On Monogenic Nondeterministic Automata*

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Abstract

A finite automaton is said to be directable if it has an input word, a directing word, which takes it from every state into the same state. For nondeterministic (n.d.) automata, directability can be generalized in several ways, three such notions, D1-, D2-, and D3-directability, are used. In this paper, we consider monogenic n.d. automata, and for each $i = 1, 2, 3$, we present sharp bounds for the maximal lengths of the shortest Di-directing words.

1 Introduction

An input word w is called a *directing* (or *synchronizing) word* of an automaton $\mathcal{A}$ if it takes $\mathcal{A}$ from every state to the same state. Directable automata have been studied extensively. In the famous paper of Cerńy [4] it was conjectured that the shortest directing word of an n-state directable automaton has length at most $(n-1)^2$. The best known upper bound on the length of the shortest directing words is $(n^3 - n)/6$ (see [5] and [7]). The same problem was investigated for several subclasses of automata. We do not list here these results but we just mention the most recent paper on the subclass of monotonic automata [1]. Further results on subclasses are mentioned in that paper, and in the papers listed in its references.

Directable n.d. automata have been obtained a fewer interest. Directability to n.d. automata can be extended in several meaningful ways. The following three nonequivalent definitions are introduced and studied in [11]. An input word w of an n.d. automaton $\mathcal{A}$ is said to be

(1) D1-*directing* if it takes $\mathcal{A}$ from every state to the same singleton set,

(2) D2-*directing* if it takes $\mathcal{A}$ from every state to the same fixed set A', where $\emptyset \subseteq A' \subseteq A$,

(3) D3-*directing* if there is a state c such that $c \in aw$, for every $a \in A$.

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The D1-directability of complete n.d. automata was investigated by Burkhard [2]. He gave a sharp exponential bound for the lengths of minimum-length D1-directing words of complete n.d. automata. Goralčik *et al.* [6] studied D1- and D3-directability and they proved that neither for D1- nor for D3-directing words, the bound can be polynomial for n.d. automata. These bounds are improved in [13], one can find an overview of the the results on directing words of n.d. automata in the book [12].

Carpi [3] considered a particular class of n.d. automata, the class of unambigous n.d. automata, and presented $O(n^3)$ bounds for the lengths of their shortest D1-directing words. Trapped n.d. automata are investigated in [8], monotonic n.d. automata are investigated in [9], and commutative n.d. automata are investigated in [10].

In this work, we study the class of *monogenic n.d. automata*, the subclass where the alphabet contains only one symbol. This class is a subclass of the commutative n.d. automata. Shortest directing words of the monogenic and commutative automata are investigated in [14] and [15]. We prove tight bounds for monogenic n.d. automata on the lengths of shortest directing words of each type.

2   Notions and notations

Let X denote a finite nonempty alphabet. The set of all finite words over X is denoted by X^* and λ denotes the empty word. The length of a word $w \in X^*$ is denoted by $|w|$.

By a *nondeterministic (n.d.) automaton* we mean a system $\mathcal{A} = (A, X)$, where A is a nonempty finite set of *states*, X is the *input alphabet*, and each input symbol $x \in X$ is realized as a binary relation $x^{\mathcal{A}}(\subseteq A \times A)$. For any $a \in A$ and $x \in X$, let

$$ax^{\mathcal{A}} = \{b : b \in A \text{ and } (a, b) \in x^{\mathcal{A}}\}.$$

Moreover, for every $B \subseteq A$, we denote by $Bx^{\mathcal{A}}$ the set $\cup \{ax^{\mathcal{A}} : a \in B\}$. Now, for any word $w \in X^*$ and $B \subseteq A$, $Bw^{\mathcal{A}}$ can be defined inductively as follows:

(1) $B\lambda^{\mathcal{A}} = B$,

(2) $Bw^{\mathcal{A}} = (Bp^{\mathcal{A}})x^{\mathcal{A}}$ for $w = px$, where $p \in X^*$ and $x \in X$.

If $w = x_1 \dots x_m$ and $a \in A$, then let $aw^{\mathcal{A}} = \{a\}w^{\mathcal{A}}$. This yields that $w^{\mathcal{A}} = x_1^{\mathcal{A}} \dots x_m^{\mathcal{A}}$. If there is no danger of confusion, then we write simply aw and Bw for $aw^{\mathcal{A}}$ and $Bw^{\mathcal{A}}$, respectively.

Following [11], we define the directability of n.d. automata as follows. Let $\mathcal{A} = (A, X)$ be an n.d. automaton. For any word $w \in X^*$, let us consider the following conditions:

(D1) $(\exists c \in A)(\forall a \in A)(aw = \{c\})$,
(D2) $(\forall a, b \in A)(aw = bw)$,

(D3) $(\exists c \in A)(\forall a \in A)(c \in aw)$.

