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Weighted and Unweighted Trace Automata

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Abstract

We reprove Droste & Gastin's characterisation from [3] of the behaviors of weighted trace automata by certain rational expressions. This proof shows how to derive their result on *weighted* trace automata as a corollary to the *unweighted* counterpart shown by Ochmański.

Keywords: weighted automata, Mazurkiewicz traces

1 Introduction

A large body of theoretical computer science deals with properties of languages as sets of finite words. These words can be understood as the sequence of events performed by some system. This modelling works fine for sequential systems because of the linear nature of words. Mazurkiewicz [11] proposed a generalization of words nowadays called Mazurkiewicz traces that allows to also model some concurrency. Since its introduction, much work has been devoted to the transfer of results on word languages to trace languages (cf. [6]). One such result is Kleene's theorem [8] equating the recognizable and the rational languages. Ochmański [12] succeeded in transferring this result to trace languages showing that the recognizable trace languages are precisely the c-rational ones.

For sequential systems, it is not just interesting to ask whether a particular word is generated, but also to know the number of different ways it can be generated. This question developed into the theory of weighted automata and formal power series (cf. [14, 9, 1, 5]). A fundamental result is Schützenberger's theorem [13], equating the behaviors of weighted automata with the set of rational formal power series.

These two distinct generalizations of Kleene's theorem were re-joint by Droste & Gastin [3] who investigated weighted trace automata and formal power series over partially commuting variables.

The theorems by Kleene, by Schützenberger, by Ochmański, and by Droste & Gastin characterize the recognizable languages, formal power series, trace languages, or formal power series over partially commuting variables by certain rational operations. All the proofs follow the line of Kleene's proof (namely showing the

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closure of recognizable objects under the respective operations) albeit with nontrivial additions. An exception to this is the recent proof of the result by Droste & Gastin that Berstel & Reutenauer gave in [2]: they extend Brzozowski's derivations to also handle weights and partial commutation.

In this paper, we present another alternative proof of Droste & Gastin's characterisation of the behavior of weighted trace automata. The *novelty* lies in the fact that we derive their result as a corollary to Ochmański's theorem. In other words, we derive a result on *weighted* trace automata from a theorem on *unweighted* trace automata. This refines the methodology introduced in [10] where I similarly derived Schützenberger's theorem from Kleene's theorem.

The idea is as follows (see later sections for missing definitions): If $\mathcal{A}$ is a weighted trace automaton, then the set of paths from some initial to some final state forms a regular language L. We define conditions (T1-3) that formalize the relation between this language and the behavior of $\mathcal{A}$ (see Lemma 4.1). From a rational expression for L, we get an mc-rational expression for the behavior of $\mathcal{A}$ (see proof of Theorem 4.1). Conversely, let E be some c- or mc-expression. Then we construct a language L (Sections 3.2 and 3.3) satisfying (T1-3). The crucial point is that the language L gives rise to a weighted trace automaton whose behavior equals the semantics of E (Prop. 3.1).

2 Definitions

2.1 Weighted trace automata and their behavior

A structure $(S, +, \cdot, 0, 1)$ is a *semiring* if $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ a monoid, $\cdot$ is both left- and right-distributive over $+$, and $0 \cdot k = k \cdot 0 = 0$ for all $k \in S$. If there is no ambiguity, we denote a semiring just by S. A semiring is *commutative* if $(S, \cdot, 1)$ is commutative; it is *idempotent* if $k + k = k$ for all $k \in S$.

An *independence alphabet* is a pair (Σ, I) where Σ is some alphabet and $I \subseteq \Sigma^2$ is an irreflexive and symmetric *independence relation*. Then $D = \Sigma^2 \setminus I$ is the complementary *dependence relation* which is reflexive and symmetric. Then $\sim$ denotes the least congruence relation on the free semigroup Σ^+ with $ab \sim ba$ for all $a, b \in \Sigma$ with $(a, b) \in I$. The quotient $\mathbb{M}^+(\Sigma, I) = \Sigma^+/\sim$ is the *trace semigroup generated by* (Σ, I)[1]; its elements are equivalence classes $[u]$ of words $u \in \Sigma^+$. Note that the semigroups $\mathbb{M}^+(\Sigma, \emptyset)$ and $(\Sigma^+, \cdot)$ are naturally isomorphic and we will identify the element $[u] = \{u\}$ of $\mathbb{M}^+(\Sigma, \emptyset)$ with the word $u \in \Sigma^+$. A language $L \subseteq \Sigma^+$ is *I-closed* if $u \sim v$ and $v \in L$ imply $u \in L$, i.e., $L = \bigcup_{v \in L} [v]$. Similarly, a function $\mu : \Sigma^+ \to X$ to some set X is *I-closed* if $u \sim v$ implies $\mu(u) = \mu(v)$. For $u \in \Sigma^+$, let $\mathrm{alph}(u)$ denote the alphabet of the word u, i.e., the set of letters occurring in u. Then $u \sim v$ implies $\mathrm{alph}(u) = \mathrm{alph}(v)$ which allows to set $\mathrm{alph}([u]) = \mathrm{alph}(u)$.

[1] Droste & Gastin work in the trace monoid $\mathbb{M}(\Sigma, I) = \mathbb{M}^+(\Sigma, I) \cup \{1\}$, but all their results hold with minor and obvious changes also in the trace semigroup (see Remark 2.1). We prefer to work in this trace semigroup since this eliminates the repetitive special handling of the unit element.

A *weighted trace automaton* over the independence alphabet (Σ, I) is a tuple $\mathcal{A} = (Q, \Sigma, \lambda, \mu, \gamma)$ where Q is a finite and nonempty set of *states*, Σ is some alphabet, $\lambda \in S^{1\times Q}$ is a row vector, $\mu : \Sigma^+ \to (S^{Q\times Q}, \cdot)$ is an I-closed (semigroup-) homomorphism, and $\gamma \in S^{Q\times 1}$ is a column vector. Its (Σ, I)*-behavior* $\|\mathcal{A}\|_{(\Sigma,I)}$ is a mapping from $\mathbb{M}^+(\Sigma, I)$ into S given by $\|\mathcal{A}\|_{(\Sigma,I)}([u]) = \lambda \cdot \mu(u) \cdot \gamma$ for all $u \in \Sigma^+$ (since $\mu(u) = \mu(v)$ for $u \sim v$, this is well-defined). Note that every weighted trace automaton over (Σ, I) is also a weighted trace automaton over $(\Sigma, \emptyset)$. If (Σ, I) is clear from the context, the *trace behavior of* $\mathcal{A}$ is the function $\|\mathcal{A}\|_{\mathrm{T}} = \|\mathcal{A}\|_{(\Sigma,I)} : \mathbb{M}^+(\Sigma, I) \to S$ and the *word behavior of* $\mathcal{A}$ is the function $\|\mathcal{A}\|_{\mathrm{W}} = \|\mathcal{A}\|_{(\Sigma,\emptyset)} : \Sigma^+ \to S$. Then the trace and the word-behaviors are directly related by $\|\mathcal{A}\|_{\mathrm{T}}([u]) = \|\mathcal{A}\|_{\mathrm{W}}(u)$ for all $u \in \Sigma^+$.

For $p, q \in Q$ and $a \in \Sigma$, we say (p, a, q) is a *transition* of $\mathcal{A}$ if $\mu(a)_{p,q} \neq 0$. A *path of length* m is a sequence $U = (p_i, a_i, p_{i+1})_{1\leq i\leq m}$ of transitions; its label is the word $\pi(U) = a_1 a_2 \ldots a_m$ and its weight is $c(U) = \prod_{1\leq i<m} \mu(a_i)_{p_i,p_{i+1}}$. Then the trace behavior of $\mathcal{A}$ can also be described in terms of these paths, namely we have

$$\|\mathcal{A}\|_{\mathrm{T}}([u]) = \sum \left(\lambda(\iota) \cdot c(U) \cdot \gamma(f) \;\middle|\; \begin{array}{l} \iota, f \in Q, U \text{ is a path from} \\ \iota \text{ to } f \text{ with } \pi(U) = u \end{array} \right) \tag{1}$$

for any $u \in \Sigma^+$ ([7, Cor. VI.6.2]) since $\|\mathcal{A}\|_{\mathrm{T}}([u]) = \|\mathcal{A}\|_{\mathrm{W}}(u)$.

