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Homomorphisms Preserving Types of Density*

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Abstract

The concept of density in a free monoid can be generalized from the infix relation to arbitrary relations. Many of the properties known for density can be established over these more general notions of densities. In this paper, we investigate homomorphisms which preserve different types of density. We demonstrate a strict hierarchy between families of homomorphisms which preserve density over different types of relations. However, as with the case of endomorphisms, a similar hierarchy for weak-coding homomorphisms collapses. We also present an algorithm to decide whether a homomorphism preserves density over any relation which satisfies some natural conditions.
Keywords: density, homomorphisms, coding theory, formal language theory

1 Introduction

A language is *dense* if every word over the alphabet is the infix of some word in the language. One can use other relations ϱ in place of the infix relation to define other types of density. Then, a language could be ϱ-dense, with traditionally density being a special case where $\varrho = \leq_{\mathrm{i}}$, the infix order. These types of densities arise naturally in the theory of codes (see [4]). Indeed, the notions of density, residues, ideals, closure, independence and maximality can be defined for arbitrary relations and properties can be established over these more general notions [3].

We will especially examine those relations which are important for the theory of codes. In particular, the length, prefix, suffix, infix, embedding, outfix, division, commutation and power relations. These produce the following families of codes, respectively, when examining independent sets [4, 5]: block codes (also called uniform codes), prefix codes, suffix codes, infix codes, hypercodes, outfix codes, 2-ps-codes, 2-codes and a superset of the 2-codes. Here, we will examine the same relations with respect to density.

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A homomorphism α from X^* to Y^* is said to preserve density if $\alpha(L)$ is dense over Y for every dense language L over X. For our purposes, we will say that a homomorphism α preserves ϱ-density if $\alpha(L)$ is ϱ-dense over Y for every ϱ-dense language L over X. The work in [3] mainly concerns determining which endomorphisms (homomorphisms where $X = Y$) preserve different types of densities. It is shown there that if ϱ is reflexive, transitive and compatible with homomorphisms (that is, $(x, y) \in \varrho$ implies $(\alpha(x), \alpha(y)) \in \varrho$) where $\varrho \subseteq \omega_n$, for some n (a large relation containing many standard relations), then a homomorphisms α preserves ϱ-density if and only if α restricted to X is a permutation of X. Many types of densities apply here including density defined with the prefix, suffix, infix, power, commutation and division relations. Hence, one gets "regular" density as a special case. In [3] it is left as an open problem to study the same problem over arbitrary homomorphisms, that is, homomorphisms where the alphabets X and Y can differ.

In this paper, we tackle the problem for weak-coding homomorphisms, and also for arbitrary homomorphisms. If ϱ is alphabet preserving ($(x, y) \in \varrho$ implies the set of letters of x is a subset of the letters of y), reflexive, and compatible with a homomorphism α from X^* to Y^*, then α preserves density if and only if every letter of Y appears in $\alpha(X)$. For arbitrary homomorphisms, the situation turns out to be quite a bit more complex. Indeed, the family of homomorphisms which preserves different types of densities forms a sometimes strict, sometimes collapsing hierarchy established in Theorem 3. The property used to separate, or collapse families in the hierarchy is given in Definition 8. This says that two relations ϱ_1 and ϱ_2 are *densely equivalent* if for every finite language L, L^* is ϱ_1-dense if and only if L^* is ϱ_2-dense. Then, Theorem 2 establishes that two relations which are transitive, reflexive and compatible with arbitrary homomorphisms are densely equivalent if and only if the families of homomorphisms preserving both types of density are identical. Hence, we can determine where the hierarchies collapse or are strict by deciding whether the two relations are densely equivalent. In addition, in Section 6 we show that we can decide if a given homomorphism preserves ϱ-density for any of prefix, suffix, infix, embedded, equality, division, commutation or power density.

2 Preliminaries and notation

In this section, we define the mathematical preliminaries necessary for this paper.

The symbol $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a set S, let $|S|$ denote the cardinality of S. Let S and T be sets and α a mapping of S into T. For a subset S' of S, $\alpha|_{S'}$ denotes the restriction of α to S'.

For a binary relation $\varrho \subseteq S \times T$, the set $\operatorname{dom}(\varrho) = \{s \mid s \in S, \exists t \in T, (s, t) \in \varrho\}$ is the *domain* of ϱ. Moreover, $\varrho^{-1} = \{(t, s) \mid (s, t) \in \varrho\}$ is the *inverse* of ϱ, and, for $s \in S$, $\varrho(s) = \{t \mid t \in T, (s, t) \in \varrho\}$. Consequently, $\varrho^{-1}(t) = \{s \mid s \in S, (s, t) \in \varrho\}$ for $t \in T$. In the sequel, for a binary relation $\varrho \subseteq S \times T, x \in S, y \in T$, we use, interchangeably, the notation $(x, y) \in \varrho$ and $x \ \varrho \ y$.

Let Γ be a countably infinite alphabet. In this paper, a finite alphabet will be any finite $X \subseteq \Gamma$. Furthermore, X and Y will be finite alphabets throughout this paper. Then X^* is the set of all *words* over X including the *empty word* λ. Let $w \in X^*$ and $a \in X$, then $|w|_a$ is the number of occurrences of a in w and $|w| = \sum_{a \in X} |w|_a$ is the length of w. A language over X is a subset of X^* and a language is any set whereby there exists an alphabet X such that $L \subseteq X^*$. For a language L over X, $\mathrm{alph}(L)$ is the set of all $a \in X$ with $|w|_a > 0$ for some $w \in L$. For a word $w \in X^*$, we define the *reversal* of w, denoted w^R by $w^R = w$ if $w = \lambda$ and $w^R = a_n \cdots a_1$ if $w = a_1 \cdots a_n, n \geq 1, a_i \in X, 1 \leq i \leq n$.

Let X, Y be finite alphabets. Then a function α from X^* to Y^* such that $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x, y \in X^*$ is called a homomorphism. A homomorphism $\alpha : X^* \to Y^*$ is called a coding if $|\alpha(a)| = 1$ for every $a \in X$. Also, α is called a weak coding if $|\alpha(a)| \leq 1$ for every $a \in X$. Moreover, α is called an endomorphism if $X = Y$.

Let $L, R \subseteq \Sigma^*$. We denote by $R^{-1}L = \{z \in \Sigma^* \mid yz \in L \text{ for some } y \in R\}$ and $LR^{-1} = \{z \in \Sigma^* \mid zy \in L \text{ for some } y \in R\}$.

3 Relations

We will use certain relations frequently which are important for code-related languages [5, 4].

Example 1. Let w and v be arbitrary words in Γ^*.

1. *Embedding order:* $w \leq_{\mathrm{e}} v$ if and only if there exist $n \in \mathbb{N}_0$, $w_1, \ldots, w_n$ and $v_0, v_1, \ldots, v_n$ in Γ^* such that $w = w_1 w_2 \cdots w_n$ and $v = v_0 w_1 v_1 w_2 \cdots w_n v_n$.
2. *Length order:* $w \leq_{\mathrm{u}} v$ if and only if $w = v$ or $|w| < |v|$.
3. *Prefix order:* $w \leq_{\mathrm{p}} v$ if and only if $v = wx$ for some $x \in \Gamma^*$.
4. *Suffix order:* $w \leq_{\mathrm{s}} v$ if and only if $v = xw$ for some $x \in \Gamma^*$.
5. *Outfix relation:* $w \leq_{\mathrm{o}} v$ if and only if there are $w_1, u, w_2 \in \Gamma^*$ such that $v = w_1 u w_2$ and $w = w_1 w_2$.
6. *Infix order:* $w \leq_{\mathrm{i}} v$ if and only if $v = xwy$ for some $x, y \in \Gamma^*$.
7. *Division order:* $w \leq_{\mathrm{d}} v$ if and only if $v = wx = yw$ for some $x, y \in \Gamma^*$.
8. *Commute order:* $w \leq_{\mathrm{c}} v$ if and only if $v = xw = wx$ for some $x \in \Gamma^*$.
9. *Power order:* $w \leq_{\mathrm{f}} v$ if and only if $v = w^n$ for some $n \geq 1$.
10. *Equality order:* $w =_{\mathrm{e}} v$ if and only if $w = v$.