For any $i = 1, 2, 3$, if w satisfies Di, then w is called a Di-*directing word* of $\mathcal{A}$ and in this case $\mathcal{A}$ is said to be Di-*directable*. Let us denote by $\mathrm{D}_i(\mathcal{A})$ the set of Di-directing words of $\mathcal{A}$. Moreover, let $\mathbf{Dir}(i)$ denote the classes of Di-directable n.d. automata. Now, we can define the following functions. For any $i = 1, 2, 3$ and $\mathcal{A} = (A, X) \in \mathbf{Dir}(i)$, let

$$\mathrm{d}_i(\mathcal{A}) = \min\{|w| : w \in \mathrm{D}_i(\mathcal{A})\},$$

$$\mathrm{d}_i(n) = \max\{\mathrm{d}_i(\mathcal{A}) : \mathcal{A} \in \mathbf{Dir}(i) \ \& \ |A| = n\}.$$

The functions $\mathrm{d}_i(n)$, $i = 1, 2, 3$, are studied in [11] and [13], where lower and upper bounds depending on n are presented for them. Similar functions can be defined for any class of n.d. automata. For a class $\mathbf{K}$ of n.d. automata, let

$$\mathrm{d}_i^{\mathbf{K}}(n) = \max\{\mathrm{d}_i(\mathcal{A}) : \mathcal{A} \in \mathbf{Dir}(i) \cap \mathbf{K} \ \& \ |A| = n\}.$$

3 Monogenic n.d. automata

In what follows, we study the case when the considered class is $\mathbf{MG}$, the class of monogenic n.d. automata. For the class $\mathbf{C}$ of commutative automata it is shown in [10] that $\mathrm{d}_1^{\mathbf{C}}(n) = (n-1)$. Since every monogenic n.d. automaton is commutative we obtain that $\mathrm{d}_1^{\mathbf{MG}}(n) \leq (n-1)$. Moreover, the n.d. automaton which proves in [10] that $\mathrm{d}_1^{\mathbf{C}}(n) \geq (n-1)$ is a monogenic one and thus we obtain immediately the following corollary.

Corollary 1. *For any* $n \geq 1$, $\mathrm{d}_1^{\mathbf{MG}}(n) = (n-1)$.

For the $D2$-directable monogenic n.d. automaton we have the following result.

Theorem 1. *For any* $n \geq 2$, $\mathrm{d}_2^{\mathbf{MG}}(n) = (n-1)^2 + 1$.

Proof. To prove that $\mathrm{d}_2^{\mathbf{MG}}(n) \geq (n-1)^2 + 1$ we can use the same n.d. automaton which was used in [10]. For the sake of completeness we recall the definition of the automaton here. The set of states is $S = \{1, \ldots, n\}$, there is one letter in the alphabet denoted by x, and it is defined as follows: $1x = \{1, 2\}$, $ix = \{i+1\}$ for $1 < i < n$, and $nx = \{1\}$. It is easy to see that the shortest $D2$-directing word of this n.d. automaton has length $(n-1)^2 + 1$.

Now we prove that $\mathrm{d}_2^{\mathbf{MG}}(n) \leq (n-1)^2 + 1$. We prove it by induction on n. If $n = 2$ then the statement is obviously valid. Let $n \geq 2$ and suppose that the inequality is valid for each $i < n$. Consider an arbitrary monogenic $D2$-directable n.d. automaton with n states. Let denote the set of states by $S = \{1, \ldots, n\}$ and the letter in the alphabet by x. Let m be the length of the shortest $D2$-directing word. This means that $ix^m = jx^m$ for each $i, j \in S$.

Suppose first that $Sx \subset S$. Then consider the n.d. automaton (Sx, x). This is a $D2$-directable monogenic n.d. automaton with less than n states. Thus its shortest $D2$-directing word has length at most $(n-2)^2 + 1$. Therefore, the original n.d.

automaton has a $D2$-directing word with length at most $(n-2)^2 + 2 \leq (n-1)^2 + 1$ and this proves the statement in this case.

Therefore, we can suppose that $Sx = S$. This yields that $Sx^k = S$ for each k. Thus $ix^m = Sx^m = S$ for each $i \in S$. Let $i \in S$ be arbitrary and consider the sequence of sets $\{i\}, ix, ix^2 \dots$. If $ix^k = S$, then $ix^l = S$ for each $l \geq k$. Now suppose that $ix^k = ix^l$, $k < l$. Then the sequence of the sets becomes a periodic sequence from ix^k with the period $k - l$, and thus this case is only possible if $ix^k = ix^l = S$.

