variables. The set of all n-ary terms of type τ, or terms of type τ over X_n, is denoted by $T_\tau(X_n)$, and by $T_\tau(X_\omega) = \cup_{n\geqslant 1} T_\tau(X_n)$ we denote the set of all terms of type τ. For any term $t \in T_\tau(X)$ we denote by $hg(t)$ the height of the term t. A pseudovariety V of type τ is a class of finite algebras of type τ closed under formation of homomorphic images, subalgebras and finitary direct products. It is well-known that pseudovarieties are defined by filters of equations [2], and that the set $\mathcal{L}^{ps}(\tau)$ of all pseudovarieties of type τ forms a complete lattice. A pseudovariety defined by equations is called an equational pseudovariety. In the sequel, we will consider a non-trivial pseudovariety V of type τ. Let $\mathcal{L}^{ps}(V)$ denote the complete lattice of all subpseudovarieties of V. Given two algebras $\mathbf{A}$ and $\mathbf{B}$, we say that $\mathbf{A}$ *divides* $\mathbf{B}$, and we write $\mathbf{A} \preceq \mathbf{B}$, if $\mathbf{A}$ is a homomorphic image of a subalgebra of $\mathbf{B}$. By $Pol_n\mathbf{A}$ we denote the set of all n-ary polynomial operations of the algebra $\mathbf{A}$. In the next two sections we give the necessary definitions and results which will be used to achieve the main results.

2 Eilenberg-type correspondences

The Eilenberg and Thérien results were generalized by Almeida and Steinby using a more general framework that included both cases: varieties of string languages and varieties of tree languages. They considered sets of the finitely generated V-free algebras $\mathbf{F}_nV$ [1]. When V is the pseudovariety of all monoids, the subsets of $\mathbf{F}_nV$ are string languages, and when V is the pseudovariety of all finite algebras of type τ we have the tree languages case.

Let $\mathbf{A}$ be an algebra and $L \subseteq A$ any subset of A. The *syntactic congruence* of L on $\mathbf{A}$ is the relation $\sim_L$ given by

$$a \sim_L b \quad \text{iff} \quad p(a) \in L \Leftrightarrow p(b) \in L,$$

to every unary polynomial operation $p \in Pol_1(\mathbf{A})$, with $a, b \in A$.

The relation $\sim_L$ is the greatest congruence of $\mathbf{A}$ for which L is the union of classes.

The *syntactic algebra* $\mathbf{A}/L$ of the subset L of A is the quotient algebra $\mathbf{A}/\sim_L$, and the homomorphism $\varphi_L : \mathbf{A} \to \mathbf{A}/L$ is called the *syntactic homomorphism* of L. We say that an algebra is syntactic if it is isomorphic to the syntactic algebra of some subset of some algebra.

[1]We follow the somewhat nonstandard definition of V-free algebras from [1] which does not require that a V-free algebra be itself in V.

An operation on subsets of an algebra $\mathbf{A}$ of the form $L \mapsto p^{-1}L := \{a \in A : p(a) \in L\}$, where $p \in Pol_1\mathbf{A}$ is an unary polynomial operation of $\mathbf{A}$, and $L \subseteq A$ is a subset of A, will be called *cancellation*. Another operation on subsets is $L \mapsto \varphi^{-1}L := \varphi^{-1}(L)$ where $\varphi : \mathbf{A} \to \mathbf{B}$ is a homomorphism, and will be referred to as *inverse homomorphism*.

Let $\mathbf{A}$ be an algebra of type τ. A subset $L \subseteq A$ of A is called *V-recognizable* if there exists an algebra $\mathbf{B} \in V$, a homomorphism $\varphi : \mathbf{A} \to \mathbf{B}$, and a subset $K \subseteq B$ such that $L = \varphi^{-1}(K)$. In this case, we say that the triple $\langle \mathbf{B}, \varphi, K\rangle$ recognizes L, or simply that L is recognized by $\mathbf{A}$.

We are only interested in the V-recognizable subsets of the finitely generated V-free algebras $\mathbf{F}_nV$. For any $n \geqslant 1$, let Rec_nV denote the set of all V-recognizable subsets of $\mathbf{F}_nV$. We will refer to the elements of Rec_nV as *V-languages*. By a field of subsets we mean a Boolean subalgebra of the power set, with the usual set operations.

A *variety of V-languages* is a sequence $\mathscr{V} = (\mathscr{V}_n)_{n\geqslant 1}$ such that

L.1) $\mathscr{V}_n$ is a field of subsets of Rec_n;

L.2) $\mathscr{V}_n$ is closed under cancellation;

L.3) $\mathscr{V}$ is closed under inverse homomorphisms $\mathbf{F}_nV \to \mathbf{F}_mV$.

The lattice of all varieties of V-languages is represented by $\mathcal{VL}(V)$.