Each of these relations is reflexive and all are transitive except the outfix relation. They are ordered by inclusion as follows [3]:

$$=_{\mathrm{e}} \subsetneq \leq_{\mathrm{f}} \subsetneq \leq_{\mathrm{c}} \subsetneq \leq_{\mathrm{d}} \subsetneq \left\{ \begin{matrix} \leq_{\mathrm{p}} \\ \leq_{\mathrm{s}} \end{matrix} \right\} \subsetneq \left\{ \begin{matrix} \leq_{\mathrm{i}} \\ \leq_{\mathrm{o}} \end{matrix} \right\} \subsetneq \leq_{\mathrm{e}} \subsetneq \leq_{\mathrm{u}}.$$

We also consider the infinite chain

$$\omega_1 \subsetneq \omega_2 \subsetneq \cdots \subsetneq w_n \subsetneq \leq_{\mathrm{e}}$$

of binary relations such that $\omega_1 = \leq_{\mathrm{i}}$ and $\leq_{\mathrm{i}} \cup \leq_{\mathrm{o}} \subsetneq \omega_n$ for $n > 1$ which is defined as follows:

Definition 1. *Let $n \in \mathbb{N}$. For $u, v \in \Gamma^*$, let $(u, v) \in \omega_n$ if and only if*

$$\exists u_1, u_2, \ldots, u_n, v_0, v_1, \ldots, v_n \, (u = u_1 u_2 \cdots u_n \wedge v = v_0 u_1 v_1 u_2 \cdots u_n v_n).$$

Note that $\lim_{n\to\infty} \omega_n = \bigcup_{n=1}^{\infty} \omega_n = \leq_{\mathrm{e}}$. That is, the transitive closure of any ω_n is $\leq_{\mathrm{e}}$.

We also use the following property of relations with respect to homomorphisms:

Definition 2. *Let ϱ be a binary relation on Γ^* and let α be a homomorphism from X^* to Y^*. The relation ϱ is* compatible *with α if, for all $x, y \in X^*$, the inclusion $(x, y) \in \varrho$ implies $(\alpha(x), \alpha(y)) \in \varrho$.*

All of the relations listed in Example 1, except $\leq_{\mathrm{u}}$ are compatible with any homomorphism of X^* to Y^*.

We define the following property of relations which is useful for characterizing weak-coding homomorphisms preserving density:

Definition 3. *For a relation ϱ on Γ^*, we say that ϱ is* alphabet preserving *if $(x, y) \in \varrho$ implies* $\mathrm{alph}(x) \subseteq \mathrm{alph}(y)$.

All of the relations listed in Example 1, except $\leq_{\mathrm{u}}$ are alphabet preserving.

4 Densities

Next, we give some definitions from [3] used to describe different types of densities.

Let S be an arbitrary, fixed, non-empty set. Most of the results in this paper concern the special case where S is a free monoid; however, we give the definition in the same generality as [3].

Definition 4. *Let ϱ be a binary relation on S and let $L \subseteq S$. The set L is said to be ϱ-dense if, for every $x \in S$, there is a $y \in L$ such that $(x, y) \in \varrho$.*

For $S = X^*$ and $\varrho = \leq_{\mathrm{i}}$, we arrive at the usual notion of density. Next, we define the property which is studied extensively in the sequel.

Definition 5. *Let ϱ be a relation on Γ^* and let α be a homomorphism of X^* into Y^*. Then α is said to* preserve ϱ-density *if, for any $L \subseteq X^*$, $\alpha(L)$ is ϱ-dense over Y^* whenever L is ϱ-dense over X^*.*

We would also like to be able to compare the families of homomorphisms which preserve different types of density. Naturally, each homomorphism α from X^* into Y^* can be represented by a set of ordered pairs, $(w, \alpha(w))$ for each $w \in X^*$.

Definition 6. *Let ϱ be a binary relation on Γ^*. Then we denote the family of homomorphisms preserving ϱ-density by $H(\varrho)$. We denote the family of weak-coding homomorphisms preserving ϱ-density by $W(\varrho)$. We denote the family of endomorphisms preserving ϱ-density by $E(\varrho)$.*

It follows from Corollary 6.9 of [3] that an endomorphism preserves ϱ-density, for any $\varrho \in \{=_\mathrm{e}, \leq_\mathrm{f}, \leq_\mathrm{c}, \leq_\mathrm{d}, \leq_\mathrm{p}, \leq_\mathrm{s}, \leq_\mathrm{i}\}$ if and only if $\varrho|_X$ is a permutation of X. Hence, we can immediately establish the following collapsed hierarchy:

Theorem 1. *[3]* $E(=_\mathrm{e}) = E(\leq_\mathrm{f}) = E(\leq_\mathrm{c}) = E(\leq_\mathrm{d}) = E(\leq_\mathrm{p}) = E(\leq_\mathrm{s}) = E(\leq_\mathrm{i})$.

We will show in this paper that this collapsing of the hierarchy will not hold true for arbitrary homomorphisms.

In addition, we use the following definition.

Definition 7. *Let $\alpha : X^* \to Y^*$ be a homomorphism. Then we define* $\mathrm{im}|_X(\alpha) = \{\alpha(a) \mid a \in X\}$ *and* $\max(\alpha) = \max\{|\alpha(a)| \mid a \in X\}$. *In addition, for each $b \in Y$, let $\mu_\alpha(b)$ be the smallest member of $\mathbb{N}_0$ such that $b^{\mu_\alpha(b)} \in \mathrm{im}|_X(\alpha)$ and let $\mu_\alpha(Y) = \max\{\mu_\alpha(b) \mid b \in Y\}$.*

5 Homomorphisms preserving density

We start by including some results from [3] which we use throughout the paper. Again, we will only provide results when S is a free monoid.

Lemma 1. *[3]*

1. *Let $L_1 \subseteq L_2 \subseteq S$ and let ϱ be a binary relation on S. If L_1 is ϱ-dense then L_2 is ϱ-dense.*
2. *Let ϱ_1 and ϱ_2 be two binary relations on S such that $\varrho_1 \subseteq \varrho_2$ and let $L \subseteq S$. If L is ϱ_1-dense then it is ϱ_2-dense.*

Proposition 1. *[3] Let $\alpha : X^* \to Y^*$ be a homomorphism and let ϱ be a binary relation on Γ^* and contained in ω_n for some $n \in \mathbb{N}$. If $\alpha(X^*)$ is ϱ-dense then the following statements hold true:*

1. *For every $a \in Y$, there is an element $b \in X$ and a positive integer $k_{a,b}$ such that $\alpha(b) = a^{k_{a,b}}$.*
2. *$|Y| \leq |X|$.*

We can rephrase condition (1) above by stating that for every $a \in Y$, it follows that $\mu_\alpha(a) > 0$. Condition (2) implies that for most of the standard binary relations, homomorphisms will only preserve that type of density if the target alphabet is no larger then the domain alphabet.

Next, we show that, for coding or weak coding homomorphisms, preserving density essentially amounts to examining alphabets.

Lemma 2. *Let ϱ be an alphabet preserving binary relation on Γ^* such that there exists some ϱ-dense language over X^* and also let $\alpha : X^* \to Y^*$ be a homomorphism which preserves ϱ-density. Then* $\mathrm{alph}(\alpha(X^*)) = Y$ *and* $\mathrm{alph}(\alpha(X)) = Y$.

Proof. Let L be ϱ-dense over X^*. Then X^* must be ϱ-dense by Lemma 1(1). Thus, for every $u \in X^*$, there exists $v \in X^*$ such that $(u, v) \in \varrho$ and $\mathrm{alph}(u) \subseteq \mathrm{alph}(v)$. Therefore, for every $u' \in Y^*$, there exists $v' \in \alpha(X^*)$ such that $(u', v') \in \varrho$. In particular, for every $y \in Y$, there exists $v' \in \alpha(X^*)$ such that $(y, v') \in \varrho$. But $\mathrm{alph}(y) \subseteq \mathrm{alph}(v')$. Hence, $y \in \mathrm{alph}(\alpha(X^*))$ for every $y \in Y$. □

Lemma 3. *Let ϱ be a binary relation on Γ^* and assume that there exists some ϱ-dense language over X^* and let $\alpha : X^* \to Y^*$ be a weak coding homomorphism. If* $\mathrm{alph}(\alpha(X^*)) = Y$ *and ϱ is compatible with α, then α preserves ϱ-density.*

Proof. Let L be a ϱ-dense language over X^* and let $y = a_1 a_2 \cdots a_n \in Y^*$ with $a_i \in Y, 1 \le i \le n$ (the case where $y = \lambda$ is similar). Since $\mathrm{alph}(\alpha(X^*)) = Y$ and α is a weak coding homomorphism, there must exist $b_1, \ldots, b_n \in X$ where $\alpha(b_1 \cdots b_n) = a_1 \cdots a_n$. However, since L is ϱ-dense, there must exist $v \in L$ such that $(b_1 \cdots b_n, v) \in \varrho$. But ϱ is compatible with α, so $(\alpha(b_1 \cdots b_n), \alpha(v)) = (a_1 \cdots a_n, \alpha(v)) \in \varrho$ and $\alpha(v) \in \alpha(L)$. Hence, $\alpha(L)$ is ϱ-dense. □

We sum up the two previous lemmas as follows:

Proposition 2. *Let ϱ be an alphabet preserving binary relation on Γ^* such that there exists some ϱ-dense language over X^*. Also, let $\alpha : X^* \to Y^*$ be a weak coding homomorphism (or indeed a coding homomorphism) whereby ϱ is compatible with α. Then α preserves ϱ-density if and only if* $\mathrm{alph}(\alpha(X^*)) = Y$ *and this holds true if and only if* $\mathrm{alph}(\alpha(X)) = Y$.