Mappings s from $\mathbb{M}^+(\Sigma, I)$ into a semiring S can be considered as formal power series in partially commuting variables (fps for short). In this context, one usually writes $(s, [u])$ for the value $s([u])$ and $S\langle\langle \mathbb{M}^+(\Sigma, I)\rangle\rangle$ for the set of all formal power series. For $s, t \in S\langle\langle \mathbb{M}^+(\Sigma, I)\rangle\rangle$ and $A \subseteq \Sigma$, we next define formal power series $s+t$, $s \cdot t$, s^+, and $(s)_A$. To this aim, let $x \in \mathbb{M}^+(\Sigma, I)$ and set

$$(s + t, x) = (s, x) + (t, x) \qquad (s \cdot t, x) = \sum_{\substack{y,z\in\mathbb{M}^+(\Sigma,I)\\ x=yz}} (s, y) \cdot (t, z)$$

$$((s)_A, x) = \begin{cases} (s, x) & \text{if } \mathrm{alph}(x) = A \\ 0 & \text{otherwise} \end{cases} \qquad (s^+, x) = \sum_{1\leq i\leq |x|} (s^i, x)$$

where s^i denotes the i^{th} power of the formal power series s.

An *expression* is a term using the constants ka for $k \in S$ and $a \in \Sigma$, the binary operations $+$ and $\cdot$, and the unary operations $(\ )_A$ and $^+$. Any such expression E can be interpreted as a fps $[\![E]\!]_{(\Sigma,I)} \in S\langle\langle \mathbb{M}^+(\Sigma, I)\rangle\rangle$, the (Σ, I)*-semantics of* E. More formally, we defined inductively

$$([\![ka]\!]_{(\Sigma,I)}, x) = \begin{cases} k & \text{if } x = [a] \\ 0 & \text{otherwise} \end{cases} \qquad ([\![E + F]\!]_{(\Sigma,I)}, x) = ([\![E]\!]_{(\Sigma,I)}, x) + ([\![F]\!]_{(\Sigma,I)}, x)$$

$$([\![E^+]\!]_{(\Sigma,I)}, x) = ([\![E]\!]^+_{(\Sigma,I)}, x) \qquad ([\![(E)_A]\!]_{(\Sigma,I)}, x) = (([\![E]\!]_{(\Sigma,I)})_A, x)$$

and

$$([\![E \cdot F]\!]_{(\Sigma,I)}, x) = \sum_{\substack{y,z\in\mathbb{M}^+(\Sigma,I)\\ x=yz}} ([\![E]\!]_{(\Sigma,I)}, y) \cdot ([\![F]\!]_{(\Sigma,I)}, z)$$

for $A \subseteq \Sigma$ and $x \in \mathbb{M}^+(\Sigma, I)$. Usually, the independence alphabet (Σ, I) will be clear from the context. Therefore, we write $[\![E]\!]_{\mathrm{T}}$ for $[\![E]\!]_{(\Sigma,I)}$ and call it the *trace semantics* of E, and $[\![E]\!]_{\mathrm{W}}$ for $[\![E]\!]_{(\Sigma,\emptyset)}$, the *word semantics* of E.

With these notions, we have the following theorem by Schützenberger.

Theorem 2.1 ([13, 7])**.** *Let S be a semiring, Σ an alphabet, and $s \in S\langle\!\langle \Sigma^+ \rangle\!\rangle$. Then s is the word behavior of some weighted trace automaton iff it is the word semantics of some expression.*

Remark 2.1. Schützenberger [13] considers only the case of the semiring of integers, the general result can be found in Eilenberg's book [7]. Both these authors deal with formal power series over the free monoid Σ^*, i.e., also include the empty word. But the result holds likewise for the free semigroup of nonempty finite words. This follows easily from the following observations (with the obvious definitions [3, 5]).

1. A mapping $s : \mathbb{M}(\Sigma, I) \to S$ is recognizable in the sense of [13, 3] iff $s \restriction \mathbb{M}^+(\Sigma, I) \in S\langle\!\langle \mathbb{M}^+(\Sigma, I) \rangle\!\rangle$ is the trace behavior of some weighted trace automaton.

2. A mapping $s : \mathbb{M}(\Sigma, I) \to S$ is rational in the sense of [13, 3] iff $s \restriction (\mathbb{M}^+(\Sigma, I)$ is the trace semantics of some expression. The only difficulty in the inductive verification of this claim concerns multiplication. This problem can be solved since
$$(s \cdot t) \restriction \mathbb{M}^+(\Sigma, I) = (s, [\varepsilon]) \cdot t' + s' \cdot (t, [\varepsilon]) + s' \cdot t'$$
holds for all $s, t : \mathbb{M}(\Sigma, I) \to S$ with $s' = s \restriction \mathbb{M}^+(\Sigma, I)$ and $t' = t \restriction \mathbb{M}^+(\Sigma, I)$.

In addition, Schützenberger and Eilenberg do not allow the operation $(\ )_A$ in their expressions. Thus, in one sense, their result is stronger: any weighted automaton can be translated into an equivalent expression that does not use the operation $(\ )_A$. On the other hand, it follows from [7, Prop. VI.7.1] that even with the operation $(\ )_A$, we can only describe behaviors of weighted automata since the language $\{u \in \Sigma^+ \mid \mathrm{alph}(u) = A\}$ is regular.

Note that the mapping $\varphi : \Sigma^+ \to \mathbb{M}^+(\Sigma, I) : u \mapsto [u]$ is a semigroup homomorphism. From φ, we define another mapping $\overline{\varphi} : S\langle\!\langle \Sigma^+ \rangle\!\rangle \to S\langle\!\langle \mathbb{M}^+(\Sigma, I) \rangle\!\rangle$ setting (for all $x \in \mathbb{M}^+(\Sigma, I)$)
$$(\overline{\varphi}(s), x) = \sum_{u \in \varphi^{-1}(x)} (s, u) \ .$$

Then direct calculations show that $\overline{\varphi}$ commutes with the operations $+$, $\cdot$, $(\ )_A$, and $^+$. By induction, it follows that the word and the trace semantics of an expression are closely related:

Proposition 2.1. *Let S be some semiring, (Σ, I) an independence alphabet, and E an expression. Then $\overline{\varphi}([\![E]\!]_{\mathrm{W}}) = [\![E]\!]_{\mathrm{T}}$, i.e., for every $u \in \Sigma^+$, we have $([\![E]\!]_{\mathrm{T}}, [u]) = \sum_{v \in [u]} ([\![E]\!]_{\mathrm{W}}, v)$.*

A language $K \subseteq \mathbb{M}^+(\Sigma, I)$ is *mono-alphabetic* if $\mathrm{alph}(x) = \mathrm{alph}(y)$ for every $x, y \in K$; a formal power series $t \in S\langle\langle \mathbb{M}^+(\Sigma) \rangle\rangle$ is *mono-alphabetic* if its support $\{x \in \mathbb{M}^+(\Sigma, I) \mid (t, x) \neq 0\}$ is mono-alphabetic.

A set $B \subseteq \Sigma$ is *I-connected*, if $(B, D \cap B^2)$ is a connected graph; a word $u \in \Sigma^+$ is I-connected if $\mathrm{alph}(u)$ is I-connected; a language $L \subseteq \Sigma^+$ is I-connected if any of its elements is I-connected. Finally, a formal power series $t \in S\langle\langle \mathbb{M}^+(\Sigma) \rangle\rangle$ is I-connected if its support is I-connected.