We represent by Con_nV the set of all congruences θ on $\mathbf{F}_nV$ such that $\mathbf{F}_nV/\theta \in V$ which we will call V-congruences. Let $\varphi : \mathbf{A} \to \mathbf{B}$ be a homomorphism and θ a congruence on $\mathbf{B}$. Then we have the homomorphism $\varphi\times\varphi : \mathbf{A}\times\mathbf{A} \to \mathbf{B}\times\mathbf{B}$ defined by $(\varphi\times\varphi)(a,b) = (\varphi(a),\varphi(b))$, for all $(a,b) \in A\times A$. Because θ is a subalgebra of $\mathbf{B}\times\mathbf{B}$, then $(\varphi\times\varphi)^{-1}\theta$ is a subalgebra of $\mathbf{A}\times\mathbf{A}$, and it is easy to prove that it is a congruence on $\mathbf{A}$.

A *variety of V-congruence filters* is a sequence $\Gamma = (\Gamma_n)_{n\geqslant 1}$ such that

C.1) Γ_n is a filter of V-congruences on $Con\mathbf{F}_nV$ contained in Con_nV;

C.2) If $\varphi : \mathbf{F}_mV \to \mathbf{F}_nV$ is a homomorphism and $\theta \in \Gamma_n$, then $(\varphi\times\varphi)^{-1}\theta \in \Gamma_m$.

To simplify the terminology, we will call a variety of V-congruence filters just a variety of V-congruences, and will represent the lattice of all varieties of V-congruence filters by $\mathcal{VC}(V)$.

Proposition 1. [1] *The correspondences*

$$(\ )^{\ell} : \mathcal{L}^{ps}(V) \to \mathcal{VL}(V) \qquad W \mapsto W^{\ell} = (\{L \subseteq F_nV : \mathbf{F}_nV/L \in W\}_n)_{n\geqslant 1}$$

and

$$(\ )^{a} : \mathcal{VL}(V) \to \mathcal{L}^{ps}(V) \qquad \mathscr{V} \mapsto \mathscr{V}^{a} = V_f\{\mathbf{A}/L \in V : L \in \mathscr{V}_n, n \geqslant 1\}$$

are mutually inverse lattice isomorphisms.

Proposition 2. [1] *The correspondences*

$$(\ )^c : \mathcal{L}^{ps}(V) \to \mathcal{VC}(V) \qquad W \mapsto W^c = (\{\theta \in Con_n V : \mathbf{F}_n V/\theta \in W\}_n)_{n\geqslant 1}$$

and

$$(\ )^a : \mathcal{VC}(V) \to \mathcal{L}^{ps}(V) \qquad \Gamma \mapsto \Gamma^a = V_f\{\mathbf{F}_n V/\theta : \theta \in \Gamma_n, n \geqslant 1\}$$

are mutually inverse lattice isomorphisms.

Where $V_f\{K\}$ denotes the pseudovariety generated by the class of finite algebras K. From the above two isomorphisms we get the next result.

Theorem 1. [1] *The lattices $\mathcal{L}^{ps}(V)$, $\mathcal{VL}(V)$ and $\mathcal{VC}(V)$ are all complete and isomorphic to each other.*

3 Hypersubstitutions and M-solid pseudovarieties

A mapping $\sigma : \{f_i : i \in I\} \to T_\tau(X_\omega)$, which assigns to every n_i-ary operation symbol f_i an n_i-ary term $\sigma(f_i)$, will be called a *hypersubstitution* of type τ. If σ is a hypersubstitution, then we can think of σ as mapping each term of the form $f_i(x_1, \ldots, x_{n_i})$ to the n_i-ary term $\sigma(f_i)$. This means that any hypersubstitution σ induces a unique map $\hat{\sigma}$ on the set $T_\tau(X)$ of all terms of type τ over X, as follows:

(1) $\hat{\sigma}[x] := x$, for all $x \in X$;

(2) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] = \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for the term $f_i(t_1, \ldots, t_{n_i})$.

We denote by $Hyp(\tau)$ the set of all hypersubstitutions of type τ. We can define a composition operation $\circ_h$ on hypersubstitutions by $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$, for $\sigma_1, \sigma_2 \in Hyp(\tau)$. Considering the identity hypersubstitution σ_{id}, the set of all hypersubstitutions form a monoid $\mathbf{Hyp}(\tau) = \langle Hyp(\tau); \circ_h, \sigma_{id}\rangle$. In the sequel, let $M \subseteq Hyp(\tau)$ be a submonoid of hypersubstitutions.

Hypersubstitutions can be applied to an equation $t \approx s \in Eq(\tau)$ to produce a new equation $\hat{\sigma}[t] \approx \hat{\sigma}[s]$. From an algebra $\mathbf{A} = \langle A, (f_i)_{i\in I}\rangle$ of type τ and a hypersubstitution $\sigma \in M$ it is possible to construct a new algebra $\sigma[\mathbf{A}] = \langle A; (\sigma(f_i)^{\mathbf{A}})_{i\in I}\rangle$ called the *M-derived algebra* of $\mathbf{A}$ by σ.