Since every reflexive relation will allow X^* to be ϱ-dense, we get the following:

Corollary 1. *Let ϱ be an alphabet preserving, reflexive binary relation on Γ^*. Also, let $\alpha : X^* \to Y^*$ be a weak coding homomorphism (or indeed a coding homomorphism) whereby ϱ is compatible with α. Then α preserves ϱ-density if and only if* $\mathrm{alph}(\alpha(X^*)) = Y$ *and this holds true if and only if* $\mathrm{alph}(\alpha(X)) = Y$.

The conditions of compatibility and alphabet preserving relations as above apply to a large variety of specific relations including the classical notion of density.

Corollary 2. *For $\varrho \in \{=_{\mathrm{e}}, \le_{\mathrm{f}}, \le_{\mathrm{c}}, \le_{\mathrm{d}}, \le_{\mathrm{p}}, \le_{\mathrm{s}}, \le_{\mathrm{i}}, \le_{\mathrm{o}}, \le_{\mathrm{e}}\}$, a weak coding homomorphism (or indeed a coding homomorphism) $\alpha : X^* \to Y^*$ preserves ϱ-density if and only if* $\mathrm{alph}(\alpha(X^*)) = Y$ *and this holds true if and only if* $\mathrm{alph}(\alpha(X)) = Y$.

Consequently, the hierarchy for weak-coding homomorphisms completely collapses.

Corollary 3. $W(=_{\mathrm{e}}) = W(\le_{\mathrm{f}}) = W(\le_{\mathrm{c}}) = W(\le_{\mathrm{d}}) = W(\le_{\mathrm{p}}) = W(\le_{\mathrm{s}}) = W(\le_{\mathrm{i}}) = W(\le_{\mathrm{o}}) = W(\le_{\mathrm{e}})$.

The following is essentially an extension of results in [3] from endomorphisms to arbitrary homomorphisms. It says that under certain conditions, determining whether $\alpha(X^*)$ is dense is equivalent to determining whether α preserves density.

Proposition 3. *Let $\alpha : X^* \to Y^*$ be a homomorphism and let ϱ be a binary relation on Γ^*. Then the following statements are true.*

1. *If ϱ is transitive and compatible with α and if $\alpha(X^*)$ is ϱ-dense over Y^* then, for every $L \subseteq X^*$ which is ϱ-dense over X^*, also $\alpha(L)$ is ϱ-dense over Y^*.*
2. *If there is an $L \subseteq X^*$ such that $\alpha(L)$ is ϱ-dense over Y^*, then $\alpha(X^*)$ is ϱ-dense over Y^*.*

Proof. (1) Let $y \in Y^*$. As $\alpha(X^*)$ is ϱ-dense, there exists $z \in \alpha(X^*)$ with $(y, z) \in \varrho$. Let $z' \in \alpha^{-1}(z)$. As L is ϱ-dense, there exists $x' \in L$ with $(z', x') \in \varrho$. Let $y' = \alpha(x')$. Hence $y' \in \alpha(L)$ and by compatibility, $(z, y') \in \varrho$. Since $(y, z), (z, y') \in \varrho$, then $(y, y') \in \varrho$ by transitivity.

(2) Let $L \subseteq X^*$ be such that $\alpha(L)$ is ϱ-dense over Y^*. Since $\alpha(L) \subseteq \alpha(X^*)$, then Lemma 1(1) implies that $\alpha(X^*)$ is ϱ-dense over Y^*. □

We use this to show that, under certain natural conditions, determining whether a homomorphism preserves density amounts to checking whether a single regular language is dense.

Proposition 4. *Let $\alpha : X^* \to Y^*$ be a homomorphism, let ϱ be a binary relation on Γ^* which is transitive and compatible with α and assume that there exists $L \subseteq X^*$ which is ϱ-dense over X^*. Then the following conditions are equivalent:*

1. *α preserves ϱ-density.*
2. *$\alpha(X^*)$ is ϱ-dense over Y^*.*
3. *$\mathrm{im}|_X(\alpha)^*$ is ϱ-dense over Y^*.*

Proof. (1) $\Rightarrow$ (2) is true by Propositon 3(2).

(2) $\Leftarrow$ (1) is true by Proposition 3(1).

(2) $\Leftrightarrow$ (3) is true because $\mathrm{im}|_X(\alpha) = \alpha(X)$ and so $\mathrm{im}|_X(\alpha)^* = (\alpha(X))^* = \alpha(X^*)$. □

Furthermore, if ϱ is also reflexive, then $(x, x) \in \varrho$ for every $x \in X^*$ and so X^* is ϱ-dense over X^* and we can simplify the proposition above.

Corollary 4. *Let $\alpha : X^* \to Y^*$ be a homomorphism and let ϱ be a binary relation on Γ^* which is transitive, reflexive and compatible with α. Then X^* is ϱ-dense over X^* and also the following conditions are equivalent:*

1. *α preserves ϱ-density.*
2. *$\alpha(X^*)$ is ϱ-dense over Y^*.*

3. $\mathrm{im}|_X(\alpha)^*$ *is* ϱ*-dense over* Y^*.

We would like to be able to study, formally, the families of homomorphisms preserving density.

Definition 8. *Let* ϱ_1, ϱ_2 *be two binary relations on* Γ^**. If, for every finite language* L*,* L^* *is* ϱ_1*-dense implies that* L^* *is* ϱ_2*-dense, then we say* ϱ_1 *is* densely smaller *than* ϱ_2*. If, for every finite language* L*,* L^* *is* ϱ_1*-dense if and only if* L^* *is* ϱ_2*-dense, then we say that* ϱ_1 *and* ϱ_2 *are* densely equivalent*.*

This property is the key to studying families of homomorphisms preserving ϱ-density where ϱ is reflexive, transitive and compatible with arbitrary homomorphisms. We will use it to collapse or separate the different families preserving density.

Theorem 2. *Let* ϱ_1, ϱ_2 *be two binary relations on* Γ^* *which are transitive, reflexive and compatible with arbitrary homomorphisms. Then the following are true:*

1. $\varrho_1 \subseteq \varrho_2$ *implies* $H(\varrho_1) \subseteq H(\varrho_2)$.

2. $H(\varrho_1) \subseteq H(\varrho_2)$ *if and only if* ϱ_1 *is densely smaller than* ϱ_2.

3. $H(\varrho_1) = H(\varrho_2)$ *if and only if* ϱ_1 *is densely equivalent to* ϱ_2.

Proof. (1) For the first statement, let $\alpha : X^* \to Y^*$ be a homomorphism which preserves ϱ_1-density. Then $\mathrm{im}|_X(\alpha)^*$ is ϱ_1-dense over Y^*, by Corollary 4. Then $\mathrm{im}|_X(\alpha)^*$ is ϱ_2-dense over Y^* by Lemma 1(2). Hence, α preserves ϱ_2-density, again by Corollary 4.

(2) Assume that $H(\varrho_1) \subseteq H(\varrho_2)$. Let $L = \{w_1, \ldots, w_n\}$ be a finite language and assume L^* is ϱ_1-dense. Let $a_1, \ldots, a_n$ be n distinct symbols of Γ. Consider the homomorphism α defined by mapping a_i to w_i for each i, $1 \le i \le n$. Then $\mathrm{im}|_X(\alpha) = L$. Thus, α must preserve ϱ_1-density since $\mathrm{im}|_X(\alpha)^* = L^*$ is ϱ_1-dense, ϱ_1 is compatible with arbitrary homomorphisms by assumption, and by Corollary 4. By the assumption, α must also preserve ϱ_2-density, and thus L^* must be ϱ_2-dense, again by Corollary 4.

Conversely, assume that ϱ_1 is densely smaller than ϱ_2. Let $\alpha : X^* \to Y^*$ be a homomorphism which preserves ϱ_1-density. Then $\mathrm{im}|_X(\alpha)^*$ is ϱ_1-dense by Corollary 4 and is thus ϱ_2-dense since ϱ_1 is densely smaller than ϱ_2. Hence, α preserves ϱ_2-density, again by Corollary 4.

(3) Immediate from (2). □

So, by the first statement of the previous theorem, we can set up a hierarchy among all relations in Example 1 which are transitive, reflexive and compatible with arbitrary homomorphisms as follows:

Corollary 5. *For* $z \in \{\mathrm{p}, \mathrm{s}\}$, $H(=_{\mathrm{e}}) \subseteq H(\le_{\mathrm{f}}) \subseteq H(\le_{\mathrm{c}}) \subseteq H(\le_{\mathrm{d}}) \subseteq H(\le_z) \subseteq H(\le_{\mathrm{i}}) \subseteq H(\le_{\mathrm{e}})$.

Proposition 5. *Let α be a homomorphism from X^* into Y^*. Then α preserves $=_{\mathrm{e}}$-density if and only if $Y \subseteq \mathrm{im}|_X(\alpha)$.*

Proof. Assume that α preserves $=_{\mathrm{e}}$-density. Thus, for every $w \in Y^*$, $w \in \mathrm{im}|_X(\alpha)^*$ by Corollary 4. Suppose $a \in Y$ such that $a \notin \mathrm{im}|_X(\alpha)$. But $(a, x) \in =_{\mathrm{e}}$ implies $x = a \in \mathrm{im}|_X(\alpha)$, a contradiction. Hence $Y \subseteq \mathrm{im}|_X(\alpha)$.