Droste & Gastin [3, page 52] consider mc-rational and c-rational formal power series that are the semantics of mc-rational and c-rational expressions defined as follows: A *c-rational expression (over* (Σ, I)*)* is an expression E not using the unary operation $(\,)_A$ such that $[\![F]\!]_{\mathrm{T}}$ is I-connected for all sub-expressions F^+ of E. If, in addition, $[\![F]\!]_{\mathrm{T}}$ is mono-alphabetic and I-connected for all subexpressions F^+, the expression is *mc-rational*.

3 From expressions to automata

Given an expression E over (Σ, I), we want to construct a weighted trace automaton $\mathcal{A}$ with $[\![E]\!]_{\mathrm{T}} = \|\mathcal{A}\|_{\mathrm{T}}$. Recall that $[\![E]\!]_{\mathrm{T}} = \overline{\varphi}([\![E]\!]_{\mathrm{W}})$ by Prop. 2.1. Hence, we will first describe a condition on a series $s \in S\langle\langle \Sigma^+ \rangle\rangle$ implying that $\overline{\varphi}(s)$ is the trace behavior of some weighted trace automaton over (Σ, I) (Prop. 3.1). Droste & Gastin [3, Example 39] showed that the fps $[\![1a + 1b]\!]_{\mathrm{T}}$ over the semiring of natural numbers is not the behavior of any weighted trace automaton provided $(a, b) \in I$. Hence, we cannot hope for $[\![E]\!]_{\mathrm{T}}$ to satisfy the condition for each and every expression E, but we prove it for mild extensions of c-rational and mc-rational expressions.

3.1 The condition

Let (Σ, I_Σ) and (Γ, I_Γ) be independence alphabets. A function $\pi : \Gamma \to \Sigma$ is a *projection of independence alphabets* if $(A, B) \in I_\Gamma \iff (\pi(A), \pi(B)) \in I_\Sigma$ for all $A, B \in \Gamma$.

Lemma 3.1. *Let S be a commutative semiring and $\pi : (\Gamma, I_\Gamma) \to (\Sigma, I_\Sigma)$ be a projection of independence alphabets. Furthermore, let $K \subseteq \Gamma^+$ be an I_Γ-closed regular language and $c : \Gamma^+ \to (S, \cdot)$ be a homomorphism. Then there exists a weighted trace automaton $\mathcal{A}$ such that we have for all $u \in \Sigma^+$*

$$(\|\mathcal{A}\|_{\mathrm{W}}, u) = \sum (c(U) \mid U \in K \cap \pi^{-1}(u)) \ .$$

Proof. Let $\mathcal{B} = (Q, \Gamma, \iota, \delta, F)$ denote the minimal deterministic finite automaton with $\mathcal{L}(\mathcal{B}) = K$. Then we have $\delta(q, AB) = \delta(q, BA)$ for every $q \in Q$ and $A, B \in \Gamma$ with $(A, B) \in I_\Gamma$ since $\mathcal{B}$ is minimal and its language K is I_Γ-closed.

From $\mathcal{B}$, we construct a weighted finite automaton $\mathcal{A} = (Q, \Sigma, \lambda, \mu, \gamma)$ on the set of states Q of $\mathcal{B}$ setting

- $\lambda(q) = 1$ for $q = \iota$ and $\lambda(q) = 0$ for $q \neq \iota$

- $\gamma(q) = 1$ for $q \in F$ and $\gamma(q) = 0$ for $q \notin F$
- $\mu(a)_{q_1,q_2} = \sum(c(A) \mid A \in \Gamma$ with $\pi(A) = a$ and $q_2 = \delta(q_1, A))$ for every $a \in \Sigma$ and $q_1, q_2 \in Q$.

We first verify that $\mathcal{A}$ is indeed a weighted trace automaton, i.e., that $\mu(ab) = \mu(ba)$ for all $a, b \in \Sigma$ with $(a, b) \in I_\Sigma$. For this, let $q_1, q_3 \in Q$. Then

$$\begin{aligned}
\mu(ab)_{q_1,q_3} &= \sum_{q_2 \in Q} \mu(a)_{q_1,q_2} \cdot \mu(b)_{q_2,q_3} \\
&= \sum_{q_2 \in Q} \sum \left(c(A) \cdot c(B) \,\middle|\, \begin{array}{l} A, B \in \Gamma, \pi(A) = a, \pi(B) = b, \\ q_2 = \delta(q_1, A), q_3 = \delta(q_2, B) \end{array} \right) \\
&\overset{(i)}{=} \sum \left(c(A) \cdot c(B) \,\middle|\, \begin{array}{l} A, B \in \Gamma, \pi(A) = a, \pi(B) = b, \\ q_3 = \delta(q_1, AB) \end{array} \right) \\
&\overset{(ii)}{=} \sum \left(c(B) \cdot c(A) \,\middle|\, \begin{array}{l} A, B \in \Gamma, \pi(A) = a, \pi(B) = b, \\ q_3 = \delta(q_1, BA) \end{array} \right) \\
&= \mu(ba)_{q_1,q_3} \ .
\end{aligned}$$

Equation (i) holds since $\mathcal{B}$ is a deterministic automaton. Regarding equation (ii), note that $(A, B) \in I_\Gamma$ since $(a, b) \in I_\Sigma$ and since π is a projection of independence alphabets. Then equation (ii) follows since the semiring is commutative and $\delta(q, AB) = \delta(q, BA)$ for every $q \in Q$ and $A, B \in \Gamma$ with $(A, B) \in I_\Gamma$.

It remains to verify $(\|\mathcal{A}\|_{\mathrm{W}}, u) = \sum(c(U) \mid U \in K \cap \pi^{-1}(u))$. So let $u = a_1 \dots a_n \in \Sigma^+$ with $a_i \in \Sigma$. Then we have (where all products stretch over the set of indices $1 \le i \le n$)

$$\begin{aligned}
(\|\mathcal{A}\|_{\mathrm{W}}, u) &= \sum_{p,q \in Q} \lambda(p) \cdot \mu(u)_{p,q} \cdot \gamma(q) = \sum_{f \in F} \mu(u)_{\iota,f} \\
&= \sum \left(\prod \mu(a_i)_{q_{i-1},q_i} \mid q_0 = \iota, q_1, q_2, \dots, q_{n-1} \in Q, q_n \in F \right) \\
&= \sum_{\substack{q_0 = \iota \\ q_1, \dots, q_{n-1} \in Q \\ q_n \in F}} \prod \sum (c(A_i) \mid A_i \in \Gamma, \pi(A_i) = a_i, q_{i+1} = \delta(q_i, A_i)) \\
&= \sum \left(\prod c(A_i) \,\middle|\, \begin{array}{l} \iota = q_0, q_1, \dots, q_{n-1} \in Q, q_n \in F, \\ \text{for } 1 \le i \le n : A_i \in \Gamma, \pi(A_i) = a_i, q_{i+1} = \delta(q_i, A_i) \end{array} \right) \\
&= \sum (c(U) \mid U \in \Gamma^+ \text{ with } \delta(\iota, U) \in F \text{ and } \pi(U) = u) \\
&= \sum (c(U) \mid U \in K \cap \pi^{-1}(u)) \ .
\end{aligned}$$

□

For a language $L \subseteq \Gamma^+$, we write $[L]$ for the set $\bigcup_{U \in L} [U] = \{V \in \Gamma^+ \mid \exists U \in L : U \sim V\}$.

Proposition 3.1. *Let S be a commutative semiring and $\pi : (\Gamma, I_\Gamma) \to (\Sigma, I_\Sigma)$ be a projection of independence alphabets, let $c : \Gamma^+ \to (S, \cdot)$ be a homomorphism, $L \subseteq \Gamma^+$ a language, and $s \in S\langle\langle \Sigma^+ \rangle\rangle$ a fps such that*

(T1) $[L]$ *is regular,*

(T2) $(s,u) = \sum(c(U) \mid U \in L \cap \pi^{-1}(u))$ *for all* $u \in \Sigma^+$*, and*

(T3) $\sum(c(U) \mid U \in [L] \cap \pi^{-1}(u)) = \sum(c(V) \mid V \in L \cap [\pi^{-1}(u)])$ *for all* $u \in \Sigma^+$*.*

Then there exists a weighted trace automaton $\mathcal{A}$ over (Σ, I_Σ) such that $\|\mathcal{A}\|_{\mathrm{T}} = \overline{\varphi}(s)$.

