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Limited Codes Associated with Petri Nets

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Abstract

The purpose of this paper is to investigate the relationship between limited codes and Petri nets. The set M of all positive firing sequences which start from the positive initial marking μ of a Petri net and reach μ itself forms a pure monoid M whose base is a bifix code. Especially, the set of all elements in M which pass through only positive markings forms a submonoid N of M. Also N has a remarkable property that N is pure. Our main interest is in the base D of N. The family of pure monoids contains the family of very pure monoids, and the base of a very pure monoid is a circular code. Therefore, we can expect that D may be a limited code. In this paper, we examine "small" Petri nets and discuss under what conditions D is limited.

Keywords: free monoid, Petri net, code, prefix code, circular code, limited code

1 Introduction

Let A be an alphabet, A^* the free monoid over A, and 1 the empty word. Let $A^+ = A^* - \{1\}$. A word $v \in A^*$ is a *right factor* of a word $u \in A^*$ if there is a word $w \in A^*$ such that $u = wv$. The right factor v of u is called *proper* if $v \neq u$. For a word $w \in A^*$ and a letter $x \in A$ we let $|w|_x$ denote the number of x in w. The length $|w|$ of w is the number of letters in w.

A non-empty subset C of A^+ is said to be a *code* if for $x_1, ..., x_p, y_1, ..., y_q \in C$, $p, q \geq 1$,

$$x_1 \cdots x_p{=}y_1 \cdots y_q \implies p{=}q,\ x_1{=}y_1,\ \ldots,\ x_p{=}y_p.$$

A subset M of A^* is a *submonoid* of A^* if $M^2 \subseteq M$ and $1 \in M$. Every submonoid M of a free monoid has a unique minimal set of generators $C = (M-\{1\})-(M-\{1\})^2$. C is called the *base* of M. A submonoid M is *right unitary* in A^* if for all $u, v \in A^*$,

$$u,\ uv \in M \Longrightarrow v \in M.$$

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M is called *left unitary* in A^* if it satisfies the dual condition. A submonoid M is *biunitary* if it is both left and right unitary.

Definition 1.1. Let M be a submonoid of a free monoid A^*, and C its base. If $CA^+ \cap C = \emptyset$, (resp. $A^+C \cap C = \emptyset$), then C is called a *prefix* (resp. *suffix*) code over A. C is called a *bifix* code if it is a prefix and suffix code.
A submonoid M of A^* is right unitary (resp. biunitary) if and only if its minimal set of generator is a prefix code (resp. bifix code) ([1, p.46],[3, p.108]).

Definition 1.2. A Petri net is a 4-tuple, $PN = (P, A, W, \mu_0)$ where $P = \{p_1, p_2, \ldots, p_m\}$ is a finite set of places, $A = \{t_1, t_2, \ldots, t_n\}$ is a finite set of transitions such that $P \cap A = \emptyset$ and $P \cup A \neq \emptyset$, $W : (P \times A) \cup (A \times P) \to \{1, 2, \ldots\}$ is a weight function, $\mu_0 : P \to \{0, 1, 2, \ldots\}$ is the initial marking.

Let $t \in A$, and let $\cdot t = \{p \in P | (p, t) \in P \times A\}$ and $t\cdot = \{p \in P | (t, p) \in A \times P\}$. In this paper we shall assume that a Petri net has no isolated transitions, i.e., no t such that $\cdot t \cup t\cdot = \emptyset$. A transition t is said to be *enabled* in a marking μ_0, if $W(p, t) \leq \mu_0(p)$ for all $p \in \cdot t$. A firing of an enabled transition t removes $W(p, t)$ tokens from each input place $p \in \cdot t$, and adds $W(t, p)$ tokens to each output place $p \in t\cdot$. A firing of an enabled transition t in μ_0 produces a new marking μ_1

$$\mu_1(p) = \mu_0(p) - W(p, t) + W(t, p)$$

for any $p \in P$, denoted by $\mu_1 = \delta(\mu, t)$. A string $w = t_1 t_2 \ldots t_r$, $t_i \in A$, of transitions is said to be a $(firing)$ *sequence* from μ_0 if there exist markings $\mu_i, 1 \leq i \leq r$, such that $\delta(\mu_{i-1}, t_i) = \mu_i$ for all i, $1 \leq i \leq r$. In this case, μ_r is reachable from μ_0 by w and we write $\delta(\mu_0, w) = \mu_r$. The set of all possible markings reachable from μ_0 is denoted by $Re(\mu_0)$, and the set of all possible sequences from μ_0 is denoted by $Seq(\mu_0)$. The function $\delta : Re(\mu_0) \times A \to Re(\mu_0)$ is called a next-state function of a Petri net PN [5.p.23]. We note that the above condition for $r = 0$ is understood to be $\mu_0 \in Re(\mu_0)$. A marking μ is said to be *positive* if $\mu(p) > 0$ for all $p \in P$. A sequence $t_1 t_2 \ldots t_n \in Seq(\mu_0)$, $t_i \in A$, is called a positive sequence from μ_0 if $\delta(\mu_0, t_1 t_2 \ldots t_i)$ is positive for all i, $1 \leq i \leq n$. The set of all positive sequences from μ_0 is denoted by $PSeq(\mu_0)$.
Let $P = \{p_1, p_2, \cdots, p_n\}$. A marking μ can be represented by a vector $\mu = (\mu(p_1), \mu(p_2), \cdots, \mu(p_n))$. For every $t \in A$ the vector Δt is defined by

$$\Delta t = (\Delta t(p_1), \Delta t(p_2), \ldots, \Delta t(p_n)),\ n = |P|,$$

where $\Delta t(p) = -W(p, t) + W(t, p)$. For a sequence $w = t_1 t_2 \ldots t_n \in Seq(\mu_0)$ and $p \in P$, $\Delta w = \sum_{i=1}^{n} \Delta t_i$ and $\Delta w(p)$ is a p-th component of a vector Δw, i.e., $\Delta w(p) = \sum_{i=1}^{n} \Delta t_i(p)$. Note that if $\delta(\mu_0, w) = \mu_1$, $w \in Seq(\mu_0)$, then $\mu_1 = \mu_0 + \Delta w$ as a vector.