Conversely, assume $Y \subseteq \mathrm{im}|_X(\alpha)$. Let $w \in Y^*$. Then $w \in \mathrm{im}|_X(\alpha)^*$ and $(w, w) \in =_{\mathrm{e}}$, and thus α preserves $=_{\mathrm{e}}$-density. □

Further, for the case of the embedding relation, we get the following simple characterization.

Proposition 6. *Let $\alpha : X^* \to Y^*$ be a homomorphism and let ϱ be a binary relation on Γ^* which is alphabet preserving, transitive and compatible with α such that $\leq_{\mathrm{e}} \subseteq \varrho$. Then the following are equivalent:*

1. *α preserves $\leq_{\mathrm{e}}$-density.*
2. *α preserves ϱ-density.*
3. $\mathrm{alph}(\mathrm{im}|_X(\alpha)) = Y$.

Proof. (1) $\Rightarrow$ (2) It must be true that $\mathrm{im}|_X(\alpha)^*$ is $\leq_{\mathrm{e}}$-dense over Y^* and thus $\mathrm{im}|_X(\alpha)^*$ is ϱ-dense over Y^* by Lemma 1(2). Consequently, α preserves ϱ-density by Corollary 4 and the fact that ϱ must be reflexive since $\leq_{\mathrm{e}}$ is and $\leq_{\mathrm{e}} \subseteq \varrho$.

(2) $\Rightarrow$ (3) This is immediate by Lemma 2 and the fact that ϱ must be reflexive and hence there must exist some ϱ-dense language over X^*.

(3) $\Rightarrow$ (1) Assume $\mathrm{alph}(\mathrm{im}|_X(\alpha)^*) = Y$. Let $w \in Y^*$ with $w = a_1 \cdots a_n, a_i \in Y, 1 \leq i \leq n$. For each a_i, there exists $x_i \in \mathrm{im}|_X(\alpha)$ such that $a_i \in \mathrm{alph}(x_i)$. Let $v = x_1 \cdots x_n \in \mathrm{im}|_X(\alpha)^*$. Also, $w \leq_{\mathrm{e}} v$ and so $\mathrm{im}|_X(\alpha)^*$ is $\leq_{\mathrm{e}}$-dense over Y^*. Hence, α preserves $\leq_{\mathrm{e}}$-density by Corollary 4. □

Using the division relation is identical to using both the prefix and suffix relations.

Proposition 7. *Let $\alpha : X^* \to Y^*$ be a homomorphism. Then α preserves $\leq_{\mathrm{d}}$-density if and only if α preserves both $\leq_{\mathrm{p}}$ and $\leq_{\mathrm{s}}$ density. Thus, $H(\leq_{\mathrm{d}}) = (H(\leq_{\mathrm{p}}) \cap H(\leq_{\mathrm{s}}))$.*

Proof. Assume that α preserves $\leq_{\mathrm{d}}$-density. Thus, $\mathrm{im}|_X(\alpha)^*$ is $\leq_{\mathrm{d}}$-dense over Y^* by Corollary 4. However, $\leq_{\mathrm{d}} \subseteq \leq_{\mathrm{p}}$ and $\leq_{\mathrm{d}} \subseteq \leq_{\mathrm{s}}$ and by Lemma 1(2), $\mathrm{im}|_X(\alpha)^*$ is $\leq_{\mathrm{p}}$-dense and also $\leq_{\mathrm{s}}$-dense. Again, using Corollary 4, α-preserves $\leq_{\mathrm{p}}$-density and also $\leq_{\mathrm{s}}$-density.

Assume that α preserves both $\leq_{\mathrm{s}}$-density and $\leq_{\mathrm{p}}$-density. Therefore, $\mathrm{im}|_X(\alpha)^*$ is $\leq_{\mathrm{s}}$-dense and also $\leq_{\mathrm{p}}$-dense by Corollary 4. Let $w \in Y^*$. Then, there exists $u_1 \in \mathrm{im}|_X(\alpha)^*$ and $u_2 \in \mathrm{im}|_X(\alpha)^*$ such that $w \leq_{\mathrm{p}} u_1$ and $w \leq_{\mathrm{s}} u_2$. Hence, there exists $x, y \in Y^*$ such that $u_1 = wx$ and $u_2 = yw$. Moreover, u_1 and u_2 are in $\mathrm{im}|_X(\alpha)^*$ and so $u_1 u_2 \in \mathrm{im}|_X(\alpha)^*$. Indeed, $w \leq_{\mathrm{d}} u_1 u_2$ and so $\mathrm{im}|_X(\alpha)^*$ is $\leq_{\mathrm{d}}$-dense over Y^* and α preserves $\leq_{\mathrm{d}}$-density by Corollary 4. □

We can collapse part of the hierarchy of Corollary 5 using the commutation and division relations as seen by the following proposition.

Proposition 8. *Let* $\alpha : X^* \to Y^*$ *be a homomorphism. Then* α *preserves* $\leq_{\mathrm{c}}$*-density if and only if* α *preserves* $\leq_{\mathrm{f}}$*-density. Thus,* $H(\leq_{\mathrm{f}}) = H(\leq_{\mathrm{c}})$*.*

Proof. Assume that α preserves $\leq_{\mathrm{c}}$-density. Hence, $\mathrm{im}|_X(\alpha)^*$ is $\leq_{\mathrm{c}}$-dense over Y^*, by Proposition 4. Let $w \in Y^*$. Then, there exists $v \in \mathrm{im}|_X(\alpha)^*$ such that $w \leq_{\mathrm{c}} v$; that is, there exists $x \in Y^*$ such that $v = wx = xw$. If $w = \lambda$, then $\lambda \leq_{\mathrm{f}} \lambda \in \mathrm{im}|_X(\alpha)^*$. If $x = \lambda$, then $w = v$ and $w \leq_{\mathrm{f}} v$. Assume then, that $w \neq \lambda$ and $x \neq \lambda$. It is well-known (see for example Lemma 1.7 of [5]) that for two words r, s with $r \neq \lambda$ and $s \neq \lambda$, if $rs = sr$, then r and s are powers of a common word. Thus, there exists $u \in Y^*$ such that $x = u^n, w = u^m$ and hence $v = u^{n+m}$ with $u \in Y^+$ and $n, m \in \mathbb{N}_0$. Since $v = u^{m+n} \in \mathrm{im}|_X(\alpha)^*$, we obtain $v' = v^m = u^{m(m+n)} = w^{m+n} \in \mathrm{im}|_X(\alpha)^*$. Indeed, $w \leq_{\mathrm{f}} v' \in \mathrm{im}|_X(\alpha)^*$ and so $\mathrm{im}|_X(\alpha)^*$ is $\leq_{\mathrm{f}}$-dense over Y^* and α preserves $\leq_{\mathrm{f}}$-density by Corollary 4.

Assume that α preserves $\leq_{\mathrm{f}}$-density. Thus, $\mathrm{im}|_X(\alpha)^*$ is $\leq_{\mathrm{f}}$-dense over Y^* by Corollary 4. However, $\leq_{\mathrm{f}} \subseteq \leq_{\mathrm{c}}$ and by Lemma 1(2), $\mathrm{im}|_X(\alpha)^*$ is $\leq_{\mathrm{c}}$-dense. Again, by Corollary 4, α preserves $\leq_{\mathrm{c}}$-density. □

This shows that the converse of Theorem 2(1) is not true because $\leq_{\mathrm{f}} \subsetneq \leq_{\mathrm{c}}$. We observe the following with respect to the difference between prefix and suffix density which will become useful in separating some parts of the hierarchy.

Proposition 9. *Let* $L \subseteq Y^*$*. Then* L *is* $\leq_{\mathrm{p}}$*-dense if and only if* L^R *is* $\leq_{\mathrm{s}}$*-dense. Equivalently,* L *is* $\leq_{\mathrm{s}}$*-dense if and only if* L^R *is* $\leq_{\mathrm{p}}$*-dense.*

Proof. Suppose L is $\leq_{\mathrm{p}}$-dense. Let $w \in Y^*$ and consider w^R. Indeed, $w^R \leq_{\mathrm{p}} v$ for some $v \in L$; that is, $v = w^r x, x \in Y^*$. Then $v^R = x^R w \in L^R$. Furthermore, $(w, v^r) \in \leq_{\mathrm{s}}$.

Conversely, suppose L^R is $\leq_{\mathrm{s}}$-dense. Let $w \in Y^*$ and consider w^R. Then $w^R \leq_{\mathrm{s}} v^R$ for some $v \in L$; that is, $v^R = xw^R$ for some $x \in Y^*$. Indeed, $v = wx^R$ and $(w, v) \in \leq_{\mathrm{p}}$. □

We now turn to separating the hierarchy of Corollary 5 between the infix and embedding relations. The following two families can be separated by showing that they are not densely equivalent using the language $\{aba\}$.