Proof. Note that the language $[L]$ is I_Γ-closed. Hence we can apply Lemma 3.1 which yields a weighted trace automaton $\mathcal{A}$. Then we have for any $u \in \Sigma^+$

$$\begin{aligned}
(\|\mathcal{A}\|_{\mathrm{T}}, [u]) = (\|\mathcal{A}\|_{\mathrm{W}}, u) & \\
&= \sum(c(U) \mid U \in [L] \cap \pi^{-1}(u)) && \text{by Lemma 3.1} \\
&= \sum(c(V) \mid V \in L \cap [\pi^{-1}(u)]) && \text{by } (T3) \\
&= \sum_{v \sim u} \sum(c(V) \mid V \in L \cap \pi^{-1}(v)) && \text{since } [\pi^{-1}(u)] = \pi^{-1}([u]) \\
&= \sum_{v \sim u} (s, v) && \text{by } (T2) \\
&= (\overline{\varphi}(s), u) && \text{by definition of } \overline{\varphi}.
\end{aligned}$$

□

3.2 c-expressions

Throughout this section, we fix an independence alphabet (Σ, I_Σ). Let CONN denote the set of I_Σ-connected subsets of Σ.

For $s \in S\langle\langle \Sigma^+ \rangle\rangle$ and $t \in S\langle\langle \mathbb{M}^+(\Sigma, I_\Sigma) \rangle\rangle$, define

$$s^{\mathrm{c}+} = \left[\sum_{A \in \mathrm{CONN}} (s)_A \right]^+ \text{ and } t^{\mathrm{c}+} = \left[\sum_{A \in \mathrm{CONN}} (t)_A \right]^+ .$$

Suppose that, for all $x \in \mathbb{M}^+(\Sigma, I_\Sigma)$ with $(s,x) \neq 0$, we have $\mathrm{alph}(x) \in \mathrm{CONN}$. Then it is immediate that $s^+ = s^{\mathrm{c}+}$.

Definition 3.1. *A* c-expression *is a term using the constants ka for $k \in S$ and $a \in \Sigma$, the binary operations $+$ and $\cdot$, and the unary operation ${}^{\mathrm{c}+}$.*

Since ${}^{\mathrm{c}+}$ can be expressed in terms of the operations of expressions, c-expressions are special expressions and we will handle them as expressions. In particular, the word and trace semantics of c-expressions are inherited from those of expressions.

Remark 3.1. Let E be some c-rational expression (i.e., $[\![E]\!]_{\mathrm{T}}$ is a c-rational formal power series as defined by Droste & Gastin [3]). Replacing, in E, any occurrence of $^+$ with $^{c+}$ then results in an equivalent c-expression E', i.e., $[\![E]\!]_{\mathrm{T}} = [\![E']\!]_{\mathrm{T}}$. Hence, any c-rational formal power series is the trace semantics of some c-expression.

The rest of this section is devoted to the proof of the following

Theorem 3.1 (cf. [3, Thm. 1(c)])**.** *Let S be a commutative and idempotent semiring and E a c-expression. Then there exists a weighted trace automaton $\mathcal{A}$ over (Σ, I_Σ) such that $[\![E]\!]_{\mathrm{T}} = \|\mathcal{A}\|_{\mathrm{T}}$.*

The proof will be based on Prop. 3.1. More precisely, we will first replace, in the expression E, every appearance of ka with a new letter (k,a) and $^{c+}$ with $^+$. This results in a rational expression whose language L, together with the functions π and c given by $\pi(k,a) = a$ and $c(k,a) = k$, satisfies (T1-3) for $s = [\![E]\!]_{\mathrm{W}}$ (see below). Then, from Prop. 3.1, we obtain a weighted trace automaton $\mathcal{A}$ with $\|\mathcal{A}\|_{\mathrm{T}} = \overline{\varphi}([\![E]\!]_{\mathrm{W}})$ which equals $[\![E]\!]_{\mathrm{T}}$ by Prop. 2.1.

3.2.1 The construction

Since we want to use Prop. 3.1, we have to construct an independence alphabet (Γ, I_Γ), a projection of independence alphabets $\pi : (\Gamma, I_\Gamma) \to (\Sigma, I_\Sigma)$, a homomorphism $c : \Gamma^+ \to (S, \cdot)$, and a language $L \subseteq \Gamma^+$ such that (T1-3) hold.

- Γ is the set of all pairs $(k,a) \in S \times \Sigma$ such that the constant ka appears in the c-expression E.
- For $(k,a), (\ell,b) \in \Gamma$, we set $((k,a),(\ell,b)) \in I_\Gamma$ iff $(a,b) \in I_\Sigma$.
- For $(k,a) \in \Gamma$, let $\pi(k,a) = a$. This defines a projection of independence alphabets $\pi : (\Gamma, I_\Gamma) \to (\Sigma, I_\Sigma)$.
- Let $c : \Gamma^+ \to (S, \cdot)$ be the homomorphism defined by $c(k,a) = k$ for $(k,a) \in \Gamma$.

From a c-expression, we define inductively a language over Γ as follows:

$$[\![ka]\!]' = \{(k,a)\} \qquad [\![E_1 + E_2]\!]' = [\![E_1]\!]' \cup [\![E_2]\!]'$$
$$[\![E_1 \cdot E_2]\!]' = [\![E_1]\!]'[\![E_2]\!]' \qquad [\![E_1^{c+}]\!]' = \{U \in [\![E_1]\!]' \mid U \text{ is connected}\}^+$$

The language $L \subseteq \Gamma^+$ that we need for the application of Prop. 3.1 is $L = [\![E]\!]'$.

3.2.2 Verification of (T1-3)

Note that the language L is constructed from the singletons by union, concatenation, intersection with the set of connected words, and iteration $^+$. In this construction, iteration is only applied to connected languages. Hence, by [12], the language $[L]$ is regular. This verifies (T1).

Property (T2) is verified inductively along the construction of the c-expression E, i.e., we prove

$$(\llbracket E_1 \rrbracket_{\mathrm{W}}, u) = \sum (c(U) \mid U \in \llbracket E_1 \rrbracket' \cap \pi^{-1}(u)) \tag{2}$$

for all $u \in \Sigma^+$ and for all sub-expressions E_1 of E:

- for $E_1 = ka$, we have $(\llbracket E_1 \rrbracket_{\mathrm{W}}, u) = k$ for $u = a$ and $(\llbracket E_1 \rrbracket_{\mathrm{W}}, u) = 0$ otherwise. On the other hand, $\llbracket E_1 \rrbracket' = \{(k, a)\}$ proving Eq. (2).
- Provided Eq. (2) holds for the c-expressions E_1 and E_2, we obtain

$$\begin{aligned}(\llbracket E_1 + E_2 \rrbracket_{\mathrm{W}}, u) &= (\llbracket E_1 \rrbracket_{\mathrm{W}}, u) + (\llbracket E_2 \rrbracket_{\mathrm{W}}, u) \\ &= \sum (c(U) \mid U \in \llbracket E_1 \rrbracket', \pi(U) = u) \\ &\qquad + \sum (c(U) \mid U \in \llbracket E_2 \rrbracket', \pi(U) = u)\end{aligned}$$

since the semiring S is idempotent, this last expression equals

$$\begin{aligned}&= \sum (c(U) \mid U \in \llbracket E_1 \rrbracket' \cup \llbracket E_2 \rrbracket', \pi(U) = u) \\ &= \sum (c(U) \mid U \in \llbracket E_1 + E_2 \rrbracket', \pi(U) = u).\end{aligned}$$

Furthermore,

$$\begin{aligned}(\llbracket E_1 \cdot E_2 \rrbracket_{\mathrm{W}}, u) &= \sum ((\llbracket E_1 \rrbracket_{\mathrm{W}}, v) \cdot (\llbracket E_2 \rrbracket_{\mathrm{W}}, w) \mid v, w \in \Sigma^+, u = vw) \\ &= \sum (c(V) \cdot c(W) \mid V \in \llbracket E_1 \rrbracket', W \in \llbracket E_2 \rrbracket', u = \pi(V)\pi(W)) \\ &\overset{(*)}{=} \sum (c(U) \mid U \in \llbracket E_1 \cdot E_2 \rrbracket', u = \pi(U)) \ .\end{aligned}$$

Regarding the equation (*), note that $(V, W) \mapsto VW$ is a surjection from the set of pairs $\{(V, W) \in \llbracket E_1 \rrbracket' \times \llbracket E_2 \rrbracket' \mid \pi(V)\pi(W) = u\}$ onto the set of words $\{U \in \llbracket E_1 \cdot E_2 \rrbracket' \mid \pi(U) = u\} : (V, W) \mapsto VW$ and that $c(VW) = c(V)c(W)$. Then (*) holds since the semiring S is idempotent.