Let p be the smallest positive value with the property $i \in ix^p$. Since $i \in ix^m$ is valid, such p exists. Then we have $i \in ix^p$. Furthermore, $\{i\} \neq ix^p$ therefore, $|ix^p| \geq 2$. On the other hand by $i \in ix^p$ it also holds that $ix^{qp} \subseteq ix^{(q+1)p}$ and this yields that if $ix^{qp} \neq S$ then $|ix^{(q+1)p}| > |ix^{qp}|$. Thus we obtain that $ix^{(n-1)p} = S$.

Now consider the following sets. Let $H_j = \cup_{k=1}^{j} ix^k$. Then $H_j \subseteq H_{j+1}$ for each j. Furthermore, if $H_j = H_{j+1}$ for some j then $H_j = H_k$ for each $k \geq j$, therefore, this is only possible in the case when $H_j = S$.

Let r be the smallest positive value with the property $|ix^r| \geq 2$. Consider the following two cases.

Case I. Suppose that $ix^r \cap H_{r-1} = \emptyset$. In this case $|H_r| \geq |H_{r-1}| + 2$, thus we obtain that $H_{n-1} = S$. This yields that $p \leq n-1$ and it follows that $(n-1)p < (n-1)^2 + 1$.

Case II. Suppose that there exists j such that $j \in ix^r \cap H_{r-1}$. Then there exists $s < r$ such that $ix^s = \{j\}$. Then for each $t \geq 0$ we have $ix^{s+t(r-s)} \subseteq ix^{s+(t+1)(r-s)}$. Since these sets can be equal only in the case when they are equal to S we obtain that $ix^{s+(n-1)(r-s)} = S$. On the other hand $r \leq n$ and $s \geq 1$ thus we obtain that $s + (n-1)(r-s) \leq (n-1)^2 + 1$.

For the $D3$-directable monogenic n.d. automaton we have the following result.

Theorem 2. *For any $n \geq 1$, $d_3^{\mathbf{MG}}(n) = n^2 - 3n + 3$.*

To prove that $d_3^{\mathbf{MG}}(n) \geq n^2 - 3n + 3$ we can use the same n.d. automaton which was used in the $D2$-directable case. In [12] it is shown that the shortest $D3$-directing word of this n.d. automaton has length $n^2 - 3n + 3$.

Now we prove that $d_3^{\mathbf{MG}}(n) \leq n^2 - 3n + 3$. Consider an arbitrary monogenic $D3$-directable n.d. automaton with n states. Let denote the set of states by $S = \{1, \dots, n\}$ and the letter in the alphabet by x. Let m be the length of the shortest $D3$-directing word. Then there exists a state i with the property $i \in jx^m$ for each $j \in S$.

Define the following n.d. automaton. Let $\mathcal{B} = (S, y)$, where the transition y is defined by the rule $jy = \{k \in S \ : \ j \in kx\}$. Then we obtain by induction that $jy^p = \{k \in S \mid j \in kx^p\}$ Therefore, $iy^m = S$. Moreover, it holds that $jy^p \neq S$ for all $p < m$ and $j \in S$ since otherwise we would obtain a shorter $D3$-directing word than a^m.

Now we can use a similar technique to finish the proof as we did in the case of $D2$-directability. Consider the sequence of sets $\{i\}, iy, iy^2 \dots, iy^m$. This sequence contains different sets and $iy^m = S$. Let us observe that $|iy| \geq 2$, otherwise by $\{iy\}y^{m-1} = S$ we would obtain a contradiction.

Let p be the smallest positive value with the property $i \in iy^p$. Then we have $iy \subseteq iy^{p+1}$. Furthermore, $\{iy\} \neq iy^{p+1}$ and therefore, $|iy^{p+1}| \geq 3$. On the other hand by $iy \subseteq iy^{p+1}$ it also holds that $iy^{qp+1} \subseteq iy^{(q+1)p+1}$ and this yields that if $iy^{qp+1} \neq S$ then $|iy^{(q+1)p+1}| > |iy^{qp+1}|$. Thus we obtain that $iy^{(n-2)p+1} = S$.

If $i \in iy$ then $p = 1$. Otherwise consider the sets $H_j = \cup_{k=1}^{j} iy^k$. Then $H_j \subset H_{j+1}$ for each j, if $H_j \neq S$. Since $|H_1| \geq 2$ we obtain that $H_{n-1} = S$. This yields that $p \leq n - 1$.

Therefore, we obtained that $ix^{(n-2)(n-1)+1} = S$ which proves that $m \leq (n-2)(n-1) + 1 = n^2 - 3n + 3$.

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