It is easy to see that, if a map φ is a homomorphism $\varphi : \mathbf{A} \to \mathbf{B}$, then it is also a homomorphism $\varphi : \sigma[\mathbf{A}] \to \sigma[\mathbf{B}]$, for all $\sigma \in Hyp(\tau)$. Related to congruences, if $\theta \in Con\mathbf{A}$ is a congruence on $\mathbf{A}$, then θ is also a congruence on $\sigma[\mathbf{A}]$, and $\sigma[\mathbf{A}/\theta] = \sigma[\mathbf{A}]/\theta$.

Definition 1. *A pseudovariety V of algebras of type τ is called M-*solid *if it is closed under M-derived algebras.*

Clearly, the pseudovarieties $Alg_f(\tau)$ of all finite algebras and $I(\tau)$ of all trivial algebras are, respectively, the greatest and smallest M-solid pseudovarieties of type τ. We will represent the set of all M-solid pseudovarieties of type τ by $\mathcal{S}^M_{ps}(\tau)$.

Theorem 2. [4] *For every monoid $M \subseteq Hyp(\tau)$ of hypersubstitutions the set $\mathcal{S}^M_{ps}(\tau)$ is a complete sublattice of the lattice $\mathcal{L}_{ps}(\tau)$ of all pseudovarieties of type τ.*

At this point, using Eilenberg-type correspondences we just know that the complete sublattice $\mathcal{S}^{ps}_M(V) := \mathcal{S}^M_{ps} \cap \mathcal{L}_{ps}(V)$ of all M-solid subpseudovarieties of V corresponds to a complete sublattice of $\mathcal{VL}(V)$ and another complete sublattice of $\mathcal{VC}(V)$. In the next section we give a characterization of this sublattices.

4 M-solid varieties of languages and congruences

We start with the definition of a special kind of hypersubstitution introduced by Płonka in [10].

Definition 2. *Let K be a class of algebras of type τ. A hypersubstitution $\sigma \in Hyp(\tau)$ is called* K-proper *if for all $t \approx s \in Id(K)$ we have $\hat{\sigma}[t] \approx \hat{\sigma}[s] \in Id(K)$. Let $P(K)$ be the set of all K-proper hypersubstitutions.*

We have that, for any class K of algebras, $\mathbf{P}(K) = \langle P(K); \circ_h, \sigma_{id}\rangle$ is a submonoid of $\mathbf{Hyp}(\tau)$. When K is an equational pseudovariety, $P(K)$ is the greatest submonoid of hypersubstitutions such that K is $P(K)$-solid.

The notion of semi-weak homomorphism of an algebra is introduced by Kolibiar in [9].

Definition 3. *Let $\mathbf{A}$ and $\mathbf{B}$ be algebras of type τ. A mapping $h : A \to B$ is called a* semi-weak homomorphism *if there exists a hypersubstitution $\sigma \in Hyp(\tau)$ such that h is a homomorphism of $\mathbf{A}$ into $\sigma[\mathbf{B}]$. In this case, we write $h : \mathbf{A} \xrightarrow{sw} \mathbf{B}$. A semi-weak homomorphism $h : \mathbf{A} \xrightarrow{sw} \mathbf{A}$ is called a* semi-weak endomorphism *of $\mathbf{A}$. We say that $h : \mathbf{A} \to \sigma[\mathbf{B}]$ is an M-semi-weak homomorphism, if $\sigma \in M$.*

Clearly, the extension of a hypersubstitution $\hat{\sigma}$ is a semi-weak endomorphism of $\mathbf{T}_\tau(X)$, and also, any usual homomorphism is a semi-weak homomorphism. We have the following fact.

Proposition 3. *Let $\sigma \in P(V)$ be a V-proper hypersubstitution. Then $\hat{\sigma} : \mathbf{F}_n V \to \mathbf{F}_n V$ is a semi-weak endomorphism, for any $n \geqslant 1$.*

Proof. Let $\mathbf{T}_\tau(X_n)$ be the algebra of n-ary terms of type τ. We have the homomorphism $\hat{\sigma} : \mathbf{T}_\tau(X_n) \to \sigma[\mathbf{T}_\tau(X_n)]$. The V-free algebra $\mathbf{F}_n V$ is given by $\mathbf{F}_n V \cong \mathbf{T}_\tau(X_n)/\theta_V(X_n)$, where the congruence $\theta_V(X_n) := \{(t,s) \in T_\tau(X_n) \times T_\tau(X_n) : t \approx s \in Id(V)\}$ is given by all the equations over X_n satisfied by V. We have that $\theta_V(X_n)$ is also a congruence on $\sigma[\mathbf{T}_\tau(X_n)]$ and that $\sigma[\mathbf{T}_\tau(X_n)]/\theta_V(X_n) = \sigma[\mathbf{T}_\tau(X_n)/\theta_V(X_n)]$. Since the equations satisfied by V are preserved by V-proper hypersubstitutions, we have the homomorphism $\hat{\sigma} : \mathbf{F}_n V \to \sigma[\mathbf{F}_n V]$. Hence, $\hat{\sigma} : \mathbf{F}_n V \xrightarrow{sw} \mathbf{F}_n V$ is an $P(V)$-semi-weak endomorphism. □

Now we give the definition of an M-solid variety of languages.