- Now suppose Eq. (2) holds for the c-expression F and let $u \in \Sigma^+$. Then we have

$$\begin{aligned}(\llbracket F^{\mathrm{c}+} \rrbracket_{\mathrm{W}}, u) &= \left(\left[\sum_{A \in \mathrm{CONN}} (\llbracket F \rrbracket_{\mathrm{W}})_A \right]^+, u \right) \\ &= \sum \left(\prod_{1 \le j \le i} (\llbracket F \rrbracket_{\mathrm{W}}, u_j) \;\middle|\; \begin{array}{l} 1 \le i \le |u|, u = u_1 u_2 \dots u_i, \\ u_1, \dots, u_i \in \Sigma^+ \text{ with} \\ \mathrm{alph}(u_j) \in \mathrm{CONN} \end{array} \right) \\ &= \sum_{\substack{1 \le i \le |u|, u = u_1 u_2 \dots u_i, \\ u_1, \dots, u_i \in \Sigma^+, \mathrm{alph}(u_j) \in \mathrm{CONN}}} \prod_{1 \le j \le i} \sum (c(U_j) \mid U_j \in \llbracket F \rrbracket', \pi(U_j) = u_j)\end{aligned}$$

$$= \sum_{\substack{1\le i\le |u|, u=u_1u_2\dots u_i,\\ u_1,\dots,u_i\in\Sigma^+,\mathrm{alph}(u_j)\in\mathrm{CONN}}} \sum \big(c(U_1U_2\dots U_i) \mid U_j \in [\![F]\!]', \pi(U_j)=u_j\big)$$

$$= \sum \left(c(U_1U_2\dots U_i) \,\middle|\, \begin{array}{l} 1\le i\le |u|, u=u_1u_2\dots u_i,\\ u_1,\dots,u_i\in\Sigma^+, \mathrm{alph}(u_j)\in\mathrm{CONN}\\ U_j\in[\![F]\!]', \pi(U_j)=u_j \end{array} \right)$$

since $U_j \in \Gamma^+$ is I_Γ-connected iff $\pi(U_j)\in\Sigma^+$ is I_Σ-connected, we can continue

$$= \sum \left(c(U_1U_2\dots U_i) \,\middle|\, \begin{array}{l} 1\le i\le |u|\\ U_1,\dots,U_i\in[\![F]\!]' \ I_\Gamma\text{-connected}\\ \pi(U_1U_2\dots U_i)=u \end{array} \right)$$

$$\stackrel{(*)}{=} \sum \big(c(U) \mid U\in[\![F^{\mathrm{c}+}]\!]', \pi(U)=u\big)\,.$$

Here, the equation $(*)$ holds since $(U_1,U_2,\dots,U_i)\mapsto(U_1U_2\dots U_i)$ is a surjection from the set

$$\left\{ (U_1,\dots,U_i) \,\middle|\, \begin{array}{l} 1\le i\le |u|\\ U_1,\dots,U_i\in[\![F]\!]' \ I_\Gamma\text{-connected}\\ \pi(U_1U_2\dots U_i)=u \end{array} \right\}$$

onto the set

$$\{U\in[\![F^{\mathrm{c}+}]\!]' \mid \pi(U)=u\}.$$

This finishes the verification of (T2).

Finally, we verify (T3). So let $u\in\Sigma^+$ and $V\in L\cap[\pi^{-1}(u)]$. Then there exists $U\in\pi^{-1}(u)$ with $V\sim U$ implying $U\in[L]\cap\pi^{-1}(u)$. Hence there is a function $f_u: L\cap[\pi^{-1}(u)]\to[L]\cap\pi^{-1}(u)$ with $f_u(V)\sim V$ and therefore $c(f_u(V))=c(V)$ since $(S,\cdot)$ is commutative. This function is even surjective: if $U\in[L]\cap\pi^{-1}(u)$, then there exists at least one word $V\in L$ with $U\sim V$ and therefore $V\in L\cap[\pi^{-1}(u)]$. Since π is a projection of independence alphabets, this implies $\pi(V)\sim\pi(U)=u$. Hence we have $f_u(V)\sim V\sim U$ and $\pi(f_u(V))=u=\pi(U)$ which implies $U=f_u(V)$. Since the semiring S is idempotent, this ensures (T3).

Since we successfully verified (T1-3), Thm. 3.1 follows from Prop. 3.1.

3.3 mc-expressions

Again, we fix an independence alphabet (Σ, I_Σ) and let CONN denote the set of I_Σ-connected subsets of Σ.

For $s\in S\langle\!\langle\Sigma^+\rangle\!\rangle$ and $t\in S\langle\!\langle\mathbb{M}^+(\Sigma,I_\Sigma)\rangle\!\rangle$, define

$$s^{\mathrm{mc}+}=\sum_{A\in\mathrm{CONN}}[(s)_A]^+ \ \text{ and } t^{\mathrm{mc}+}=\sum_{A\in\mathrm{CONN}}[(t)_A]^+ \ .$$

Suppose there exists $A\in\mathrm{CONN}$ such that, for all $x\in\mathbb{M}^+(\Sigma,I_\Sigma)$ with $(s,x)\neq 0$, we have $\mathrm{alph}(x)=A$. Then it is immediate that $s^+=s^{\mathrm{mc}+}$.

Definition 3.2. *An* mc-expression *is a term using the constants ka for $k \in S$ and $a \in \Sigma$, the binary operations $+$ and $\cdot$, and the unary operation* $^{\mathrm{mc}+}$.

Since $^{\mathrm{mc}+}$ can be expressed in terms of the operations of expressions, mc-expressions are special expressions and we will handle them as expressions. In particular, the word and trace semantics of mc-expressions are inherited from those of expressions.

Remark 3.2. Let E be some mc-rational expression (i.e., $[\![E]\!]_{\mathrm{T}}$ is an mc-rational formal power series as defined by Droste & Gastin [3]). Replacing, in E, any occurrence of $^+$ with $^{\mathrm{mc}+}$ results in an equivalent mc-expression E', i.e., $[\![E]\!]_{\mathrm{T}} = [\![E']\!]_{\mathrm{T}}$. Hence, any mc-rational formal power series is the trace semantics of some mc-expression.

The rest of this section is devoted to the proof of the following

Theorem 3.2 (cf. [3, Thm. 1(b)])**.** *Let S be some commutative semiring and E some mc-expression. Then there exists a weighted trace automaton $\mathcal{A}$ such that $[\![E]\!]_{\mathrm{T}} = \|\mathcal{A}\|_{\mathrm{T}}$.*

3.3.1 The construction

The first idea is to proceed analogously to the proof of Thm. 3.1, i.e., to first replace, in the mc-expression E, every appearance of ka with a new letter (k, a) and $^{\mathrm{mc}+}$ with $^+$. The resulting expression describes a language L. Furthermore, we would set $\pi(k, a) = a$ and $c(k, a) = k$ for $(k, a) \in \Gamma$. Since the semiring S is not assumed to be idempotent anymore, verification of (T2) with $s = [\![E]\!]_{\mathrm{W}}$ causes problems that are best explained by the following two examples using the semiring $\mathcal{N} = (\mathbb{N}, +, \cdot, 0, 1)$ of natural numbers.