2 Some codes related to Petri nets

For a Petri net $PN = (P, A, W, \mu)$ and a subset $X \subseteq Re(\mu)$ we can define a deterministic automaton $A(PN)$ as follows: $Re(\mu)$, A, $\delta : Re(\mu) \times A \to Re(\mu)$, μ, and X, are regarded as the state set, the input set, the next-state function, the initial state, and the final set of $A(PN)$, respectively (For basic concepts of automata, refer to [1,p.10]). By using such automata, in [9] we defined four kinds of prefix codes and examined fundamental properties of these codes.
The set

$$Stab(PN) = \{w \mid w \in Seq(\mu) \text{ and } \delta(\mu, w) = \mu\}$$

forms a submonoid of A^*. If $Stab(PN) \neq \{1\}$, then we denote the base of $Stab(PN)$ by $S(PN)$. Since $S(PN)A^+ \cap S(PN) = \emptyset$, $S(PN)$ is a prefix code over A.

A submonoid M of A^* is called *pure* [6] if for all $x \in A^*$ and $n \geq 1$,

$$x^n \in M \Longrightarrow x \in M.$$

A subsemigroup H of a semigroup S is *extractable* in S [8, p.191] if

$$x, y \in S,\ z \in H,\ xzy \in H \Longrightarrow xy \in H.$$

Proposition 2.1. $Stab(PN)$ is an extractable pure monoid.

Proof. It is clear that $Stab(PN)$ is right unitary. Let $y, xy \in Stab(PN)$. Then x is a sequence from the initial marking μ. Since $\Delta y = 0$(the zero vector) and $\Delta(xy) = \Delta x + \Delta y = 0$, we have $x \in Stab(PN)$. Thus $Stab(PN)$ is left unitary. Therefore $Stab(PN)$ is biunitary.

Assume that $x^n \in Stab(PN), n \geq 1$. Then it is obvious that x is a squence from μ. Since $\Delta(x^n)\Delta x = 0$, we have $\Delta x = 0$DThus $x \in Stab(PN)$, and $Stab(PN)$ is pure.

Let $x,\ y \in A^*$ and $z,\ xzy \in Stab(PN)$. If $x = 1$, then $z,\ zy \in Stab(PN)$. Since $Stab(PN)$ is biunitary, we have $y \in Stab(PN)$ and $xy \in Stab(PN)$. Similarly $y = 1$ implies $xy \in Stab(PN)$. Suppose that $x, y \in A^+$. y is a sequence from $\mu + \Delta xz = \mu + \Delta x$. Thus xy is a squence from μ. From $\mu + \Delta(xzy) = \mu + \Delta(xy) = \mu$ we have $xy \in Stab(PN)$. □

Definition 2.1. Let $PN = (P, A, W, \mu)$ be a Petri net with a positive marking μ. Define the subset $D(PN)$ as the set of all positive sequence w of $S(PN)$.

Since $D(PN)$ is a subset of a bifix code $S(PN)$, also $D(PN)$ is a bifix code over A if $D(PN) \neq \emptyset$. By the same argument mentioned above, we have

Proposition 2.2. If $D(PN) \neq \emptyset$, then $D(PN)^*$ is an extractable pure monoid.
Example 2.1. Let $PN = (\{p, q\}, \{a, b\}, W, \mu)$ be a Petri net defined by $W(a, p) = W(p, b) = W(q, a) = W(b, q) = 1$, $\mu(p) = \mu(q) = 2$. Then $D(PN) =$

$\{ab, ba\}$, therefore $\{ab, ba\}^*$ is pure [1, p.324].

Proposition 2.3. If $x, y \in A^+$, z, $xzy \in D(PN)$, then $xz^*y \subseteq D(PN)$.

Proof Let $x, y \in A^+$, $z, xzy \in D(PN)$ and μ an initial marking of PN.

First we show that $xy \in D(PN)$. x is a positive sequence from μ, and y is a positive sequence from $\mu + \Delta(xz) = \mu + \Delta x$. Therefore $xy \in PSeq(\mu)$. Since $\Delta(xzy) = \Delta(xy) = 0$, we have that $xy \in D(PN)^*$, so that $xy = d_1 \cdots d_m$ for some $d_i \in D(PN)$, $m \geq 1$. We have the following three cases.
Case (a). $m = 1, xy = d_1 \in D(PN)$
Case (b). $m \geq 2$, $x = d_1 u$ for some $u \in A^*$.
Case (c). $m \geq 2$, $d_1 = xu, y = uv, v = d_2 \cdots d_m$ for some $u, v \in A^*$.
In Case (b) we have $d_1, xzy = d_1 uzy \in D(PN)$, but it dose not occur since $D(PN)$ is a prefix code. In Case (c), if $u = 1$, then $d_1, xzy = d_1 zy \in D(PN)$DThis contradicts the fact that $D(PN)$ is a prefix code. Therefore $u \neq 1$DIt follows that both d_m and $xzy = xzud_2 \cdots d_m$ are elements of $D(PN)$. This contradicts the fact that $D(PN)$ is a suffix code. Therefore only Case (a) is possible to occur. Thus $xy \in D(PN)$.

Next we show that $x, y \in A^+$, z, $xzy \in D(PN)$ implies $xz^2y \in D(PN)$. Since z is a positive sequence from $\mu + \Delta x = \mu + \Delta(xz)$, we have $xz^2 \in PSeq(\mu)$. Since y is a positive sequence from $\mu + \Delta(xz) = \mu + \Delta(xz^2)$, we have $xz^2y \in PSeq(\mu)$. Therefore, from $\Delta(xz^2y) = \Delta(xzy) = 0$ we have $xz^2y \in D(PN)^*$DThus

$$xz^2y = d_1 \cdots d_m,\ d_i \in D(PN),\ m \geq 1.$$

We have the following four cases.
Case 1. $m = 1$, $xz^2y = d_1 \in D(PN)$,
Case 2. $m \geq 2$, $d_1 = xz^2u$, $y = ud_2 \cdots d_m$ for some $u \in A^*$,
Case 3. $m \geq 2$, $d_1 = xzu$, $z = uv$, $vy = d_2 \cdots d_m$ for some $u, v \in A^*$,
Case 4. $m \geq 2$, $d_1 = xu$, $z = uv$, $vzy = d_2 \cdots d_m$, $u, v \in A^*$ for some $u, v \in A^*$,
Case 5. $m \geq 2$, $x = d_1 u$, for some $u \in A^*$.

In Case 2, $d_m, xzy = xzud_2 \cdots d_m \in D(PN)$. Thus Case 2 cannot occur since $D(PN)$ is a suffix code. In Case 3, $d_m \in D(PN)$, and $xzy = xuvy = xud_2 \cdots d_m \in D(PN)$. However Case 3 cannot occur since $D(PN)$ is a suffix code. Since $D(PN)$ is a prefix code, Case 4 and Case 5 cannot occur. Therefore only Case 1 is possible to occur. Thus $xz^2y \in D(PN)$.