Definition 4. *Let $\mathscr{V} = (\mathscr{V}_n)_{n\geqslant 1}$ be a variety of V-languages. The variety $\mathscr{V}$ is called an M-*solid variety of V-languages *if for all $n, m \geqslant 1$, and all M-semi-weak homomorphism $h : \mathbf{F}_m V \to \mathbf{F}_n V$ and all $L \in \mathscr{V}_n$ we have $h^{-1}(L) \in \mathscr{V}_m$.*

It is easy to see that the trivial variety of V-languages $LtriV = (\{\emptyset, F_n V\})_n$ is M-solid, for any M. Using the Eilenberg-type correspondences between pseudovarieties and varieties of languages we have the following result.

Proposition 4. *Let W be an M-solid subpseudovariety of V. Then, W^ℓ is an M-solid variety of V-languages.*

Proof. Let W be an M-solid subpseudovariety of V. From Proposition 1 we have that W^ℓ is a variety of V-languages. For $n, m \geqslant 1$, let $h : \mathbf{F}_m V \to \mathbf{F}_n V$ be an M-semi-weak homomorphism, and $L \in W^\ell_n$. We want to show that $h^{-1}(L) \in W^\ell_m$. Since $L \in W^\ell_n$, then $\mathbf{F}_n V/L \in W$. So, L is W-recognizable, and there exists a finite algebra $\mathbf{A} \in W$, a homomorphism $\varphi : \mathbf{F}_n V \to \mathbf{A}$, and a set $K \subseteq A$ such that $\varphi^{-1}(K) = L$. We have the homomorphism $h : \mathbf{F}_m V \to \sigma[\mathbf{F}_n V]$ and as well the homomorphism $\varphi : \sigma[\mathbf{F}_n V] \to \sigma[\mathbf{A}]$, for a hypersubstitution $\sigma \in M$. Then, $h^{-1}(L) = h^{-1}(\varphi^{-1}(K)) = (\varphi \circ h)^{-1}(K)$. So, $h^{-1}(L)$ is recognized by $\langle \sigma[\mathbf{A}], \varphi \circ h, K\rangle$. As a consequence $\mathbf{F}_m V/h^{-1}(L) \preceq \sigma[\mathbf{A}]$. Since W is M-solid, we have that $\sigma[\mathbf{A}] \in W$ and then $\mathbf{F}_m V/h^{-1}(L) \in W$. Hence $h^{-1}(L) \in W^\ell_m$. This proves that W^ℓ is an M-solid variety of V-languages. □

From the last proposition we conclude the following results.

Corollary 1. *Let $L \in Rec_n V$ be a V-recognizable language and $h : \mathbf{F}_m V \to \mathbf{F}_n V$ an M-semi-weak homomorphism. Then $h^{-1}(L)$ is recognized by $\sigma[\mathbf{A}]$ for some hypersubstitution $\sigma \in M$.*

Corollary 2. *Let $\mathscr{V}$ be a variety of V-languages. If the pseudovariety $\mathscr{V}^a$ is M-solid then $\mathscr{V}$ is an M-solid variety of V-languages.*

As claimed in [12] every subdirectly irreducible algebra is a syntactic algebra. Hence, each finite algebra of V can be represented as a subdirect product of a finite number of syntactic algebras of some subsets, that are V-recognized by the algebra. So, as remarked in [11], for any variety $\mathscr{V}$ of V-languages and any finite algebra $\mathbf{A}$, if for all n, all V-languages in $T_\tau(X_n)$ recognized by $\mathbf{A}$ are in $\mathscr{V}_n$, then $\mathbf{A} \in \mathscr{V}^a$.

Proposition 5. *Let V be an equational pseudovariety. If $\mathscr{V}$ is an M-solid variety of V-languages, then the pseudovariety $\mathscr{V}^a$ is $M \bigcap P(V)$-solid.*

Proof. Let $\mathbf{A} \in \mathscr{V}^a$ be any finite algebra and $\sigma \in M \bigcap P(V)$ a hypersubstitution. We will prove that $\sigma[\mathbf{A}] \in \mathscr{V}^a$. From the previous remark, we have only to prove that all V-languages recognized by $\sigma[\mathbf{A}]$ are in $\mathscr{V}$. For any $n \geqslant 1$, let $L \in Rec_n V$ be a V-language recognized by $\sigma[\mathbf{A}]$. Hence, there exists a homomorphism $\varphi : \mathbf{F}_n V \to \sigma[\mathbf{A}]$ and a subset $K \subseteq A$ such that $\varphi^{-1}(K) = L$. Since $\sigma \in P(V)$, then $\sigma[\mathbf{A}] \in V$ and $\hat{\sigma} : \mathbf{F}_n V \to \mathbf{F}_n V$ is an M-semi-weak endomorphism of $\mathbf{F}_n V$.