- The mc-expression $E = 1a + 1a$ would be transformed into the rational expression $(1, a) + (1, a)$, i.e., $L = \{(1, a)\}$. With $u = a$, the left hand side in (T2) then equals 2, the right-hand side is just 1.

- The mc-expression $E = ((1a)^{\mathrm{mc}+})^{\mathrm{mc}+}$ would be transformed into $((1, a)^+)^+$, i.e., $L = \{(1, a)\}^+$. Note that $[\![(1a)^{\mathrm{mc}+}]\!]_{\mathrm{W}}$ is the characteristic function of $\{a\}^+$, hence

$$([\![E]\!]_{\mathrm{W}}, aa) = ([\![(1a)^{\mathrm{mc}+}]\!]_{\mathrm{W}}, aa) + ([\![(1a)^{\mathrm{mc}+}]\!]_{\mathrm{W}}, a) \cdot ([\![(1a)^{\mathrm{mc}+}]\!]_{\mathrm{W}}, a) = 2 .$$

  On the other hand, the right-hand side of (T2) yields 1 (with $u = aa$).

The sole reason for these problems is that the Boolean semiring is idempotent while the semiring S can be arbitrary. The first of these problems can be solved by replacing the constants in E with pairwise distinct new letters. A solution to the second problem is based on the observation that

$$[\![E^+]\!]_{\mathrm{W}} = [\![E + (E \cdot E)^+ + (E \cdot E)^+ \cdot E]\!]_{\mathrm{W}} .$$

To perform this programme formally, we define a relation Red between mc-expressions E over Σ, alphabets Γ, languages $L \subseteq \Gamma^+$, functions $\pi : \Gamma \to \Sigma$, and homomorphisms $c : \Gamma^+ \to (S, \cdot)$. Set $(E, \Gamma, L, \pi, c) \in \mathrm{Red}$ iff

1. $E = ka$, $\Gamma = \{\perp\}$ for some letter $\perp$, $L = \{\perp\}$, $\pi(\perp) = a$, and $c(\perp) = k$, or
2. there exist $(E_i, \Gamma_i, L_i, \pi_i, c_i) \in \mathrm{Red}$ for $i = 0, 1$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma = \Gamma_0 \cup \Gamma_1$, $\pi = \pi_0 \cup \pi_1$, $c{\restriction}\Gamma = c_0{\restriction}\Gamma_0 \cup c_1{\restriction}\Gamma_1$, and one of the following holds
   a) $E = E_0 + E_1$ and $L = L_0 \cup L_1$,
   b) $E = E_0 \cdot E_1$ and $L = L_0 \cdot L_1$, or
   c) $E = E_0^{\mathrm{mc}+}$, $E_0 = E_1$, and
   $$L = \bigcup_{A \in \mathrm{CONN}} L_1^A \cup (L_1^A \cdot L_0^A)^+ \cup (L_1^A \cdot L_0^A)^+ \cdot L_1^A$$
   where L_i^A is the set of words $U \in L_i$ with $\pi(\mathrm{alph}(U)) = A$.

Let $(E, \Gamma, L, \pi, c) \in \mathrm{Red}$. Then one can show by induction that $L \subseteq \Gamma^+$ is a regular language, the only nontrivial case is $E = E_0^+$ where one has to observe that L^A is regular as soon as L is regular. Furthermore, the binary relation

$$I_\Gamma = \{(A, B) \in \Gamma \mid (\pi(A), \pi(B)) \in I_\Sigma\}$$

is the only independence relation on Γ such that $\pi : (\Gamma, I_\Gamma) \to (\Sigma, I_\Sigma)$ is a projection of independence alphabets. In the following, we will always assume Γ to be equipped with this independence relation.

3.3.2 Verification of (T1-3)

Lemma 3.2. *For* $(E, \Gamma, L, \pi, c) \in \mathrm{Red}$*, the language* $[L] \subseteq \Gamma^+$ *is regular.*

Proof. We proceed by induction along the construction of the mc-expression E. By [12], the base case $E = ka$ as well as the inductive arguments for the cases $E = E_0 + E_1$ and $E = E_0 \cdot E_1$ are immediate. So assume $E = E_0^{\mathrm{mc}+}$, $E_0 = E_1$, and $(E_i, \Gamma^i, L_i, \pi_i, c_i) \in \mathrm{Red}$. Let $U \in L_1^A \cdot L_0^A$ for some $A \in \mathrm{CONN}$. Then $U = V_1 V_0$ for some words $V_i \in L_i^A$. Hence $\pi(\mathrm{alph}(V_i)) = A$ implying $\pi(\mathrm{alph}(U)) = A$. Since $A \in \mathrm{CONN}$, the set $\mathrm{alph}(U)$ is I_Γ-connected. Hence the language $L_1^A \cdot L_0^A$ is connected. Now, from [12], we obtain that $[L]$ is regular. □

Lemma 3.3. *For* $(E, \Gamma, L, \pi, c) \in \mathrm{Red}$*, the following holds for all* $u \in \Sigma^+$*:*

$$([\![E]\!]_{\mathrm{W}}, u) = \sum (c(U) \mid U \in L \cap \pi^{-1}(u)) \ .$$

Proof. The lemma is shown by induction on the construction of E. The base case $E = ka$ is obvious. Now suppose that the lemma has been shown for the tuples $(E_i, \Gamma_i, L_i, \pi_i, c_i) \in \mathrm{Red}$ $(i = 0, 1)$. Furthermore, assume $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma = \Gamma_0 \cup \Gamma_1$, $\pi = \pi_0 \cup \pi_1$, and $c{\restriction}\Gamma = c_0{\restriction}\Gamma_0 \cup c_1{\restriction}\Gamma_1$.

First suppose $E = E_0 + E_1$ and $L = L_0 \cup L_1$. Then we have

$$\begin{aligned}
([\![E]\!]_{\mathrm{W}}, u) &= ([\![E_0]\!]_{\mathrm{W}}, u) + ([\![E_1]\!]_{\mathrm{W}}, u) \\
&= \sum (c_0(U) \mid U \in L_0 \cap \pi_0^{-1}(u)) + \sum (c_1(U) \mid U \in L_1 \cap \pi_1^{-1}(u))
\end{aligned}$$

Since the alphabets Γ_0 and Γ_1 are disjoint, so are the languages L_0 and L_1. Furthermore, c_i agrees with c on Γ_i^+ and similarly for π_i. Hence we can continue

$$\begin{aligned}
&= \sum (c(U) \mid U \in (L_0 \cup L_1) \cap \pi^{-1}(u)) \\
&= \sum (c(U) \mid U \in L \cap \pi^{-1}(u)) \ .
\end{aligned}$$

Next let $E = E_0 \cdot E_1$ and $L = L_0 \cdot L_1$. Then we have

$$\begin{aligned}
([\![E]\!]_{\mathrm{W}}, u) &= \sum_{u=vw} ([\![E_0]\!]_{\mathrm{W}}, v) \cdot ([\![E_1]\!]_{\mathrm{W}}, w) \\
&= \sum_{u=vw} \left(\begin{array}{l} \sum (c_0(V) \mid V \in L_0 \cap \pi_0^{-1}(v)) \\ \cdot \sum (c_1(W) \mid W \in L_1 \cap \pi_1^{-1}(w)) \end{array} \right) \\
&= \sum \left(c_0(V) \cdot c_1(W) \;\middle|\; \begin{array}{l} u = vw, V \in L_0 \cap \pi_0^{-1}(v), \\ W \in L_1 \cap \pi_1^{-1}(w) \end{array} \right)
\end{aligned}$$