Proposition 10. $H(\leq_{\mathrm{i}}) \subsetneq H(\leq_{\mathrm{e}})$*.*

Proof. Consider the language $L_1 = \{aba\}$ and let α be a homomorphism that maps a to aba. Then α preserves $\leq_{\mathrm{e}}$-density by Proposition 6.

Suppose that L_1^* is $\leq_{\mathrm{i}}$-dense. Let $w = bb$. Clearly, bb is not an infix of any word in L_1^*.

Since L_1^* is $\leq_{\mathrm{e}}$-dense but L_1^* is not $\leq_{\mathrm{i}}$-dense, it follows from Theorem 2(3) that $H(\leq_{\mathrm{i}}) \subsetneq H(\leq_{\mathrm{e}})$. □

In the following, we are able to separate the the homomorphisms which preserve both prefix and suffix density with those that preserve each of prefix and suffix individually. Moreover, prefix and suffix are both incomparable.

For the proofs which follow, for $n \in \mathbb{N}_0$, let $\pi(n)$ be 0 if n is even and 1 otherwise.

Proposition 11. *For* $z \in \{\mathrm{p}, \mathrm{s}\}$,

$$(H(\leq_{\mathrm{p}}) \cap H(\leq_{\mathrm{s}})) \subsetneq H(\leq_z) \subsetneq (H(\leq_{\mathrm{p}}) \cup H(\leq_{\mathrm{s}})).$$

Also, $H(\leq_{\mathrm{p}}) \not\subseteq H(\leq_{\mathrm{s}})$ *and* $H(\leq_{\mathrm{s}}) \not\subseteq H(\leq_{\mathrm{p}})$.

Proof. Consider the language $L = \{a^2, b, ab\}$ and let $Y = \{a, b\}$. We want to show that L^* is $\leq_{\mathrm{p}}$-dense over Y^*. Let $w \in Y^*$. If $w \in \{\lambda\} \cup a^* \cup b^*$ then there exists $v \in L^*$ such that $w \leq_{\mathrm{p}} v$. Otherwise,

$$w = a^{n_1} b^{m_1} a^{n_2} b^{m_2} \cdots a^{n_k} b^{m_k},$$

with $n_1, m_k \geq 0, n_2, \ldots, n_k, m_1, \ldots, m_{k-1} > 0$. Consider,

$$v = (a^2)^{\lfloor n_1/2 \rfloor} (ab)^{\pi(n_1)} (b)^{m_1 - \pi(n_1)} \cdots (a^2)^{\lfloor n_k/2 \rfloor} (ab)^{\lfloor n_k \rfloor} (b)^{n_k - \pi(n_k)}.$$

Indeed, $w \leq_{\mathrm{p}} v$. Thus, L^* is $\leq_{\mathrm{p}}$-dense over $\{a, b\}^*$.

We would like to show that L^* is not $\leq_{\mathrm{s}}$-dense over $\{a, b\}^*$. Suppose otherwise. Consider the word $w = ba$. Then there exists $v = u_1 \cdots u_k$ with $ba \leq_{\mathrm{s}} v$ and $k \geq 1$. Since ba ends with the letter a, necessarily $u_k = a^2$, but $a^2 \neq ba$, a contradiction.

By Theorem 2, this shows that the $\leq_{\mathrm{p}}$ relation is not densely smaller than the $\leq_{\mathrm{s}}$ relation and that $H(\leq_{\mathrm{p}}) \not\subseteq H(\leq_{\mathrm{s}})$. Furthermore, consider the language L^R. By Proposition 9, L^R is $\leq_{\mathrm{s}}$-dense but is not $\leq_{\mathrm{p}}$-dense. Therefore, by Theorem 2, this shows that the $\leq_{\mathrm{s}}$ relation is not densely smaller than the $\leq_{\mathrm{p}}$ relation and that $H(\leq_{\mathrm{s}}) \not\subseteq H(\leq_{\mathrm{p}})$. Therefore, $(H(\leq_{\mathrm{p}}) \cap H(\leq_{\mathrm{s}})) \subsetneq H(\leq_{\mathrm{p}})$ and $(H(\leq_{\mathrm{p}}) \cap H(\leq_{\mathrm{s}})) \subsetneq H(\leq_{\mathrm{s}})$. In addition, $H(\leq_{\mathrm{p}}) \subsetneq (H(\leq_{\mathrm{p}}) \cup H(\leq_{\mathrm{s}}))$ and $H(\leq_{\mathrm{s}}) \subsetneq (H(\leq_{\mathrm{p}}) \cup H(\leq_{\mathrm{s}}))$. □

We still need to separate $=_{\mathrm{e}}$-density from $\leq_{\mathrm{f}}$-density.

Proposition 12. $H(=_{\mathrm{e}}) \subsetneq H(\leq_{\mathrm{f}})$

Proof. Let $L = \{a^2, b, ba, ab\}$. and let $Y = \{a, b\}$. First we define a homomorphism α which maps a to a^2, b to b, c to ba and d to ab. Indeed, α does not preserve $=_{\mathrm{e}}$-density by Proposition 5 and since $a \notin \mathrm{im}|_X(\alpha)$.

We now want to show that $L^* = \mathrm{im}|_X(\alpha)^*$ is $\leq_{\mathrm{f}}$-dense and hence α preserves $\leq_{\mathrm{f}}$-density. Let $w \in Y^*$. If $w \in \{\lambda\} \cup a^* \cup b^*$, then there exists $v \in L^*$ with $w \leq_{\mathrm{f}} v$. Otherwise,

$$w = b^{n_1} a^{m_1} b^{n_2} a^{m_2} \cdots b^{n_k} a^{m_k},$$

with $k \geq 1, n_1, m_k \geq 0, m_1, \ldots, m_{k-1}, n_2, \ldots, n_k > 0$.

Case 1: Assume that m_k is even. Then we rewrite

$$\begin{aligned} w \quad = \quad & (b)^{n_1} (a^2)^{\lfloor m_1/2 \rfloor} (ab)^{\pi(m_1)} (b)^{n_2 - \pi(m_2)} \cdots \\ & \cdots (a^2)^{\lfloor m_{k-1}/2 \rfloor} (ab)^{\pi(m_{k-1})} (b)^{n_k - \pi(m_{k-1})} (a^2)^{m_k/2}. \end{aligned}$$

Furthermore, $(w, w) \in \leq_{\rm f}$.

Case 2: Assume that $n_1 > 0$ and m_k is odd. Then we rewrite

$$\begin{aligned} w &= (b)^{n_1-\pi(m_1)}(ba)^{\pi(m_1)}(a^2)^{\lfloor m_1/2 \rfloor} \cdots \\ &\quad \cdots (b)^{n_k-\pi(m_k)}(ba)^{\pi(m_k)}(a^2)^{\lfloor m_k/2 \rfloor}. \end{aligned}$$

Furthermore, $(w, w) \in \leq_{\rm f}$.

Case 3: Assume that $n_1 = 0$, m_k is odd and m_1 is even. Then we rewrite

$$\begin{aligned} w &= (a^2)^{m_1/2}(b)^{n_2-\pi(m_2)}(ba)^{\pi(m_2)}(a^2)^{\lfloor m_2/2 \rfloor} \cdots \\ &\quad \cdots (b)^{n_k-\pi(m_k)}(ba)^{\pi(m_k)}(a^2)^{\lfloor m_k/2 \rfloor}. \end{aligned}$$

Furthermore, $(w, w) \in \leq_{\rm f}$.

Case 4: Assume that $n_1 = 0$, m_k is odd and m_1 is odd. Then

$$w^2 = a^{m_1}b^{n_2}a^{m_2} \cdots b^{n_k}a^{m_k}a^{m_1}b^{n_2}a^{m_2} \cdots b^{n_k}a^{m_k}.$$

Then we rewrite

$$\begin{aligned} w^2 &= (a^2)^{\lfloor m_1/2 \rfloor}(ab)^{\pi(m_1)}(b)^{n_2-\pi(m_1)} \cdots \\ &\quad \cdots (a^2)^{\lfloor m_{k-1}/2 \rfloor}(ab)^{\pi(m_{k-1})}(b)^{n_k-\pi(m_{k-1})} \\ &\quad (a^2)^{(m_k+m_1)/2}(b)^{n_2-\pi(m_2)}(ba)^{\pi(m_2)}(a^2)^{\lfloor m_2/2 \rfloor} \cdots \\ &\quad \cdots (b)^{n_k-\pi(m_k)}(ba)^{\pi(m_k)}(a^2)^{\lfloor m_k/2 \rfloor}. \end{aligned}$$

Furthermore, $(w, w^2) \in \leq_{\rm f}$.

Therefore, $L^* = {\rm im}|_X(\alpha)^*$ is $\leq_{\rm f}$-dense and hence α preserves $\leq_{\rm f}$-density by Corollary 4.