Let $\psi : \mathbf{F}_n V \to \mathbf{A}$ be the unique homomorphism such that $\psi \circ \hat{\sigma} = \varphi$. Thus, $L = \varphi^{-1}(K) = (\psi \circ \hat{\sigma})^{-1}(K) =$
$= \hat{\sigma}^{-1}(\psi^{-1}(K))$. Since $\mathbf{A}$ recognizes the language $L' = \psi^{-1}(K)$, then $\mathbf{F}_n V/L' \preceq \mathbf{A}$, and as consequence $L' \in \mathscr{V}_n$. Since $\mathscr{V}$ is M-solid, this implies that $L = \hat{\sigma}^{-1}(L') \in \mathscr{V}_n$. We have proved that all V-languages recognized by $\sigma[\mathbf{A}]$ are in $\mathscr{V}$, and so $\sigma[\mathbf{A}] \in \mathscr{V}^a$. Hence, $\mathscr{V}^a$ is an $M \bigcap P(V)$-solid pseudovariety. □

The condition imposed on V, that V be an equational pseudovariety, isn't very restrictive because in both cases, string languages and tree languages, we are dealing with equational pseudovarieties.

The analogous definition of M-solidity for varieties of congruences is presented.

Definition 5. *Let $\Gamma = (\Gamma_n)_{n \geqslant 1}$ be a variety of V-congruences. The variety Γ is said to be M-solid if for all $n, m \geqslant 1$, M-semi-weak homomorphism $h : \mathbf{F}_m V \to \mathbf{F}_n V$ and $\theta \in \Gamma_n$, then $(h \times h)^{-1}\theta \in \Gamma_m$.*

Clearly, the trivial variety of congruences $CtriV = (\{\nabla_{F_n V}\})_n$ is M-solid.

We need these technical results to prove the next proposition.

Lemma 1. [2] *Let $\varphi : \mathbf{A} \to \mathbf{B}$ be a homomorphism, $L \subseteq B$ a subset and $\theta \in Con\mathbf{A}$ a congruence.*

i) If $\sim_L$ has a finite number of classes, then there is only a finite number of subsets that may be obtained from L by cancellation;

ii) $(\varphi \times \varphi)^{-1} \sim_L = \bigcap_{L'} \{\sim_{\varphi^{-1}L'}$: L' is obtained from L by cancellation$\}$;

iii) $\theta = \bigcap_L \{\sim_L$: L is a class of $\theta\}$.

Lemma 2. [12] *Let $\mathscr{V}$ be a variety of V-languages. For any $n \geqslant 1$ and $L \in \mathscr{V}_n$ all the $\sim_L$-classes are in $\mathscr{V}_n$.*

To connect varieties of languages and varieties of congruences we have this interesting result.

Proposition 6. *Let $\mathscr{V}$ be a variety of V-languages. Then, $\mathscr{V}$ is M-solid iff $\mathscr{V}^c$ is M-solid.*

Proof. ($\Rightarrow$) For $n, m \geqslant 1$, let $h : \mathbf{F}_m V \xrightarrow{sw} \mathbf{F}_n V$ be an M-semi-weak homomorphism and $\theta \in \mathscr{V}_n^c$. Using Lemma 1 it follows that

$$(h \times h)^{-1}\theta = \bigcap_L \bigcap_p \{\sim_{h^{-1}(p^{-1}(L))} \colon p \in Pol_1(\mathbf{F}_n V)\ L \text{ is a } \theta\text{-class}\}$$

is a finitary intersection. From Lemma 2 we conclude that all classes of θ are in $\mathscr{V}_n$. Since $\mathscr{V}$ is a variety of V-languages we have $p^{-1}(L) \in \mathscr{V}_n$ for all $p \in Pol_1(\mathbf{F}_n V)$. Moreover, $\mathscr{V}$ is M-solid so $h^{-1}(p^{-1}(L)) \in \mathscr{V}_m$ for all $p \in Pol_1(\mathbf{F}_n V)$ and for all θ-classes L. Hence, $(h \times h)^{-1}\theta \in \mathscr{V}_m^c$, which proves that $\mathscr{V}^c$ is M-solid.