Since the alphabets Γ_0 and Γ_1 are disjoint, every word U from $L = L_0 \cdot L_1$ has a unique factorization VW into factors from L_0 and L_1, resp. Hence we can continue

$$= \sum (c(U) \mid U \in L \cap \pi^{-1}(u)) \ .$$

Finally assume $E = E_0^{\mathrm{mc}+}$, $E_0 = E_1$, and $L = \bigcup_{A \in \mathrm{CONN}} L_1^A \cup (L_1^A \cdot L_0^A)^+ \cup (L_1^A \cdot L_0^A)^+ \cdot L_1^A$. Now let $u \in \Sigma^+$ with $B = \mathrm{alph}(u)$. If $B \notin \mathrm{CONN}$, then both sides of the equation from the lemma yield 0. So assume $B \in \mathrm{CONN}$. Then $([\![E]\!]_{\mathrm{W}}, u) = ((([\![E_0]\!]_{\mathrm{W}})_B)^+, u)$. In the following equations, we write π_j for $\pi_{j \bmod 2}$ and similarly L_j for $L_{j \bmod 2}$ for any $j \geq 1$. Then we get

$$\begin{aligned}
([\![E]\!]_{\mathrm{W}}, u) &= \sum_{1 \leq i \leq |u|} \sum_{\substack{u = u_1 \dots u_i \\ \pi_j(\mathrm{alph}(u_j)) = B}} \prod_{1 \leq j \leq i} ([\![E_0]\!]_{\mathrm{W}}, u_j) \\
&= \sum_{1 \leq i \leq |u|} \sum_{\substack{u = u_1 \dots u_i \\ \pi_j(\mathrm{alph}(u_j)) = B}} \prod_{1 \leq j \leq i} \sum \left(c_j(U_j) \mid U_j \in L_j \cap \pi_j^{-1}(u_j) \right) \\
&= \sum \left(c(U_1) \cdot c(U_2) \cdot \ldots c(U_i) \;\middle|\; \begin{array}{l} 1 \leq i \leq |u|, u = u_1 \dots u_i, \\ \text{for all } 1 \leq j \leq i: \\ \quad \pi(\mathrm{alph}(u_j)) = B \\ \quad U_j \in L_j \cap \pi^{-1}(u_j) \end{array} \right)
\end{aligned}$$

where we used that c and π coincide with c_i and π_i on Γ_i^+.

Since the alphabets Γ_0 and Γ_1 are disjoint, every word U from $L \cap \pi^{-1}(u)$ has a unique factorization $U_1 U_2 \dots U_i$ into alternating factors from L_1^B and L_0^B and no factorization into alternating factors from L_1^A and L_0^A for $B \neq A \subseteq \Sigma$. Hence the above expression equals $\sum(c(U) \mid U \in L \cap \pi^{-1}(u))$. $\square$

Lemma 3.4. *For* $(E, \Gamma, L, \pi, c) \in \mathrm{Red}$ *and* $u \in \Sigma^+$, *we have*

$$V_1, V_2 \in L \text{ and } V_1 \sim V_2 \implies V_1 = V_2 \ .$$

Proof. The lemma is shown by induction on the construction of E. The base case $E = ka$ is obvious. Now suppose that the lemma has been shown for the tuples $(E_i, \Gamma_i, L_i, \pi_i, c_i) \in \mathrm{Red}$ $(i = 0, 1)$. Furthermore, assume $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma = \Gamma_0 \cup \Gamma_1$, $\pi = \pi_0 \cup \pi_1$, and $c\restriction\Gamma = c_0\restriction\Gamma_0 \cup c_1\restriction\Gamma_1$

Now suppose $E = E_0 + E_1$ and $L = L_0 \cup L_1$. From $V_1 \sim V_2$, we obtain $\mathrm{alph}(V_1) = \mathrm{alph}(V_2)$. Since the languages L_1 and L_2 have disjoint alphabets, $V_1, V_2 \in L$ implies $V_1, V_2 \in L_i$ for $i = 0$ or for $i = 1$. Hence, by the induction hypothesis, $V_1 \sim V_2$ implies $V_1 = V_2$.

Next suppose $E = E_0 \cdot E_1$ and $L = L_0 \cdot L_1$. Then $V_1, V_2 \in L$ implies the existence of $V_i^j \in L_j$ for $j = 0, 1$ with $V_i = V_i^0 V_i^1$ for $i = 1, 2$. By disjointness of the alphabets, $V_1 \sim V_2$ implies $V_1^j \sim V_2^j$ for $j = 1, 2$. Hence, by the induction hypothesis, $V_i^1 = V_i^2$ and therefore $V_1 = V_2$.

Finally let $E = E_0^{\mathrm{mc}+}$, $E_0 = E_1$, and

$$L = \bigcup_{A \in \mathrm{CONN}} L_1^A \cup (L_1^A \cdot L_0^A)^+ \cup (L_1^A \cdot L_0^A)^+ \cdot L_1^A \ .$$

From $V_1 \in L$, we obtain $B = \pi(\mathrm{alph}(V_1)) \in \mathrm{CONN}$ and $V_1 = V_1^1 V_1^2 \dots V_1^{i_1}$ with $V_1^j \in L_{j \bmod 2}^B$ for all $1 \leq j \leq i_1$. From $V_1 \sim V_2$, we deduce $\mathrm{alph}(V_1) = \mathrm{alph}(V_2)$ and therefore $\pi(\mathrm{alph}(V_1)) = \pi(\mathrm{alph}(V_2))$. Hence $V_2 = V_2^1 V_2^2 \dots V_2^{i_2}$ with $V_1^j \in L_{j \bmod 2}^B$ for all $1 \leq j \leq i_2$.

For $B \subseteq \Sigma$ and $W \in \Gamma^+$, let $\mathrm{proj}_B(W)$ denote the projection of W to the letters from $\pi^{-1}(B)$, i.e., $\mathrm{proj}_B(W)$ is obtained from W by deleting all letters $\gamma \in \Gamma$ with $\pi(\gamma) \notin B$. Now assume that any two letters from $\emptyset \neq B \subseteq \Sigma$ are dependent. Then the same holds for $\pi^{-1}(B)$. Hence $V_1 \sim V_2$ implies $\mathrm{proj}_B(V_1) = \mathrm{proj}_B(V_2)$. Since $V_i^j \in L_{j \bmod 2}^A$, we have $\mathrm{proj}_B(V_i^j) \neq \varepsilon$ for all $\emptyset \neq B \subseteq A$. Since the independence alphabets are disjoint, this implies $i_1 = i_2$ and $\mathrm{proj}_B(V_1^j) = \mathrm{proj}_B(V_2^j)$ for all $1 \leq j \leq i_1$ and $\emptyset \neq B \subseteq A$ with $B \times B \subseteq D$. But this implies $V_1^j \sim V_2^j$ for all $1 \leq j \leq i_1$ and therefore, by the induction hypothesis, $V_1^j = V_2^j$. Hence, indeed, $V_1 = V_2$. $\square$

Lemma 3.5. *For* $(E, \Gamma, L, \pi, c) \in \mathrm{Red}$ *and* $u \in \Sigma^+$, *we have*

$$\sum(c(U) \mid U \in [L] \cap \pi^{-1}(u)) = \sum(c(V) \mid V \in L \cap [\pi^{-1}(u)]) \ .$$

Proof. As in the verification of (T3) from Section 3.2 (page 402), there is a surjection $f_u : L \cap \pi^{-1}([u]) \to [L] \cap \pi^{-1}(u)$ with $U \sim f_u(U)$ and therefore $c(U) = c(f_u(U))$. Hence, by Lemma 3.4, f_u is injective and therefore a weight-preserving bijection. This implies the statement. □

Proof of Thm. 3.2. Inductively, one finds a tuple $(E, \Gamma, L, \pi, c) \in \mathrm{Red}$. Let $s = [\![E]\!]_{\mathrm{W}}$. Then, by Lemmas 3.2, 3.3, and 3.5, we have (T1-3) from Prop. 3.1. Hence, there exists a weighted trace automaton $\mathcal{A}$ with $\|\mathcal{A}\|_{\mathrm{T}} = \overline{\varphi}(s)$ which equals $[\![E]\!]_{\mathrm{T}}$ by Prop. 2.1. □

4 From automata to expressions

In this section, we want to show that the trace behavior of every weighted trace automaton $\mathcal{A}$ can be described by an expression.