□

Indeed, the last case in the above proof is necessary as shown by the example where $w = aba$. If $w \in L^*$, then $w = u_1u_2$, where u_1 is necessarily ab and u_2 is a. However, $a \notin L$. That being said, $w^2 = abaaba = (ab)(aa)(ba) \in L^*$.

Note that in the proof above, it would have been immediate to show that L^* was $\leq_{\rm d}$-dense since $L = L^R$ and thus L^* is both $\leq_{\rm p}$ and $\leq_{\rm s}$ dense by Proposition 9 and thus is $\leq_{\rm d}$-dense by Proposition 11. That being said, it was not immediate that L^* was $\leq_{\rm f}$-dense.

Next, we separate the union of the homomorphisms preserving prefix and suffix density with those preserving infix density.

Proposition 13. $H(\leq_{\rm p}) \cup H(\leq_{\rm s}) \subsetneq H(\leq_{\rm i})$.

Proof. The inclusion is immediate from Corollary 5. For the strictness, consider the language $L = \{a^2, b, bab, aba, aaab, baaa\} \subseteq Y^*$ where $Y = \{a, b\}$. We first prove the following claim:

Claim 1. *For each* $n > 1, m \geq 0, z = (ba)^m ba^n \in L^*$.

Proof. First assume that n is odd and $m = 0 \mod 3$. Then

$$z = ((bab)(aba))^{m/3}(baaa)(a^2)^{\lfloor n/2 \rfloor -1}.$$

Assume that n is odd and $m = 1 \mod 3$. Then

$$z = (b)(aba)((bab)(aba))^{(m-1)/3}(a^2)^{\lfloor n/2 \rfloor}.$$

Assume that n is odd and $m = 2 \mod 3$. Then

$$z = ((bab)(aba))^{(m+1)/3}(a^2)^{\lfloor n/2 \rfloor}.$$

Assume that n is even and $m = 0 \mod 3$. Then

$$z = ((bab)(aba))^{m/3}(b)(a^2)^{n/2}.$$

Assume that n is even and $m = 1 \mod 3$. Then

$$z = (bab)((aba)(bab))^{(m-1)/3}(a^2)^{n/2}.$$

Assume that n is even and $m = 2 \mod 3$. Then

$$z = (b)(aba)((bab)(aba))^{(m-2)/3}(b)(a^2)^{n/2}.$$

□

Let $Y_\$ = Y \cup \{\$_1, \$_2, \$_3, \$_4\}$, where $\$_1, \$_2, \$_3, \$_4$ are new symbols. We will show that L^* is $\leq_{\mathrm{i}}$-dense over Y^*. We define four rewriting rules as follows:

$$\begin{array}{lll} w_1 \rightarrow_1 w_2 & \text{if and only if} & w_1 = x(ba)^m ba^n cy, w_2 = x\$_1 cy, x, y \in Y_\$^*, \\ & & m > 0, n > 1, c \in \{b, \$_1\}, x, y \in Y_\$^*, ba \not\leq_{\mathrm{s}} x. \\ w_1 \rightarrow_2 w_2 & \text{if and only if} & w_1 = xba^n cy, w_2 = x\$_2 cy, x, y \in Y_\$^*, \\ & & n > 1, c \in \{b, \$_1, \$_2\}, x, y \in Y_\$^*. \\ w_1 \rightarrow_3 w_2 & \text{if and only if} & w_1 = xabay, w_2 = x\$_3 y, x, y \in Y_\$^*. \\ w_1 \rightarrow_4 w_2 & \text{if and only if} & w_1 = xbaby, w_2 = x\$_4 y, x, y \in Y_\$^*. \end{array}$$

Furthermore, for each $i \in \{1, 2, 3, 4\}$, let $w_1 \rightarrow_i^{(*)} w_2$ if and only if there exists $n \in \mathbb{N}$ and n words $y_1, \ldots, y_n \in Y_\$^*$ such that $w_1 = y_1 \rightarrow_i y_2 \rightarrow_i \cdots \rightarrow_i y_n = w_2$ and there does not exist any $z \in Y_\$^*$ such that $y_n \rightarrow_i z$.

Let $w \in Y^*$. We would like to create $x, y \in Y^*$ such that $xwy \in L^*$. Let w_1, w_2, w_3, w' be any words in $Y_\$^*$ such that $w \rightarrow_1^{(*)} w_1 \rightarrow_2^{(*)} w_2 \rightarrow_3^{(*)} w_3 \rightarrow_4^{(*)} w'$. Let $w' = x_1 \$_{\gamma_1} x_2 \$_{\gamma_2} \cdots \$_{\gamma_{k-1}} x_k$ and $w_3 = z_1 \$_{\tau_1} z_2 \$_{\tau_2} \cdots \$_{\tau_{l-1}} z_l$ where $k, l \geq 1, x_j, z_j \in Y^*$ and all \$ symbols are in $\{\$_1, \$_2, \$_3, \$_4\}$.

Now, examining the rewriting rules, we see $(ba)^m ba^n, ba^n \in L^*, m > 0, n > 1$ by Claim 1 and also $aba, bab \in L^*$. Thus, it is sufficient to find $x, y \in Y^*$ such that $xx_1, x_2, x_3, \ldots, x_{k-1}, x_k y \in L^*$ as this implies $xwy \in L^*$.

We will show that for each $i, 2 \leq i \leq k-1$ with $k \geq 2$, it must be true that $a \notin \text{alph}(x_i)$. Suppose otherwise.

First, $a^n, n > 1$ cannot be an infix of any $z_2, \ldots, z_l$ if $l > 1$. Otherwise, $\$_j a^n$ must be an infix of w_3 for some $j \in \{1,2\}$ and the first two rewriting rules can only leave a b or a \$ symbol after a \$ symbol. In addition, $ba^n, n > 1$ cannot be an infix of z_1. Thus, $a^n, n > 1$ cannot be an infix of x_i, otherwise $\$_{\gamma_{i-1}} a^n$ must be an infix of w', γ_{i-1} can be neither 1 nor 2 and if γ_{i-1} is 3 or 4, then $ba^n \leq_{\mathrm{i}} w_3$, a contradiction.

Thus, $x_i \in \{a, ab^n a, ab^m, b^m a \mid n > 1, m \geq 1\}$ since a^n, aba, bab are not infixes of x_i for $n > 1$. Suppose that $x_i = av, v \in Y^*$. Then $\gamma_{i-1} \in \{3,4\}$. If it is 3, then $abaav \leq_{\mathrm{i}} w_2$, a contradiction. If it is 4, then $babav \leq_{\mathrm{i}} w_3$, a contradiction. Hence $x_i = b^m a, m \geq 1$ and $\$_{\gamma_{i-1}} b^m a \$_{\gamma_i} \leq_{\mathrm{i}} w'$. If $\gamma_i = 3$, then $baab \leq_{\mathrm{i}} w_3$, a contradiction. If $\gamma_i = 4$, then $aba \leq_{\mathrm{i}} w_4$. So γ_i is either 1 or 2. If it is 2, then $baba^n c \leq_{\mathrm{i}} w_1, c \in \{b, \$_1\}$. Furthermore, it cannot be 1 since $ba \leq_{\mathrm{s}} b^m a$.

Hence, for each i, $2 \leq i \leq k-1$ with $k \geq 2$, it is true that $x_i \in b^* \subseteq L^*$. Thus, we still need to verify that there exists x, y with $xx_1, x_k y \in L^*$. Indeed, x_1 must be of the form $a^{n_1} b^{n_2} a^{n_3}, n_1, n_2 \geq 0$ and n_3 either 0 or 1. If $n_3 = 0$, then x can be empty if n is even and a if n_1 is odd. Also, $n_2 > 0$ necessarily. So assume $n_2 > 0$ and $n_3 = 1$. We reach a contradiction similarly to the case above. Similarly for the case of x_1, x_k must also equal $a^{n_1} b^{n_2} a^{n_3}, n_1, n_2 \geq 0$, n_3 either 0 or 1. If $n_1 = 0$ or $n_2 = 0$, we are done. Otherwise, $\$_{\gamma_{k-1}}$ must be $\$_3$ or $\$_4$. If it is $\$_3$, then $abaa^{n_1} b \leq_{\mathrm{i}} w_2$, a contradiction. If it is $\$_4$, then $baba \leq_{\mathrm{i}} w_3$, a contradiction. Lastly, if $k = 1$ then $x \in a^* b^* a^*$, and we are done.

To show that L^* is not $\leq_{\mathrm{p}}$-dense, let $w = abba$. It is clear that there does not exist any $v \in L^*$ such that $w \leq_{\mathrm{p}} v$. Thus L^* is not $\leq_{\mathrm{p}}$-dense. Moreover, L^* is not $\leq_{\mathrm{s}}$-dense, since $L = L^R$ and by Proposition 9. Then $H(\leq_{\mathrm{p}}) \cup H(\leq_{\mathrm{s}}) \subsetneq H(\leq_{\mathrm{i}})$. □

Finally, we determine that the inclusion between $H(\leq_{\mathrm{f}})$ and $H(\leq_{\mathrm{d}})$ is strict.