($\Leftarrow$) Suppose $\mathscr{V}$ is a variety of V-languages such that $\mathscr{V}^c$ is an M-solid variety of V-congruences. For $n, m \geqslant 1$, let $L \in \mathscr{V}_n$ and $h : \mathbf{F}_m V \to \mathbf{F}_n V$ be an M-semi-weak homomorphism. We have the homomorphism $h : \mathbf{F}_m V \to \sigma[\mathbf{F}_n V]$ for some hypersubstitution $\sigma \in M$. It is easy to prove that $(h \times h)^{-1} \sim_L \subseteq \sim_{h^{-1}(L)}$. Since, $\sim_L \in \mathscr{V}_n^c$ and $\mathscr{V}^c$ is M-solid, then $(h \times h)^{-1} \sim_L \in \mathscr{V}_m^c$. Now, because $\mathscr{V}_m^c$ is a filter of congruences, we conclude that $\sim_{h^{-1}(L)} \in \mathscr{V}_m^c$. Hence, $h^{-1}(L) \in \mathscr{V}_m^{c\ell} = \mathscr{V}_m$. So we have proved that $\mathscr{V}$ is M-solid. □

Corollary 3. *A variety Γ of V-congruence filters is M-solid iff Γ^ℓ is an M-solid variety of V-languages.*

When V is the pseudovariety of all finite algebras of type τ, the V-languages are precisely the recognizable tree languages of type τ. In this case we can prove the next result.

Theorem 3. *Let $V = Alg_f(\tau)$ be the pseudovariety of all finite algebras of type τ. If W is any pseudovariety of type τ the following conditions are equivalent:*

i) W is an M-solid pseudovariety;

ii) W^ℓ is an M-solid variety of tree languages;

iii) W^c is an M-solid variety of congruences.

Proof. This proof is straightforward. We only need to use the previous results and the fact that $Hyp(\tau)$ is the set of all $Alg_f(\tau)$-proper hypersubstitutions. □

The next result shows how to obtain complete sublattices of the lattice of all varieties of tree languages $\mathcal{VL}(\tau)$ and the lattice of all varieties of congruences $\mathcal{VC}(\tau)$. Let $\mathcal{VL}_M(\tau)$ denote the set of all M-solid varieties of tree languages and by $\mathcal{VC}_M(\tau)$ we will represent the set of all M-solid varieties of congruences.

Corollary 4. *The sets $\mathcal{VL}_M(\tau)$ and $\mathcal{VC}_M(\tau)$ are complete sublattices of the complete lattices $\mathcal{VL}(\tau)$ and $\mathcal{VC}(\tau)$, respectively, and both are isomorphic to the complete lattice $\mathcal{S}_M^{ps}(\tau)$ of all M-solid pseudovarieties of type τ.*

We know that any semi-weak homomorphism $h : \mathbf{T}_\tau(X_m) \to \mathbf{T}_\tau(X_n)$ is a homomorphism $h : \mathbf{T}_\tau(X_m) \to \sigma[\mathbf{T}_\tau(X_n)]$ for some hypersubstitution $\sigma \in Hyp(\tau)$. Hence, there exists an endomorphism $\varphi : \mathbf{T}_\tau(X_n) \to \mathbf{T}_\tau(X_n)$ such that $h = \varphi \circ \hat{\sigma}$. Using this fact, in the case of tree languages, M-solid varieties of tree languages and M-solid varieties of congruences have an alternative and more simple characterization given by hypersubstitutions.

Proposition 7. *A variety $\mathscr{V}$ of tree languages is M-solid iff it is closed under inverse hypersubstitution with respect to M, i.e. iff for all $n \geqslant 1$, and any hypersubstitution $\sigma \in M$ and any $L \in \mathscr{V}_n$, we have $\hat{\sigma}^{-1}(L) \in \mathscr{V}_n$.*

A similar result holds for varieties of congruences.

Proposition 8. *Let $\Gamma \in \mathcal{VC}(\tau)$ be a variety of congruences. Then, Γ is M-solid iff for all $n \geqslant 1$, all $\theta \in \Gamma_n$ and all $\sigma \in M$, we have $(\hat{\sigma} \times \hat{\sigma})^{-1}\theta \in \Gamma_n$.*

An important notion in the theory of tree language is the tree homomorphism [13]. Semi-weak homomorphisms and the extensions of hypersubstitutions are particular cases of tree homomorphisms, and each monoid of hypersubstitutions can define a special set of tree homomorphisms. So, the M-solid varieties of tree languages can be characterized using tree homomorphisms.

5 Examples

Now we will see two well known cases of tree languages that are M-solid, for some monoids $M \subseteq Hyp(\tau)$ of hypersubstitutions.

5.1 Nilpotent algebras

An algebra $\mathbf{A}$ of type τ is called *nilpotent*, if there exists an absorbing element a_0 and an integer $k \geq 1$ such that $t^{\mathbf{A}}(a_1, \ldots, a_n) = a_0$ for all terms t in n variables with $hg(t) \geq k$ and for all n-tuples of elements $(a_1, \ldots, a_n)$, where hg is the usual height function on terms. The smallest k for which this holds is called the degree of nilpotency of $\mathbf{A}$. Let $Nil(\tau)$ be the class of all nilpotent algebras of type τ.

For each $n \geqslant 1$, let $\mathcal{N}il_n(\tau)$ consist of all finite and all cofinite tree languages in $T_\tau(X_n)$. For string languages it is well-know that finite and cofinite language are recognizable, and that they form a variety of languages. In the case of tree languages we have the following result.