Now suppose that $x, y \in A^+, z, xz^ny \in D(PN)$, $n \geq 2$. Then, xz^{n-1}, $y \in A^+$, z, $(xz^{n-1})zy \in D(PN)$. Therefore we have $(xz^{n-1})z^2y = xz^{n+1}y \in D(PN)$. □

Let C be a code over A. C is an *infix* code ([7,p.129]), if for all $x, y, z \in A^*$,

$$z,\ xzy \in C \Longrightarrow x = y = 1,$$

Proposition 2.4. If $D(PN)$ is a non-empty finite set, then $D(PN)$ is an infix code.

Proof Let $x, y \in A^*$, $z, xzy \in D(PN)$. $x = 1, y \neq 1$ or $x \neq 1, y = 1$ cannot ocuur because $D(PN)$ is a bifix code. Therefore either $x = y = 1$ or $x, y \in A^+$. By Proposition 2.3, $x, y \in A^+$ and z, $xzy \in D(PN)$ follow that $xz^*y \in D(PN)$. This contradicts the fact that $D(PN)$ is a finite set. Thus we have $x = y = 1$. □

Example 2.2. (1). Let PN=$(\{p, q, r\}, \{a, b, c, d\}, W, \mu_0)$ be a Petri net such that $W(p, a) = W(a, q) = W(q, b) = W(b, r) = W(r, c) = W(c, q) = W(q, d) = W(d, p) = 1$, $\mu_0 = (2, 1, 1)$. Then $D(PN) = a(bc)^*d$. Therefore the infinite code $D(PN)$ is infix. Thus the converse of Proposition 2.4 is false.
(2). Let $PN = (\{p\}, \{a, b\}, W, \mu_0)$ be a Petri net such that $W(a, p) = 1, W(p, b) = 1$, $\mu_0 = (1)$. Then $ab, a^2b^2 \in D(PN)$. Therefore the infinite code $D(PN)$ is not infix.

3 Limited code

A submonoid M of A^* is *very pure* if for all $u, v \in A^*$,

$$uv,\, vu \in M \;\Rightarrow\; u,\, v \in M.$$

The base of a very pure monoid is called a *circular* code.

Let $p, q \geq 0$ be two integers. A code C is called $(p,\, q)$-*limited* if for any sequence $u_0, u_1, ..., u_{p+q}$ of words in A^*, the assumptions $u_{i-1}u_i \in C^*$ $(1 \leq i \leq p+q)$ imply $u_p \in C^*$.

Any limited code is circular ([1, p.329, Proposition 2.1]). If a subset C of A^* is a bifix (1,1)-limited code, then for any $u_0, u_1, u_2 \in A^*$ such that $u_0u_1, u_1u_2 \in C^*$ we have $u_1 \in C^*$. Thus $u_0u_1, u_1, u_1u_2 \in C^*$. This imples that $u_0, u_1, u_2 \in C^*$ because C^* is biunitary. Therefore C is (p, q)-limited for all p, q with $p + q = 2$.

Let $PN_0 = (\{p\}, \{a, b\}, W, \mu_0)$ be a Petri net such that $W(a, p) = \alpha, W(p, b) = \beta$, $\mu_0 = (\lambda_p)$, $\lambda_p > 0$.

Consider the set Ω of all positive markings in PN_0;

$$\Omega = \{\mu \,|\, \mu = \mu_0 + \Delta w,\, w \in PSeq(\mu_0\}.$$

Let $g = gcd(\alpha, \beta)$ be the greatest common divisor of α and β, and let $\mathbb{N} = \{0, 1, 2, \cdots\}$ be the set of non-negative integers. Then we have
(0) $D(PN_0)$ is dense, that is, $D(PN_0) \cap A^*wA^* \neq \emptyset$ for every $w \in A^*$.
(1) If $\lambda_p < g$, then $\Omega = \{\lambda_p + ng \,|\, n \in \mathbb{N}\}$.
(2) If $\lambda_p = sg$, $s \geq 1$, $s \in \mathbb{N}$, then $\Omega = \{ng \,|\, n \geq 1, n \in \mathbb{N}\}$.
(3) If $\lambda_p = sg + t_p$, $s \geq 0$, $0 < t_p < g$, then $\Omega = \{t_p + ng \,|\, n \geq 0,\, n \in \mathbb{N}\}$.

Proposition 3.1. If $\lambda_p > gcd(\alpha, \beta)$, then $D(PN_0)$ is not circular.

Proof. Let $D = D(PN_0)$, and let $g = gcd(\alpha, \beta)$. Note that $\mu_0 = \lambda_p$. We have the following two cases:
Case 1. $g = \alpha$ or $g = \beta$. Case 2. $\alpha = \alpha' g$, $\beta = \beta' g$, $\alpha' \geq 2$, $\beta' \geq 2$, $gcd(\alpha', \beta') = 1$.

Case 1-(i). If $\alpha = gcd(\alpha, \beta), \beta = k\alpha, k > 1$, then $aa^{k-1}b$, $a^{k-1}ba \in D$ and $a \notin D^*$. Therefore D is not circular.

Case 1-(ii). If $\beta = gcd(\alpha, \beta)$, $\alpha = k\beta$, $k > 1$, then $ab^{k-1}b$, $bab^{k-1} \in D$ and $b \notin D^*$. Thus D is not circular.

Case 1-(iii). If $\alpha = gcd(\alpha, \beta), \alpha = \beta$, then $ab, ba \in D$. Consequently D is not circular.

Case 2. Since $g = gcd(\alpha, \beta)$, there exist some integers x' and y' such that $\alpha x' + \beta y' = g$.

Case 2-(i). We consider the case $\alpha x' + \beta y' = g, x' > 0, y' < 0$. We set $x = x', y = -y'$, then $\alpha x - \beta y = g, x > 0, y > 0$. Since $\alpha x = \beta y + g > \beta i$, for $i = 1, \cdots, y$, b^y is a sequence from $\lambda_p + \Delta(a^x)$, and $\lambda_p + \Delta(a^x b^y) = \lambda_p + g$. Consequently $a^x b^y$ is also a sequence from $\lambda_p + \Delta(a^x b^y)$, therefore $(a^x b^y)^2 \in PSeq(\mu_0)$. Similarly we have $(a^x b^y)^{\beta'} \in PSeq(\mu_0)$ and $\lambda_p + \Delta((a^x b^y)^{\beta'}) = \lambda_p + \beta' g$. Thus $(a^x b^y)^{\beta'} b \in D(PN_0)$. On the other hand, since $\lambda_p > g$, we have $\lambda_p + \Delta((a^x b^y)^{\beta' - 1} b) = \lambda_p - g$. It follows that $(a^x b^y)^{\beta' - 1} b \cdot a^x b^y \in D$. However $a^x b^y \notin D^*$. Thus D is not circular.