Proposition 14. $H(\leq_{\mathrm{f}}) \subsetneq (H(\leq_{\mathrm{p}}) \cap H(\leq_{\mathrm{s}}))$.

Proof. The inclusion is immediate from Theorem 2. For the strictness of the inclusion, consider the language $L = Y^4 \cup \{b\} \setminus \{baab\}$ over $Y = \{a, b\}$. We need to prove that L^* is both prefix and suffix dense. It is enough to show that it is prefix dense, since $L = L^R$ using Proposition 9.

For $w = a_1 \cdots a_m \in Y^+, m \geq 1, a_j \in Y, 1 \leq j \leq m$, let $\chi(w, n) = d_1 d_2 d_3 d_4, d_i \in Y, 1 \leq i \leq 4$, and $d_i = a_{i+n-1}$ for $1 \leq i \leq 4$, where we define $a_{m+1} = a_{m+2} = a_{m+3} = a$

Let $w \in Y^*$. If $w = \lambda$ then we can construct $v \in L^*$ such that $w \leq_{\mathrm{p}} v$. Assume then that $w \in Y^+$. Consider the two sequences $\{c_i\}_{i \in \mathbb{N}}$ over $\mathbb{N}$ and $\{u_i\}_{i \in \mathbb{N}}$ over Y^* defined as follows:

$$c_i = \begin{cases} 1, & \text{if } i = 1, \\ c_{i-1} + 4, & \text{if } i > 1, c_{i-1} + 4 \leq |w| \text{ and } \chi(w, c_{i-1}) \neq baab, \\ c_{i-1} + 1, & \text{if } i > 1, c_{i-1} + 1 \leq |w| \text{ and } \chi(w, c_{i-1}) = baab, \\ \text{undefined}, & \text{otherwise,} \end{cases}$$

and

$$u_i = \begin{cases} \chi(w, c_i), & \text{if } c_i > 0 \text{ and } \chi(w, c_i) \neq baab, \\ b, & \text{if } c_i > 0 \text{ and } \chi(w, c_i) = baab, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

For each i, $1 \leq i < l$, $|u_1 \cdots u_i| = c_{i+1} - 1$. Let l be the largest integer such that u_l is defined (which must exist by the definitions) and consider the word $v = u_1 \cdots u_l$. Thus l is also the largest integer such that c_l is defined.

Claim 2. *For each* i, $1 \leq i \leq l$, $u_1 \cdots u_l \leq_{\mathrm{p}} waaa$.

Proof. The claim follows when $i = 1$. Let j satisfy $1 \leq j < l$ and assume that $u_1 \cdots u_j \leq_{\mathrm{p}} waaa$. As noted above, $c_{j+1} = |u_1 \cdots u_j| + 1$. Consider u_{j+1} which is the same as $\chi(w, c_{j+1})$ if $\chi(w, c_{j+1}) \neq baab$ and b otherwise. If $j + 1 < l$, then $u_1 \cdots u_{j+1} \leq_{\mathrm{p}} w$ and if $j + 1 = l$, then $u_1 \cdots u_{j+1} \leq_{\mathrm{p}} waaa$. □

Thus, $u_1 \cdots u_l \leq_{\mathrm{p}} waaa$. We would like to still show that $w \leq_{\mathrm{p}} u_1 \cdots u_l$. This follows since $0 \leq |u_1 \cdots u_l| - |w| \leq 3$. Furthermore, $\chi(w, c_i) \neq baab$ is in L and b is also in L. Hence, L^* is $\leq_{\mathrm{s}}$-dense.

We now show that L^* is not $\leq_{\mathrm{f}}$-dense. Assume otherwise and let $w = baab$. Then there exists $n \in \mathbb{N}$ such that $w^n \in L^*$. That is, $(baab)^n = u_1 \cdots u_l, u_i \in L, 1 \leq i \leq l$. Necessarily, $u_1 = b$ and $u_i = aabb$ for each i, $1 \leq i \leq l$, a contradiction. □

Combining Corollary 5 with Propositions 7, 8, 10, 11, 12, 13 and 14, we get the following hierarchy which is far more detailed than the one of Corollary 5.

Theorem 3. *For* $z \in \{\mathrm{p}, \mathrm{s}\}$,

$$H(=_{\mathrm{e}}) \subsetneq H(\leq_{\mathrm{f}}) = H(\leq_{\mathrm{c}}) \subsetneq H(\leq_{\mathrm{d}}) = (H(\leq_{\mathrm{p}}) \cap H(\leq_{\mathrm{s}})) \subsetneq H(\leq_z) \subsetneq (H(\leq_{\mathrm{p}}) \cup H(\leq_{\mathrm{s}})) \subsetneq H(\leq_{\mathrm{i}}) \subsetneq H(\leq_{\mathrm{e}}).$$

Moreover, $H(\leq_{\mathrm{p}}) \not\subseteq H(\leq_{\mathrm{s}})$ *and* $H(\leq_{\mathrm{s}}) \not\subseteq H(\leq_{\mathrm{p}})$.

This is quite different from the special case for endomorphisms in Theorem 1 and for weak-coding homorphisms in Corollary 3 where the hierarchy collapses.

The property of being densely equivalent was important to establish which parts of the hierarchy collapsed and which did not. Despite this, we used ad hoc techniques in order to determine which two relations were densely equivalent. It is an open question as to whether the results of this hierarchy can be condensed into a more general and concise formulation.

6 Deciding if a homomorphism preserves density

We turn briefly to the question of deciding whether or not a homomorphism preserves different types of density. It turns out that we have already done most of the difficult work for most types. The proposition uses the construct of an a-transducer, which is essentially a nondeterminsitic gsm which can output on λ-input [1].

Proposition 15. *Let $\alpha : X^* \to Y^*$ be an effectively given homomorphism and let ϱ be a binary relation on Γ^* which is transitive, reflexive and compatible with α. Then the following conditions hold:*

1. *If it is decidable whether a regular language $L \subseteq Y^*$ is ϱ-dense over Y^*, then it is decidable whether α preserves density.*

2. *If it is decidable whether $\varrho^{-1}(L) = Y^*$ for every regular language $L \subseteq Y^*$, then it is decidable whether α preserves density.*

3. *If there is an a-transducer M_ϱ which satisfies $M_\varrho(L) = \varrho^{-1}(L)$ for every $L \subseteq Y^*$, then it is decidable whether α preserves density.*

Proof. (1) This follows from Corollary 4 and the fact that $\mathrm{im}|_X(\alpha)$ is finite.

(2) Let $L = \mathrm{im}|_X(\alpha)^*$ which is regular. We can decide whether $\varrho^{-1}(L) = Y^*$. Furthermore, $\varrho^{-1}(L) = Y^*$ if and only if for every $u \in Y^*$, there exists $v \in L$ such that $(u, v) \in \varrho$. Thus, $\mathrm{im}|_X(\alpha)^*$ is ϱ-dense if and only if $\varrho^{-1}(\mathrm{im}|_X(\alpha)^*) = Y^*$, which is decidable.

(3) If there is an a-transducer (or indeed a nondeterministic gsm mapping) M_ϱ which satisfies $M_\varrho(R) = \varrho^{-1}(R)$, for every $R \subseteq Y^*$, then $L' = M_\varrho(\mathrm{im}|_X(\alpha)^*)$ is regular since the family of regular languages is closed under arbitrary a-transductions. Further, the universe problem (given a language L, is $L = Y^*$?) is decidable for the family of regular languages and thus we can decide if $L' = Y^*$. □

As an immediate consequence, we obtain decidability for the five relations $\le_{\mathrm{e}}, \le_{\mathrm{p}}, \le_{\mathrm{s}}, \le_{\mathrm{i}}, =_{\mathrm{e}}$. For the case of the equality relation, one can decide this property trivially using a much simpler characterization of Proposition 5, whereby, one need only check whether $Y \subseteq \mathrm{im}|_X(\alpha)$ in order to determine whether or not a homomorphism α preserves $=_{\mathrm{e}}$-density. Similarly, for the embedding relation, it follows from Proposition 6 that we need only verify that $\mathrm{alph}(\mathrm{im}|_X(\alpha)) = Y$.

In addition, by Proposition 7, we know a homomorphism preserves $\le_{\mathrm{d}}$-density if and only if it preserves both prefix and suffix density. Hence, by Proposition 15, we can decide whether a homomorphism preserves $\le_{\mathrm{d}}$-density.

The problem is not so easy to decide for power and commutation density however. We need to start with the following characterization.