Proposition 9. [12] *The sequence $\mathcal{N}il(\tau) = (\mathcal{N}il_n(\tau))_{n \geqslant 1}$ is a variety of tree languages and $\mathcal{N}il(\tau)^a = Nil(\tau)$.*

A hypersubstitution σ is called regular if for each f_i, all the variables $x_1, \ldots, x_{n_i}$ occur in $\sigma(f_i)$. The set $Reg(\tau)$ of all regular hypersubstitutions is a submonoid of $Hyp(\tau)$, and applying induction over a term t we prove that if σ is a regular hypersubstitution then $hg(\hat{\sigma}[t]) \geqslant hg(t)$.

Using this lemma we can easily prove the next result about the pseudovariety $Nil(\tau)$.

Proposition 10. *The pseudovariety $Nil(\tau)$ is $Reg(\tau)$-solid.*

Proof. By proposition 9 we know already that $Nil(\tau)$ is a pseudovariety. So, we only have to show that it is closed under $Reg(\tau)$-derived algebras. Let $\mathbf{A} \in Nil(\tau)$ be a nilpotent algebra of degree k and $\sigma \in Reg(\tau)$ a regular hypersubstitution. For any $t \in T_\tau(X_\omega)$ and by induction on the height of t it is easy to prove that $t^{\sigma[\mathbf{A}]} = \hat{\sigma}[t]^{\mathbf{A}}$. Using this fact, for any $t \in T_\tau(X_\omega)$ of $hg(t) \geqslant k$ then $hg(\hat{\sigma}[t]) \geqslant hg(t) \geqslant k$, and we have $t^{\sigma[\mathbf{A}]} = \hat{\sigma}[t]^{\mathbf{A}} = a_0$. Hence, $\sigma[\mathbf{A}]$ is a nilpotent algebra of degree k, and thus $\sigma[\mathbf{A}] \in Nil(\tau)$. □

By proposition 3 we can conclude the following result.

Corollary 5. *The variety of tree languages $\mathcal{N}il(\tau)$ is $Reg(\tau)$-solid.*

5.2 Definite tree languages

A string language L is called k-definite, if a word of length greater then k is in L iff it suffix of length k is in L. The extension to tree languages of this notion is made using roots. For any term t, $root(t) = t$ if t is a variable, and $root(t) = f_i$ if $t = f_i(t_1, \ldots, t_{n_i})$, for some $i \in I$.

The k-root $R_k(t)$ of a term $t \in T_\tau(X_\omega)$ is defined as follows:

i) $R_0(t) = \varepsilon$, where ε is a special symbol which represents an empty root and $R_1(t) = root(t)$;

ii) let $k \geqslant 2$, if $hg(t) < k$ and $t = f_i(t_1, \ldots, t_{n_i})$ for some $i \in I$, then $R_k(t) = t$. If $hg(t) \geqslant k$ then $R_k(t) = f_i(R_{k-1}(t_1), \ldots, R_{k-1}(t_{n_i}))$.

Let $k \geqslant 0$ and $L \subseteq T_\tau(X_n)$. The language L is called k-definite, if for all $t, s \in T_\tau(X_n)$ such that $R_k(t) = R_k(s)$, $t \in L$ iff $s \in L$. For each $n \geqslant 0$, we define the relations $\sim_{k,n}$ in $T_\tau(X_n)$ as follows:

$$t \sim_{k,n} s \quad \text{iff} \;\; R_k(t) = R_k(s),$$

for all $t, s \in T_\tau(X_n)$. We will simply denote $t \sim_k s$, if $R_k(t) = R_k(s)$.

Let $\mathcal{D}^k(\tau) = (\mathcal{D}^k_n(\tau))_{n\geqslant 1}$ be a sequence, where $\mathcal{D}^k_n(\tau)$ is the set of all k-definite tree languages of $T_\tau(X_n)$. The sequence of all definite tree languages is $\mathcal{D}(\tau) = (\mathcal{D}_n(\tau))_{n\geqslant 1}$, where $\mathcal{D}_n(\tau) = \bigcup\{\mathcal{D}_{k,n}(\tau) : k \geqslant 0\}$ is the set of all definite tree languages of $T_\tau(X_n)$. Clearly, we can conclude the inclusions

$$\mathcal{D}^0(\tau) \subseteq \mathcal{D}^1(\tau) \subseteq \cdots \subseteq \mathcal{D}^k(\tau) \subseteq \cdots \subseteq \mathcal{D}(\tau).$$

Thus, we have the following result.

Proposition 11. [12] *For all $n \geqslant 1$ and $k \geqslant 0$ the relation $\sim_{k,n}$ is a congruence on the algebra $\mathbf{T}_\tau(X_n)$, and it has a finite number of classes.*

Let $\Gamma_{k,n} = [\sim_{k,n})$ be the principal filter generated by $\sim_{k,n}$ on the lattice $FCon\mathbf{T}_\tau(X_n)$ of all finite index congruence relations of $\mathbf{T}_\tau(X_n)$.