In the following, let $\sqsubseteq$ be some linear order on the alphabet Σ. Let $u \in \Sigma^+$. Then $[u]$ is finite and therefore contains a lexicographically minimal word that we denote $\mathrm{lexNF}(u)$ and call *the lexicographic normal form of u*. Let $\mathrm{LNF}(\Sigma)$ denote the set of words $u \in \Sigma^+$ with $u = \mathrm{lexNF}(u)$.

Lemma 4.1. *Let S be some (possibly non-commutative) semiring, (Σ, I_Σ) an independence alphabet, and $\mathcal{A} = (Q, \Sigma, \lambda, \mu, \gamma)$ some weighted trace automaton over $(\Sigma, \emptyset)$ such that, for any $u, v \in \Sigma^+$ with $u \sim v$, we have $(\|\mathcal{A}\|_{\mathrm{W}}, u) = (\|\mathcal{A}\|_{\mathrm{W}}, v)$. For $u \in \Sigma^+$ let $(s, u) = (\|\mathcal{A}\|_{\mathrm{W}}, u)$ if $u \in \mathrm{LNF}$ and 0 otherwise.*

Then there exist a projection of independence alphabets $\pi : (\Gamma, I_\Gamma) \to (\Sigma, I_\Sigma)$, a homomorphism $c : \Gamma^+ \to (S, \cdot)$, and a language $L \subseteq \Gamma^+$ of words in lexicographic normal form such that (T1-3) hold.

Note that every weighted trace automaton satisfies the above condition on $\|\mathcal{A}\|_{\mathrm{W}}$, but the condition is also satisfied by some weighted automata that are no weighted trace automaton. Hence, this lemma proves that the condition expressed in Prop. 3.1 is also necessary.

Proof. We can assume $\lambda(p), \gamma(p) \in \{0, 1\}$ for all $p \in Q$. Let Γ be the set of transitions of $\mathcal{A}$ and set $((p, a, q), (p', b, q')) \in I_\Gamma$ iff $(a, b) \in I_\Sigma$. The mapping $\pi : (\Gamma, I_\Gamma) \to (\Sigma, I_\Sigma) : (p, a, q) \mapsto a$ is a projection of independence alphabets. The homomorphism c is defined by $c(p, a, q) = \mu(a)_{p,q}$ for all $(p, a, q) \in \Gamma$.

There is some linear order $\sqsubseteq$ on Γ such that $(p, a, q) \sqsubseteq (p', a', q')$ implies $a \sqsubseteq a'$. Then $U \in \mathrm{LNF}$ iff $\pi(U) \in \mathrm{LNF}$ for all $U \in \Gamma^+$. Note that every path in $\mathcal{A}$ is a word over Γ. Then let $L \subseteq \Gamma^+$ be the set of paths in lexicographic normal form from some state $\iota \in \lambda^{-1}(1)$ to some state $f \in \gamma^{-1}(1)$ in $\mathcal{A}$. Since $L \subseteq \mathrm{LNF}$ is regular, [12] implies the regularity of $[L] \subseteq \Gamma^+$, i.e., we showed (T1).

To verify (T2), let $u \in \Sigma^+$. If $u \notin \mathrm{LNF}$, then $\pi^{-1}(u)$ does not contain any word in lexicographic normal form. Thus, in this case, $L \cap \pi^{-1}(u) = \emptyset$ implying that both sides of the equation yield 0. So let $u \in \mathrm{LNF}$. Then $L \cap \pi^{-1}(u)$ equals the set of u-labeled paths from $\lambda^{-1}(1)$ to $\gamma^{-1}(1)$. Hence, (T2) follows from Eq. (1).

As in the verification of (T3) from Section 3.2 (page 402), there is a surjection $f_u : L \cap \pi^{-1}([u]) \to [L] \cap \pi^{-1}(u)$ with $U \sim f_u(U)$ and therefore $c(U) = c(f_u(U))$. In the current setting, this surjection is even injective: If $V, W \in L \cap [\pi^{-1}(u)]$ with $f_u(V) = f_u(W)$, then $V \sim f_u(V) = f_u(W) \sim W$. But then $V, W \in L \subseteq \mathrm{LNF}$ implies $V = W$. Hence f_u is a weight-preserving bijection implying (T3). □

As a consequence, we obtain

Theorem 4.1 (cf. [3, Thm. 1(a)]). *Let S be a semiring and $\mathcal{A}$ a weighted trace automaton over the independence alphabet (Σ, I). Then there exists an mc-rational expression E such that $[\![E]\!]_{\mathrm{T}} = \|\mathcal{A}\|_{\mathrm{T}}$.*

Proof. We can apply Lemma 4.1 since the weighted trace automaton $\mathcal{A}$ satisfies the conditions of that lemma. So let π, Γ etc. be as above such that (T1-3) hold. Since L is regular, there is a regular expression E with $\mathcal{L}(E) = L$. By [4, Lemma 2.1], we can assume that the language $\mathcal{L}(F)$ is mono-alphabetic for every sub-expression F^+ of E. Since $\mathcal{L}(E) \subseteq \mathrm{LNF}$, [12] implies that the language $\mathcal{L}(F)$ is connected for every sub-expression F^+ of the rational expression E. Let the expression G be obtained from E by replacing every appearance of $A \in \Gamma$ with $c(A)\pi(A)$. Then one shows inductively along the construction of the rational expression E:

1. if $([\![G]\!]_{\mathrm{T}}, [u]) \neq 0$, then there exists $U \in \mathcal{L}(E)$ with $\pi(U) \sim u$. Recall that for any sub-expression F^+ of E, the language $\mathcal{L}(F)$ is connected and mono-alphabetic. Hence G is an mc-rational expression.
2. $([\![G]\!]_{\mathrm{W}}, v) = \sum(c(V) \mid V \in \mathcal{L}(E) \cap \pi^{-1}(v))$ which, by (T2) equals $([\![\mathcal{A}]\!]_{\mathrm{W}}, v)$ for $v \in \mathrm{LNF}$ and 0 otherwise.

Then we obtain for $u \in \Sigma^+$:

$$\begin{aligned}([\![G]\!], [u]) &= \sum_{v \in [u]} ([\![G]\!]_{\mathrm{W}}, v) \text{ from Prop. 2.1}\\ &= ([\![G]\!]_{\mathrm{W}}, \mathrm{lexNF}(u))\\ &= (\|\mathcal{A}\|_{\mathrm{W}}, u) = (\|\mathcal{A}\|_{\mathrm{T}}, [u])\end{aligned}$$

□

5 Discussion

Let S be a commutative semiring. A consequence of Theorems 3.1, 3.2, and 4.1 is the closure of the set of behaviors of weighted trace automata under addition, multiplication, and iteration $^{\mathrm{mc}+}$ (and $^{\mathrm{c}+}$ provided the semiring is idempotent): if $\mathcal{A}_1$ and $\mathcal{A}_2$ are weighted trace automata, by Thm. 4.1, there exist mc-rational expressions E_1 and E_2 with $[\![E_i]\!]_{\mathrm{T}} = \|\mathcal{A}_i\|_{\mathrm{T}}$. Since $E_1 \cdot E_2$ is another mc-rational expression (that is equivalent with some mc-expression by Remark 3.2), its trace behavior $[\![E_1 \cdot E_2]\!]_{\mathrm{T}} = [\![E_1]\!]_{\mathrm{T}} \cdot [\![E_2]\!]_{\mathrm{T}} = \|\mathcal{A}_1\|_{\mathrm{T}} \cdot \|\mathcal{A}_2\|_{\mathrm{T}}$ is the trace behavior of some weighted trace automaton $\mathcal{A}$ by Thm. 3.2 (similar arguments can be applied for the other operations mentioned above). Since all our proofs (including those referred to from the literature) are effective, the weighted trace automaton $\mathcal{A}$ is computable from $\mathcal{A}_1$ and $\mathcal{A}_2$. Even more explicit automata constructions for these operations were given by Droste & Gastin [3].

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