Case 2-(ii). We consider the case $-\alpha x + \beta y = g$ for some positive integers x and y. Then $a(a^x b^y)^j \in PSeq(\mu_0)$, $1 \leq j \leq \alpha'$. Thus $a(a^x b^y)^{\alpha'} \in D$. On the other hand, from $\lambda_p > g$ and $\alpha' \geq 2$ we have $\lambda_p + \Delta(a^x b^y) = \lambda_p - g > 0$. Thus $a^x b^y a \in PSeq(\mu_0)$. It follows that $a^x b^y a (a^x b^y)^{\alpha' - 1} \in D$. However $a^x b^y \notin D^*$. Therefore D is not circular. □

Proposition 3.2. If $\lambda_p \leq gcd(\alpha, \beta)$, then any nonempty subset of $D(PN_0)$ is (p, q)-limited for all p, q with $p + q = 2$.

Proof. Let $D = D(PN_0)$ and $g = gcd(\alpha, \beta)$. Let d be an arbitrary element in D. Then d has a proper right factor $v \neq 1, d$, because $a, b \notin D$. Let $d = uv$, $u, v \in A^+$D

First, we shall show that $\Delta v \leq -g$. Assume the contrary. Then $\Delta v > -g$, and we have $\Delta v \geq 0$ since Δv is a multiple of g. If $\Delta v = 0$, then $\Delta u = 0$ since $\Delta d = \Delta(uv) = \Delta u + \Delta v = 0$. Therefore we get $u \in D^*$. This contradicts the fact that D is a prefix code. Thus we have $\Delta v > 0$, it follows that $\Delta v \geq g$ and $\Delta u = -\Delta(v) \leq -g$. Then we have $\mu_0 + \Delta u = \lambda_p + \Delta u \leq \lambda_p - g \leq 0$, showing that $u \notin PSeq(\mu_0)$ and contradicting $d \in D$. Therefore we have prove $\Delta v \leq -g$.

Next we shall show that any nonempty subset C of D is (1,1)-limited. Note that C is a bifix code. Suppose that $u_0, u_1, u_2 \in A^*$ and $u_0 u_1$, $u_1 u_2 \in C^*$. If $u_i = 1$ for some i, $0 \leq i \leq 2$, then $u_0, u_1, u_2 \in D^*$ since C^* is biunitary. We assume that $u_i \neq 1$ for all $i, 0 \leq i \leq 2$. We may write

$$u_0 = v_0 x_1,\ u_1 = y_1 w_1,\ x_1 y_1 \in C,\ v_0,\ w_1 \in C^*.$$

If $y_1 \neq 1$ and $y_1 \notin C$, then y_1 is a proper right factor of $d_i \in D$. Therefore $\Delta(y_1) \leq -g$ as mentioned above. It follows $\lambda_p + \Delta y_1 \leq 0$, and $y_1 \notin PSeq(\mu_0)$. However, $u_1u_2 = y_1w_1u_2 \in C^* \subseteq D^*$. Thus $y_1 \in PSeq(\mu_0)$. This is a contradiction. Therefore $y_1 = 1$ or $y_1 \in C$. Thus $u_1 \in C^*$. □

Let $PN_1 = (\{p,q\},\{a,b\},W,\mu_0)$ be a Petri net such that $W(a,p) = \alpha > 0, W(p,b) = \alpha' > 0$, $W(q,a) = \beta > 0$, $W(b,q) = \beta' > 0$, $\mu_0(p) = \lambda_p$, $\mu_0(q) = \lambda_q$. We examine the code $D(PN_1)$ associated with Petri net PN_1.

Suppose that $D(PN_1) \neq \emptyset$ and $w \in D(PN_1)$. Let $n = |w|_a$ and $m = |w|_b$, then $\Delta(w)\Delta(a) + m\Delta(b) = 0$. Consequently the linear equation

$$\begin{pmatrix} \alpha & -\alpha' \\ -\beta & \beta' \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has a non-trivial solution. Thus $\alpha\beta' = \alpha'\beta$. Therefore, if $D(PN_1) \neq \emptyset$, then $PN_1 = (\{p,q\},\{a,b\},W,\mu_0)$ has the following form:
$W(a,p) = \alpha, W(p,b) = k\alpha, W(q,a) = \beta, W(b,q) = k\beta$, for some $k > 0$.
Here we assume that k is a positive integer. That is, we define a Petri net $PN_1 = (\{p,q\},\{a,b\},W,\mu_0)$ as follows:

$$\Delta(a) = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}, \quad \Delta(b) = \begin{pmatrix} -k\alpha \\ k\beta \end{pmatrix},$$

where k is a positive integer.

We define an integer M_p as follows

$$M_p = \begin{cases} \dfrac{\lambda_p}{\alpha} - 1, & \text{if } \dfrac{\lambda_p}{\alpha} \text{ is an integer,} \\[2ex] \left[\dfrac{\lambda_p}{\alpha}\right], & \text{if } \dfrac{\lambda_p}{\alpha} \text{ is not an integer,} \end{cases}$$

where [ ] is the symbol of Gauss. Similarly we difine an integer M_q as follows, $M_q = \frac{\lambda_q}{\beta} - 1$ if $\frac{\lambda_q}{\beta}$ is an integer, and $M_q = [\frac{\lambda_p}{\beta}]$ if $\frac{\lambda_q}{\beta}$ is not an integer. Note that $a, a^2, \cdots, a^{M_q} \in PSeq(\mu_0)$ and $a^{M_q+1} \notin PSeq(\mu_0)$. If $M_p \geq k$, then $b \in PSeq(\mu_0)$.

Now, we observe that $M_p + M_q \leq k-1$ implies $D(PN_1) = \emptyset$. If $M_p + M_q \leq k-1$, then $M_p \leq k-1$. It follows that $b \notin PSeq(\mu_0)$. Furthermore, if $M_q = 0$, then $a \notin PSeq(\mu_0)$ and $PSeq(\mu_0) = \{1\}$. If $M_q > 0$, we have

$$a^i \in PSeq(\mu_0),\ 1 \leq i \leq M_q, \text{ and } \ \Delta a^i(p) = \lambda_p + \alpha i \leq \lambda_p + \alpha M_q.$$

Assume that $\lambda_p + \alpha M_q - \alpha k > 0$. If $\frac{\lambda_p}{\alpha}$ is an integer, then $M_p + 1 + M_q - k > 0$. This contradicts the hypothesis. If $\lambda_p = \alpha M_p + s$ for some s, $1 \leq s < \alpha$, then

$$0 < M_p + M_q + \frac{s}{\alpha} - k < M_p + M_q + 1 - k.$$

This also contradicts our hypothesis. Therefore we get $\lambda_p + \alpha M_q - \alpha k \leq 0$, showing that b is not enabled in $\mu_0 + \Delta(a^i)$, $1 \leq i \leq M_q$. Therefore $PSeq(\mu_0) = \{1, a, a^2, \cdots, a^{M_q}\}$. Thus the condition $M_p + M_q \leq k - 1$ implies $D(PN_1) = \emptyset$.