Proposition 12. [12] *For each $k \geqslant 0$ the sequence $\Gamma_k = (\Gamma_{k,n})_{n\geqslant 1}$ is a variety of congruences. Moreover, $\Gamma^\ell_k = \mathcal{D}^k(\tau)$.*

From this last proposition we conclude that for any $k \geqslant 0$ the sequences $\mathcal{D}^k(\tau)$ and $\mathcal{D}(\tau)$ are varieties of tree languages.

A hypersubstitution σ is called a *pre-hypersubstitution* if for every $i \in I$ the term $\sigma(f_i)$ is not a variable. The set of all pre-hypersubstitutions $Pre(\tau)$ is a submonoid of $Hyp(\tau)$. The next result shows an important behavior of pre-hypersubstitutions related to preservations of k-roots.

Lemma 3. *Let $t, s \in T_\tau(X)$ and $\sigma \in Pre(\tau)$. If $t \sim_k s$ then $\hat{\sigma}[t] \sim_k \hat{\sigma}[s]$, for all $k \geqslant 0$.*

Proof. This proof is made by induction on k. The first case $k = 0$ is obvious. Let $k \geqslant 1$ and suppose it is true for $k-1$. Let $\sigma \in Pre(\tau)$ and $t, s \in T_\tau(X_n)$, such that $t \sim_k s$. If t or s are variables then they must be the same variable and clearly $\hat{\sigma}[t] \sim_k \hat{\sigma}[s]$. If not, we must have $t = f_i(t_1, \ldots, t_{n_i})$ and $s = f_i(s_1, \ldots, s_{n_i})$ for some $i \in I$ such that $t_1 \sim_{k-1} s_1, \ldots, t_{n_i} \sim_{k-1} s_{n_i}$. Hence, we have $\hat{\sigma}[t] = \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ and $\hat{\sigma}[s] = \sigma(f_i)(\hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_{n_i}])$. By induction hypothesis $\hat{\sigma}[t_1] \sim_{k-1} \hat{\sigma}[s_1], \ldots, \hat{\sigma}[t_{n_i}] \sim_{k-1} \hat{\sigma}[s_{n_i}]$. Because $\sigma(f_i)$ is not a variable, and $\sim_k$ is a congruence we can conclude that $\hat{\sigma}[t] \sim_k \hat{\sigma}[s]$. □

Proposition 13. *For each $k \geqslant 0$ the sequence $\Gamma_k = (\Gamma_{k,n})_{n \geqslant 1}$ is a pre-solid variety of congruences.*

Proof. Let $k \geqslant 0$. By proposition 8, we need to prove that for every pre-hypersubstitution $\sigma \in Pre(\tau)$ and $\theta \in \Gamma_{k,n}$, we have $(\hat{\sigma} \times \hat{\sigma})^{-1}\theta \in \Gamma_{k,n}$, for each $n \geqslant 1$. So, it is sufficient to show that $\sim_{k,n} \subseteq (\sigma \times \sigma)^{-1}\theta$. Let $t, s \in T_\tau(X_n)$ such that $t \sim_{k,n} s$. By the previous lemma we have $\hat{\sigma}[t] \sim_{k,n} \hat{\sigma}[s]$, and so $(t, s) \in (\varphi \times \varphi)^{-1} \sim_{k,n} \subseteq (\varphi \times \varphi)^{-1}\theta$. Hence, $\Gamma_k = (\Gamma_{k,n})_{n \geqslant 1}$ is a pre-solid variety of congruence filters. □

Now, we are able to state the next result.

Corollary 6. *For any $k \geqslant 0$, the varieties of tree languages $\mathcal{D}^k(\tau)$ and $\mathcal{D}(\tau)$ are pre-solid varieties of tree languages.*

This Corollary follows from Corollary 11.13 of [7].

6 Conclusion

The approaches of Ésik [7] and Steinby [13] generalize the Eilenberg-type correspondence to tree languages; they are not restricted to a fixed algebraic type. In particular, Ésik uses algebraic theories, and Steinby adds some constructions to the usual Universal Algebra. It is easy to see that the $*$-varieties and $+$-varieties of Ésik correspond to solid and pre-solid varieties of languages, respectively, as presented here. Steinby's general varieties correspond to hypersubstitutions which map the n-ary operational symbols to primitive terms $f_i(x_1, \ldots, x_n)$ (with a change in the algebraic type). We believe that hypersubstitutions and M-solid pseudovarieties are an adequate generalization of the aforementioned approaches and that they are suitable for characterizing varieties of tree languages. In order to see this, one needs to work with hypersubstitutions between different algebraic types which form a small category. Then it is easy to generalize all the other notions presented here. But, to go even further and generalize these results to positive and many-sorted varieties [11], it is necessary to work within the framework of Institutions [8].

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