Proposition 16. *Let $\alpha : X^* \to Y^*$ be a homomorphism. Then the following are equivalent:*

1. *α preserves $\le_{\mathrm{f}}$-density.*

2. *α preserves $\le_{\mathrm{c}}$-density.*

3. *For every $w \in Y^*$, there exists $v \in \mathrm{im}|_X(\alpha)^*$ and an integer n, $1 \le n \le \max(\alpha)$ such that $v = w^n$.*

4. *For every* $w \in Y^*$, *there exists an integer* n, $1 \leq n \leq \max(\alpha)$ *and either* $w \in \inf(\mathrm{im}|_X(\alpha))$ *and there exists* $v \in \mathrm{im}|_X(\alpha)^*$ *such that* $v = w^n$ *or there exists* $n+1$ *ordered pairs,*

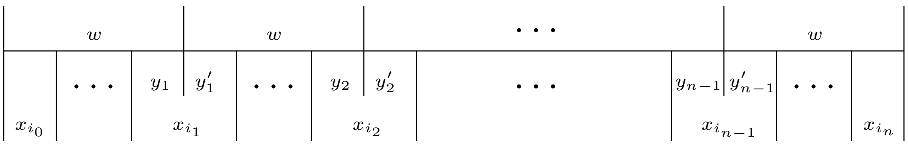
$$(y_0, y_0'), (y_1, y_1'), \ldots, (y_n, y_n'),$$

with $y_i y_i' \in \mathrm{im}|_X(\alpha)$ *for* $0 \leq i \leq n$, $y_0 = y_n' = \lambda, \{y_0', \ldots, y_{n-1}', y_1, \ldots, y_n\} \subseteq Y^+$ *and* $(y_i')^{-1} w (y_{i+1})^{-1} \in \mathrm{im}|_X(\alpha)^*$, $0 \leq i < n$.

Proof. (1) $\Leftrightarrow$ (2) Immediate from Lemma 8.

(1) $\Rightarrow$ (3) Assume that α preserves $\leq_{\mathrm{f}}$-density. Let $w \in Y^*$. Then there exists a minimal integer $n \geq 1$ and $v \in \mathrm{im}|_X(\alpha)^*$ with $v = w^n$. If $n \leq \max(\alpha)$ then we are done. Assume that $n > \max(\alpha)$. Thus, $w^n = x_1 x_2 \cdots x_m, x_i \in \mathrm{im}|_X(\alpha)$. Thus, for some integer j, $1 \leq j \leq \max(\alpha)$, there exists k_1, k_2 with $k_1 < k_2$ such that $|x_1 \cdots x_{k_1}| = l_1|w| + j$, $l_1 \in \mathbb{N}_0$ and $|x_1 \cdots x_{k_2}| = l_2|w| + j, l_2 \in \mathbb{N}_0$. Thus, consider $v' = x_1 \cdots x_{k_1} x_{k_2+1} \cdots x_m$. Then, $v' = w^{n+l_1-l_2} \in \mathrm{im}|_X(\alpha)^*$. This contradicts the minimality of n.

(3) $\Rightarrow$ (4) Let $w \in Y^*$. Then there exists $v \in \mathrm{im}|_X(\alpha)^*$ and an integer n, $1 \leq n \leq \max(\alpha)$ such that $v = w^n$ with n minimal. If $w \in \inf(\mathrm{im}|_X(\alpha))$, then we are done. Assume that $w \notin \inf(\mathrm{im}|_X(\alpha))$. Thus, $v = w^n = x_1 x_2 \cdots x_m$, $1 \leq n \leq \max(\alpha), x_i \in \mathrm{im}|_X(\alpha), 1 \leq i \leq m$ with $m > 1$. If $n = 1$, then there exists $(\lambda, x_1), (x_m, \lambda)$ such that $x_1, x_m \in \mathrm{im}|_X(\alpha)$ and $(x_1)^{-1} x_1 \cdots x_m (x_m)^{-1} \in \mathrm{im}|_X(\alpha)^*$ (m must be greater than 1 and if $m = 2$ then $(x_1)^{-1} x_1 \cdots x_m (x_m)^{-1} = \lambda \in \mathrm{im}|_X(\alpha)^*$). Assume that $n > 1$. Since $w \notin \inf(\mathrm{im}|_X(\alpha))$, there exists $i_1, i_1, \ldots, i_n$ such that $i_0 = 1 < i_1 < \cdots < i_{n-1} < i_n = m$ where $|x_1 \cdots x_{i_{j-1}}| < j|w| < |x_1 \cdots x_{i_j}|$, $1 \leq j \leq n-1$.



For each j, consider the ordered pair (y_j, y_j') where y_j is the prefix of x_{i_j} of length $j|w| - |x_1 \cdots x_{i_{j-1}}|$ and $y_j' = (y_j)^{-1} x_{i_j}$. Both $y_j \neq \lambda$ and $y_j' \neq \lambda$ by the minimality of n. So we have ordered pairs,

$$(\lambda, x_1), (y_1, y_1'), \ldots, (y_{n-1}, y_{n-1}'), (x_m, \lambda).$$

Also, let $y_0' = x_1$ and $y_n = x_m$. Indeed, $y_j y_j' \in \mathrm{im}|_X(\alpha)$ for all j, $1 \leq j \leq n-1$ and $x_1, x_m \in \mathrm{im}|_X(\alpha)$. Moreover, for each j, $1 \leq j \leq n-2, x_{i_j+1} \cdots x_{i_{j+1}-1} \in \mathrm{im}|_X(\alpha)^*, x_2 \cdots x_{i_1-1} \in \mathrm{im}|_X(\alpha)^*$ and $x_{i_{n-1}+1} \cdots x_{m-1} \in \mathrm{im}|_X(\alpha)^*$. Hence, for each k, $0 \leq k < n$, $(y_k')^{-1} w (y_{k+1})^{-1} \in \mathrm{im}|_X(\alpha)^*$.

(4) $\Rightarrow$ (1) Let $w \in Y^*$. Then there exists $1 \leq n \leq \max(\alpha)$ satisfying the stated assumptions. If $w \in \inf(\mathrm{im}|_X(\alpha))$ then there exists $v \in \mathrm{im}|_X(\alpha)^*$ with $v = w^n$, by assumption. Otherwise, there exists $n+1$ ordered pairs, $(y_0, y_0'), \ldots, (y_n, y_n')$ where $y_i y_i' \in \mathrm{im}|_X(\alpha), 0 \leq i \leq n, y_0 = y_n' = \lambda$ and $\{y_0', \ldots, y_{n-1}', y_1, \ldots, y_n\} \subseteq Y^+$ and $z_i = (y_i')^{-1} w (y_{i+1})^{-1} \in \mathrm{im}|_X(\alpha)^*, 0 \leq i < n$. Consider

$$v = y_0' z_0 y_1 y_1' z_1 y_2 \cdots y_{n-1} y_{n-1}' z_{n-1} y_n.$$

Indeed, $v \in \mathrm{im}|_X(\alpha)^*$. Furthermore, $v = w^n$. □

We can then use this characterization to show that determining whether a homomorphism preserves power density amounts to deciding the universe problem on regular languages. The proof uses an NFA which nondeterministically guesses the $n + 1$ ordered pairs in the proof above.

Proposition 17. *Let $\alpha : X^* \to Y^*$ be an effectively given homomorphism. Then we can construct a regular language L whereby $L = Y^*$ if and only if α preserves $\leq_{\mathrm{f}}$-density.*

Proof. Let M be a nondeterministic finite automata which nondeterministically guesses an integer n, $1 \leq n \leq \max(\alpha)$ and $n + 1$ ordered pairs

$$(y_0, y'_0), \ldots, (y_n, y'_n),$$

where $y_0 = y'_n = \lambda$, $y_i y'_i \in \mathrm{im}|_X(\alpha)$ for each i, $0 \leq i \leq n$ and also the set $\{y'_1, \ldots, y'_n, y_2, \ldots, y_{n+1}\} \subseteq Y^+$. Then, on input $w \in Y^* \setminus \inf(\mathrm{im}|_X(\alpha))$ (intersect Y^* with the complement of $\inf(\mathrm{im}|_X(\alpha))$ which is regular), in parallel, for each i, $0 \leq i \leq n$, M verifies that $(y'_i)^{-1} w (y_{i+1})^{-1} \in \mathrm{im}|_X(\alpha)^*$. Let $L_1 = L(M)$. Furthermore, let $L_2 = \{w \mid w \in \inf(\mathrm{im}|_X(\alpha)), w^n \in \mathrm{im}|_X(\alpha)^*, 1 \leq n \leq \max(\alpha)\}$. It is clear that L_2 is finite and can be effectively constructed. Let $L = L_1 \cup L_2$. Then L is regular and $L = Y^*$ if and only if α preserves $\leq_{\mathrm{f}}$-density, by Proposition 16. □

Combining Proposition 17 and 16, and the fact that the universe problem for regular languages is decidable [2], we get decidability for $\leq_{\mathrm{c}}$- and $\leq_{\mathrm{f}}$-density. Collecting the decidability over all relations together, we obtain:

Corollary 6. *Let $\alpha : X^* \to Y^*$ be an effectively given homomorphism. Then it is decidable whether α preserves ϱ-density where $\varrho \in \{=_{\mathrm{e}}, \leq_{\mathrm{f}}, \leq_{\mathrm{c}}, \leq_{\mathrm{d}}, \leq_{\mathrm{p}}, \leq_{\mathrm{s}}, \leq_{\mathrm{i}}, \leq_{\mathrm{e}}\}$.*

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