Lemma 3.3. Let $d = uv \in D(PN_1), u, v \in A^+$.
(1) If $M_p = 0$ and $M_q \geq k$, then $\Delta v(p) \leq -\alpha$.
(2) If $M_p \geq k$ and $M_q = 0$, then $\Delta v(q) \leq -\beta$.

Proof. (1). Suppose that $\Delta v(p) > -\alpha$. Then, since $\Delta v(p)$ is a multiple of α, we have $\Delta v(p) \geq 0$. Note that

$$\Delta v = |v|_a \Delta a + |v|_b \Delta b = \begin{pmatrix} (|v|_a - k|v|_b)\alpha \\ -(|v|_a - k|v|_b)\beta \end{pmatrix}.$$

If $\Delta v(p) = 0$, then $|v|_a - k|v|_b = 0$. Thus $\Delta v = 0$, it follows that $\Delta u = 0$ and $u \in D(PN_1)^*$. This contradicts the fact that $D(PN_1)$ is a prefix code. Thus $\Delta v(p) \geq \alpha$. It implies that $\Delta u(p) = -\Delta v(p) \leq -\alpha$. Since $M_p = 0$, $\mu_0(p) + \Delta u(p) < \lambda_p - \alpha \leq 0$. This yields $u \notin PSeq(\mu_0)$. This is a contradiction. Therefore we have $\Delta v(p) \leq -\alpha$.
(2). Proof is omitted. □

Proposition 3.4. We have
(1) If $M_p + M_q > k, M_p \geq k$ and $M_q \geq 1$, then $D(PN_1)$ is not circular.
(2) If $M_p + M_q > k, k > M_p \geq 1, M_q > 1$, then $D(PN_1)$ is not circular.
(3) If $M_p + M_q = k$, then $D(PN_1)$ is a singleton.
(4) If $M_p = 0, M_q > k$, then any nonempty subset of $D(PN_1)$ is (p, q)-limited for all p, q with $p + q = 2$.
(5) If $M_p > k, M_q = 0$, then any nonempty subset of $D(PN_1)$ is (p, q)-limited for all p, q with $p + q = 2$.

Proof. Let $D = D(PN_1)$.
(1) From $M_p \geq k$ it follows that $b \in PSeq(\mu_0)$ and $ba^k \in D$. On the other hand, from $\lambda_p > k\alpha$ we have $\lambda_p + \alpha - k\alpha > \alpha$. Thus $ab \in PSeq(\mu_0)$. Since $\lambda_p + (k-1)\beta > (k-1)\beta$, we have $aba^{k-1} \in D$. Therefore D is not circular.
(2) Let $k = M_p + r$DSince $M_p + M_q > k$, we have $M_q > r$. It follows that $a^r \in PSeq(\mu_0)$. Since $\lambda_p + r\alpha - k\alpha = \lambda_p - \alpha M_p > 0$, we have $a^r b \in PSeq(\mu_0)$. Consequently $a^r ba^{M_p} \in D$. On the other hand we have $a^{r+1}ba^{M_p - 1} \in D$ since $M_q > r$. Therefore D is not circular.
(3) First we consider the case that $M_p \geq 1, M_q \geq 1$. Since $M_p = k - M_q < k$, we have $b \notin PSeq(\mu_0)$. It is obvious that $a^i \in PSeq(\mu_0)$ for $i = 0, 1, \cdots, M_q$. If $i \leq M_q - 1$, then from $\lambda_p + i\alpha - k\alpha = \lambda_p - M_p\alpha - (M_q - i)\alpha$ and $\lambda_p - M_p\alpha \leq \alpha$, we get $\lambda_p + i\alpha - k\alpha \leq 0$, showing that $a^i b \notin PSeq(\mu_0)$, $0 \leq i \leq M_q - 1$. It is obvious that $a^{M_q} b \in PSeq(\mu_0)$ and

$$\mu_0 + \Delta(a^{M_q} b) = \begin{pmatrix} \lambda_p - \alpha M_p \\ \lambda_q + \beta M_p \end{pmatrix}.$$

For $j = 0, \cdots, k - M_q$, we have $a^{M_q}ba^j \in PSeq(\mu_0)$ since $\lambda_q + \beta M_p - \beta j \geq \lambda_q$. However $a^{M_q}ba^jb \notin PSeq(\mu_0)$ since

$$\lambda_p - \alpha M_p + \alpha j \leq \lambda_p - \alpha M_p + \alpha(k - M_q) = \lambda_p \leq k\alpha.$$

Therefore $D = \{a^{M_q}ba^{k-M_q}\}$. Similarly, if $M_p = 0, M_q = k$, then $D = \{a^kb\}$. If $M_p = k, M_q = 0$, then $D = \{ba^k\}$.
(4) Let C be a nonempty subset of D. Suppose that $w_0, w_1, w_2 \in A^*$ and $w_0w_1,\ w_1w_2 \in C^*$. If $w_i = 1$ for some i, $0 \leq i \leq 2$, then $w_0, w_1, w_2 \in C^*$ since C^* is biunitary. We assume that $w_i \neq 1$ for all $i, 0 \leq i \leq 2$. We may write

$$w_0 = u_0x_1,\ w_1 = y_1v_1,\ x_1y_1 \in C \text{ for some } x_1, y_1 \in A^*,\ u_0,\ v_1 \in C^*.$$

If either $y_1 = 1$ or $y_1 \in C$, then $w_1 \in C^*$. Assume that $y_1 \neq 1$ and $y_1 \notin C$. Then y_1 is a proper right factor of an element in D. Since $M_p \leq \alpha$, $\lambda_p + \Delta y_1 \leq 0$ by Lemma 3.3, we have $y_1 \notin PSeq(\mu_0)$. However $w_1w_2 = y_1v_1w_2 \in C^* \subset D^*$. Thus $y_1 \in PSeq(\mu_0)$. This is a contradiction. Therefore $y_1 \in C \cup \{1\}$. This yields $w_1 \in C^*$.
(5) The proof of (5) is similar to the proof of (4), therefore it is omitted. □

Remark 3.1. In the above proof for (3) we have $D = \{w\}$, $w = a^{M_q}ba^{k-M_q}$, $M_q \neq 0,\ k - M_q \neq 0$. D is (s, t)-limited for all $s,\ t \geq 0$ with $s + t = 3$. For any $n,\ m \geq 0$ the code $D = \{a^nba^{k-n}\} = \{a^nba^m\}, k + m$, is realizable as a Petri net code which is produced by the Petri net PN_1 such that $W(a,p) = W(q,a) = 1$, $W(p,b) = W(b,q) = k$, $\mu_0(p) = m + 1$, $\mu_0(q) + 1$.

Let $PN = (P, A, W, \mu_0)$ be a Petri net. By $PRe(\mu_0)$ we denote the set of all possible positive markings reachable from μ_0. For a Petri net PN we define a deterministic automaton $A(PN)$ as follows:
$PRe(\mu_0)$, A, $\delta : PRe(\mu_0) \times A \to PRe(\mu_0)$, μ_0, and $\{\mu_0\}$, are regarded as the state set, the input set, the next-state function, the initial state, and the final set of $A(PN)$, respectively.

Corollary 3.5. Let n and k be arbitrary integers such that $n > k > 1$. Define the automaton

$$\mathcal{A}_{(n,k)} = (\{1, 2, \cdots, n\}, \{a, b\}, f, 1, \{1\})$$

by $f(i, a) = i + 1$, $1 \leq i \leq n - 1$, $f(j, b) = j - k$, $k + 1 \leq j \leq n$. Then any nonempty subset of the base of language $L(\mathcal{A}_{(n,k)})$ recognized by $\mathcal{A}_{(n,k)}$ is a (p,q)-limited code for all p, q with $p + q = 2$.

Proof. We define the $PN_1 = (\{p,\ q\}, \{a,\ b\}, W, \mu_0)$ as follows:
$W(a,p) = 1, W(p,b) = k, W(b,q) = k, W(q,a) = 1,\ \mu_0(p) = 1, \mu_0(q)$. Then $M_p = 0,\ M_q - 1 \geq k$. Therefore, by Proposition 3.4, $D(PN)$ is (1,1)-limited. Since $A(PN_1)$ is isomorphic to $\mathcal{A}_{(n,k)}$ as an automaton, we have Corollary 3.5. □

Proposition 3.6. Let $PN = (\{p_1, \cdots, p_n\}, \{a_1, \cdots, a_n\}, W, \mu_0), n \geq 2$, be a Petri net such that $W(p_i, a_i) = \alpha_i, W(a_i, p_{i+1}) = \beta_i, 1 \leq i \leq n-1$, and $W(p_n, a_n) = \alpha_n, W(a_n, p_1) = \beta_n, \quad \mu_0 = (\lambda_1, \cdots, \lambda_n), \mu_0(p_i) = \lambda_i,\ 1 \leq i \leq n$. Furthermore let $g_j = gcd(\beta_{j-1}, \alpha_j), 2 \leq j \leq n$. If $\lambda_1/\alpha_1 > 1$ and $\lambda_i \leq g_i$ for all $i = 2, \cdots, n$, and if $D(PN) \neq \emptyset$, then any nonempty subset of $D(PN)$ is (p, q)-limited for all p, q with $p + q = 2$.

Proof. We set $D = D(PN)$. Since $\lambda_i \leq g_i$ for all $i = 2, \cdots, n$, we have $D \subset a_1A^+$. Let $d \in D$, $d = auv, u \in A^*, v \in A^+$DNote that v is a proper right factor of an element in D.

First we show that $\Delta v(p_i) \leq 0$ for all $i = 2, \cdots, n$. Suppose that $\Delta v(p_j) > 0$ for some $j \geq 2$. Since $\Delta v(p_j) > 0$ is a linear combination of β_{j-1} and α_j, $\Delta v(p_j)$ is a multiple of g_j. Therefore $\Delta v(p_j) > 0$ implies $\Delta(v)(p_j) \geq g_j$. Thus $-\Delta v(p_j) \leq -g_j$. On the other hand, $\Delta d = \Delta(a_1u) + \Delta v = 0$, and we have $\Delta(a_1u) = -\Delta v$. Therefore $\Delta(a_1u)(p_j) = -\Delta v(p_j) \leq -g_j$. However, $\mu_0(p_j) + \Delta(a_1u)(p_j) \leq \lambda_j - g_j \leq 0$. This contradicts the fact that $a_1u \in PSeq(\mu_0)$. Consequently we have that $\Delta v(p_i) \leq 0$ for all i, ($i \geq 2$).

Next we show that $v \notin PSeq(\mu_0)$. To prove this we show that there exists some p_t, $t \geq 2$, such that $\Delta v(p_t) \leq -g_t$. Suppose the contrary. Then $\Delta v(p_i) = 0$ for all $i \geq 2$. Let x_j be the number of the letter a_j in v, then

$$\Delta(v) = \begin{pmatrix} -\alpha_1 & 0 & \cdots & 0 & \beta_n \\ \beta_1 & -\alpha_2 & \cdots & 0 & 0 \\ 0 & \beta_2 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & \beta_{n-1} & -\alpha_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} \Delta(v)(p_1) \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

We regard the equation above as a system of linear equations. Since $D \neq \emptyset$, the determinant of a matrix $(\Delta a_1, \cdots, \Delta a_n)$ is zero. That is, $\alpha_1\alpha_2 \cdots \alpha_n = \beta_1\beta_2 \cdots \beta_n$. Since there exists a solution, we must have $\Delta v(p_1) = 0$. Consequently $\Delta(a_1u) = -\Delta v = 0$ and $a_1u \in PSeq(\mu_0)$. Therefore $a_1u \in D^*$. This contradicts the fact that D is a prefix code. Thus we have proved that $\Delta v(p_t) \leq -g_t$ for some p_t, $t \geq 2$. This means that $v \notin PSeq(\mu_0)$ since $\mu_0(p_t) + \Delta(v)(p_t) \leq \lambda_j - g_t \leq 0$.

Finally we prove that any nonempty subset C of D is (1,1)-limited. Suppose that $w_0, w_1, w_2 \in A^*$, and $w_0w_1, w_1w_2 \in C^*$. We may write

$$w_0 = u_0x_1,\ w_1 = y_1v_1,\ x_1y_1 \in C \ \text{ for some } x_1,\, y_1 \in A^*,\ u_0,\, v_1 \in C^*.$$

Note that y_1 is a right factor of an element of D. If $y_1 \neq 1$ and $y_1 \notin C$, then $y_1 \notin PSeq(\mu_0)$ as we mentioned above. Therefore $w_1w_2 = y_1v_1w_2 \notin PSeq(\mu_0)$. This contradicts our hypothesis. Thus $y_1 = 1$ or $y_1 \in C$. This shows that $w_1 \in C^*$. □

Let $PN_2 = (\{p_1, p_2\}, \{a, b, c\}, W, \mu_0)$ be a Petri net such that $W(a, p_1) = \alpha_1, W(p_1, b) = \alpha_2$, $W(b, p_2) = \beta_1, W(p_1, c) = \alpha_3, W(p_2, c) = \beta_2$, $\mu_0(p_1) = \lambda_1$, $\mu_0(p_2) = \lambda_2$.

Lemma 3.7. Let PN_2 be a Petri net mentioned above, and let $\alpha = gcd(\alpha_1, \alpha_2, \alpha_3)$, $\beta = gcd(\beta_1, \beta_2)$. Suppose that $D(PN_2) \neq \emptyset$ and $\lambda_1 \leq \alpha$, $\lambda_2 \leq \beta$. If $d \in D(PN_2)$ and v, $v \neq 1$, is a proper right factor of d, then we have one of the following:
(1) $\Delta v(p_1) \leq -\alpha,\ \Delta v(p_2) \leq -\beta$.
(2) $\Delta v(p_1) = 0,\ \Delta v(p_2) \leq -\beta$.
(3) $\Delta v(p_1) \leq -\alpha,\ \Delta v(p_2) = 0$.

Proof. Let $D = D(PN_2)$. It is obvious that b and c are not enabled in μ_0, so that $D \subset aA^*$. Let $d \in D, d = auv, u \in A^*, v \in A^+$. If $u = 1$, then $\Delta v = -\Delta a$ and (3) holds. Assume $u \neq 1$. If $\Delta v(p_1) > 0$, then $\Delta v(p_1) \geq \alpha$ because $\Delta v(p_1)$ is a multiple of α. Thus $\Delta(au)(p_1) = -\Delta v(p_1) \leq -\alpha$. Hence $\mu_0(p_1) + \Delta(au)(p_1) \leq \mu_0(p_1) - \alpha \leq 0$. This contradicts the fact that $au \in PSeq(\mu_0)$. Therefore $\Delta v(p_1) \leq 0$. Similarly we have $\Delta v(p_2) \leq 0$. If $\Delta v(p_1) = 0$ and $\Delta v(p_2) = 0$, i.e., $\Delta v = 0$, then $\Delta(au) = 0$. Since $au \in PSeq(\mu_0)$, we have $au \in D^*$, contradicting the fact that D is a prefix code. Therefore at least one of (1),(2) or (3) occurs. □

Proposition 3.8. If $D(PN_2) \neq \emptyset$ and $\lambda_1 \leq \alpha,\ \lambda_2 \leq \beta$, then any nonempty subset of $D(PN_2)$ is (p, q)-limited for all p, q with $p + q = 2$.

Proof. Let $D = D(PN_2)$. We note that for a right factor v, $v \neq 1$, of an element $d \in D$ the vector $\mu_0 + \Delta v$ is not positive by Lemma 3.7. That is, $v \notin PSeq(\mu_0)$. Let C be a nonempty subset of D. Assume $w_0, w_1, w_2 \in A^*$ and $w_0 w_1, w_1 w_2 \in C^*$. If either $w_0 = 1$ or $w_1 = 1$, then $w_1 \in C^*$. Therefore we consider the case where $w_0 \neq 1$ and $w_1 \neq 1$. Let $w_0 w_1 = d_1 \cdots d_m,\ d_i \in C, 1 \leq i \leq m$. There exist an integer i, $1 \leq i \leq m$, and u, $v \in A^*$ such that $w_0 = d_1 \cdots d_{i-1} u$, $d_i = uv$, $w_1 = v d_{i+1} \cdots d_m$. If $v \neq 1$ and $v \notin D$, then v is a proper right factor d_i. Thus $v \notin PSeq(\mu_0)$. However, from $w_1 w_2 = v d_{i+1} \cdots d_m w_2 \in C^*$, we heve $v \in PSeq(\mu_0)$. This is a contradiction. Hence $v = 1$ or $d \in C$. It follows that $w_1 \in C^*$. Thus C is (1,1)-limited. □

Let $PN_3 = (\{p, q\}, \{a, b, c\}, W, \mu_0)$ be a Petri net such that $W(a, p) = \alpha$, $W(q, a) = \beta, W(p, b) = \alpha + \beta, W(b, q) = \alpha + \beta, W(c, p) = \beta, W(q, c) = \alpha$, $\mu_0(p) = \lambda_p, \mu_0(q) = \lambda_q$.

Lemma 3.9. Let PN_3 be a Petri net mentioned above. If $\beta < \lambda_p \leq \alpha + \beta$ and $\beta < \lambda_q \leq \alpha$, then for any $u \in PSeq(\mu_0)$ we have one of the following:

$$(1)\ \Delta u = \begin{pmatrix} k(\alpha - \beta) \\ k(\alpha - \beta) \end{pmatrix},\ k \geq 0, \quad (2)\ \Delta u = \begin{pmatrix} k(\alpha - \beta) + l\alpha \\ k(\alpha - \beta) - l\beta \end{pmatrix},\ k \geq 0,\ l \geq 1,$$

(3) $\Delta u = \begin{pmatrix} k(\alpha-\beta)-l\beta \\ k(\alpha-\beta)+l\alpha \end{pmatrix}$, $k \geq 0,\ l \geq 1$.

Proof. We shall prove the lemma by induction on the length of u in $PSeq(\mu_0)$. Since $\beta < \lambda_p \leq \alpha+\beta$, and $\beta < \lambda_q \leq \alpha$, only a is enabled in μ_0. That is, a positive sequence of length 1 is only a, and Δa is of the form (2).
Since $\lambda_q - \beta < \alpha - \beta < \alpha$, c is not enabled in $\mu_0 + \Delta a$. $\Delta(a^2)$ is of the form (2), and $\Delta(ab) = (-\beta, \alpha)$ is of the form (3). c is not enabled in $\mu_0 + \Delta(a^2)$. b is not enabled in $\mu_0 + \Delta(ab)$. $\Delta(a^3)$ is of form (2). $\Delta(a^2b)$, $\Delta(aba)$ and $\Delta(abc)$ are of the form (1).

Now we suppose that for $u \in PSeq(\mu_0), |u| > 3$, the vector Δu has a form (1),(2) or (3). Let x be an element in $\{a,b,c\}$ such that $ux \in PSeq(\mu_0)$. We shall show that $\Delta(ux)$ is of the form (1),(2) or(3).

Case 1. $\Delta u = (k(\alpha-\beta), k(\alpha-\beta))$. $\Delta(ua)$ is of the form (2). If $k=0$, then both b and C are not enabled in $\mu_0 + \Delta u$. For $k \geq 1$, $\Delta(ub) = ((k-1)(\alpha-\beta)-2\beta, (k-1)(\alpha-\beta)+2\alpha)$ is of the form (3). $\Delta(uc) = ((k-1)(\alpha-\beta)+\alpha, (k-1)(\alpha-\beta)-\beta)$ is of the form (2).

Case 2. $\Delta u = (k(\alpha-\beta)+l\alpha, k(\alpha-\beta)-l\beta)$. $\Delta(ua)$ is of the form (2). If $l=1$, then $\Delta(ub)$ is of the form (3). If $l \geq 2$, then $\Delta(ub) = ((k+1)(\alpha-\beta)+(l-2)\alpha, (k+1)(\alpha-\beta)-(l-2)\beta)$ is the form (1) or (2). If $k=0$, then c is not enabled in $\mu_0 + \Delta u$. For $k \geq 1$, $\Delta(uc) = ((k-1)(\alpha-\beta)+2\alpha, (k-1)(\alpha-\beta)-2\beta)$.

Case 3. $\Delta u = (k(\alpha-\beta)-l\beta, k(\alpha-\beta)+l\alpha)$. $\Delta(ua) = ((k+1)(\alpha-\beta)-(l-1)\beta, (k+1)(\alpha-\beta)+(l-1)\alpha)$ is of the form (1) or (3). $\Delta(uc) = (k(\alpha-\beta)-(l-1)\beta, k(\alpha-\beta)+(l-1)\alpha)$. If $k=0$, then b is not enabled in $\mu_0+\Delta u$. For $k \geq 1$, $\Delta(ub) = ((k-1)(\alpha-\beta)-(l+2)\beta, (k-1)(\alpha-\beta)+(l+2)\alpha)$. Thus Lemma 3.9 is proved. □

Proposition 3.10. If $D(PN_3) \neq \emptyset$, and if $\beta < \lambda_p \leq \alpha+\beta$ and $\beta < \lambda_q \leq \alpha$, then any nonempty subset of $D(PN_3)$ is (p, q)-limited for all p, q with $p+q=2$.

Proof. Let $D = D(PN_3)$, and let v, $v \neq 1$, be a proper right factor of $d \in D$.

First we shall show that $v \notin PSeq(\mu_0)$. Let $d = uv, u, v \in A^+$, then $\Delta v = -\Delta u$. Since $u \in PSeq(\mu_0)$, by Lemma 3.9 we have one of the following:

(i) $\Delta v = \begin{pmatrix} -k(\alpha-\beta) \\ -k(\alpha-\beta) \end{pmatrix}$, $k \geq 0$, (ii) $\Delta v = \begin{pmatrix} -k(\alpha-\beta)-l\alpha \\ -k(\alpha-\beta)+l\beta \end{pmatrix}$, $k \geq 0, l \geq 1$,

(iii) $\Delta v = \begin{pmatrix} -k(\alpha-\beta)+l\beta \\ -k(\alpha-\beta)-l\alpha \end{pmatrix}$, $k \geq 0,\ l \geq 1$.

We consider Case (iii). If $v \in PSeq(\mu_0)$, then, by Lemma 3.9 we have the following three cases. Case (iii)-(1)

$$\Delta v = \begin{pmatrix} -k(\alpha-\beta)+l\beta \\ -k(\alpha-\beta)-l\alpha \end{pmatrix} = \begin{pmatrix} x(\alpha-\beta) \\ x(\alpha-\beta) \end{pmatrix},\ k \geq 0,\ l \geq 1,\ x \geq 0.$$

In this case, there is not a solution for xD
Case (iii)-(2)

$$\Delta v = \begin{pmatrix} -k(\alpha-\beta)+l\beta \\ -k(\alpha-\beta)-l\alpha \end{pmatrix} = \begin{pmatrix} x(\alpha-\beta)+y\alpha \\ x(\alpha-\beta)-y\beta \end{pmatrix},\ k \geq 0,\ l \geq 1,\ x \geq 0,\ y \geq 1.$$

In this case, only one solution of linear system is a non-positive $(x, y) = (-(k+l), l)$.
Case (iii)-(3)

$$\Delta v = \begin{pmatrix} -k(\alpha-\beta)+l\beta \\ -k(\alpha-\beta)-l\alpha \end{pmatrix} = \begin{pmatrix} x(\alpha-\beta)-y\beta \\ x(\alpha-\beta)+y\alpha \end{pmatrix},\ k \geq 0,\ l \geq 1,\ x \geq 0,\ y \geq 1.$$

In this case, only one solution of linear system is a non-positive $(x, y) = (-k, -l)$. Therefore in any cases we have $v \notin PSeq(\mu_0)$. Similarly, in Case (i) or Case (ii) we cannot write Δv in the form (1), (2) or (3) of Lemma 3.9. Therefore $v \notin PSeq(\mu_0)$.

Let C be a subset of D. Assume $w_0, w_1, w_2 \in A^*$ and $w_0w_1, w_1w_2 \in C^*$. If either $w_0 = 1$ or $w_1 = 1$, then $w_1 \in C^*$. Therefore we consider the case where $w_0 \neq 1$ and $w_1 \neq 1$. Let $w_0w_1 = d_1 \cdots d_m$, $d_j \in C, 1 \leq j \leq m$. There exist an integer i, $1 \leq i \leq m$, and $u, v \in A^*$ such that $w_0 = d_1 \cdots d_{i-1}u$, $d_i = uv$, $w_1 = vd_{i+1} \cdots d_m$. If $v \neq 1$ and $v \neq d_i$, then v is a proper right factor d_i. By using the above fact that $v \notin PSeq(\mu_0)$, we obtain $w_1 \notin PSeq(\mu_0)$. This is a contradiction. Thus we have either $v = 1$ or $v = d_i$ which implies $w_1 \in C^*$. Thus C is (1,1)-limited. □

When a submonoid of a free monoid is given, it seems complicated to judge whether the submonoid is pure or not. This is because we have to show it by the treatment of many different cases of words which belong to the submonoid. Also it doesn't seem easy to decide whether the base of a pure monoid is limited or not. Proposition 2.1 and 2.2 ensure that any submonoid generated by a code $D(PN)$ or $S(PN)$ is always pure. The proof techniques of Proposition 3.2-3.10 which use the properties of right factors of the elements in $D(PN)$ may be usable to decide whether $D(PN)$ is limited or not in other Petri nets.

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