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Sufficient Conditions for Order-Independency in Sequential Thinning*

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Abstract

The main issue of this paper is to introduce some conditions for template-based sequential thinning that are capable of producing the same skeleton for a given binary image, independent of the visiting order of object points. As an example, we introduce two order-independent thinning algorithms for 2D binary images that satisfy these conditions.

Keywords: skeleton, sequential thinning, order-independency, digital topology, topology preservation

1 Introduction

Skeleton provides a reduced-dimensional representation, which describes the general shape of objects [10]. Thinning is a widely used strategy for skeletonization that is based on an iterative peeling of the object boundary [8, 11]. Several parallel and sequential alternatives have been proposed for this method. In the sequential case, an iteration step of the thinning process is usually performed in two phases. Algorithm 1 shows the "classic" sequential thinning scheme.

Basically, a "deletable" point must not be a so-called endpoint (which is important in the view of shape preservation), and its removal must be topology-preserving. A usual way to define "deletable" points is to construct a set of matching templates $\mathcal{T}$. In this case, an object point can be considered as deletable, if it matches at least one template $T \in \mathcal{T}$. Further on, we call such object points as $\mathcal{T}$*-deletable*. Without loss of generality, we assume square templates of size $(2k+1) \times (2k+1)$ $(k \in \mathbb{N})$. The *central point* of a template is then the point with position (k, k).

As sequential thinning algorithms remove only one point at a time, topology preservation can be much easier guaranteed than in the parallel case [6, 8]. However, Algorithm 1 suffers from the problem that it may produce various skeletons for

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Algorithm 1:

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repeat
    // Phase 1: contour tracking
    mark all border points
    // Phase 2: reduction
    foreach marked point p do
        if p is "deletable" in the actual image then
            delete p
until no points are deleted
```

different visiting orders of border points. Finding order-independent strategies (i. e., algorithms that produce the same skeletons for any visiting orders) is a key problem in sequential thinning.

First Ranwez and Soille [9], then Iwanowski and Soille [3] investigated this problem. The main disadvantage of the their order-independent algorithms lies in the fact that they are basically anchor preserving reductive shrinking methods [2]. This means that, as a preprocessing step, endpoints must be previously detected as anchors. An order-independent algorithm with built-in endpoint-criterion has been proposed by Kardos, Németh, and Palágyi [4]. Their method is based on the classification of simple points. For this purpose, simple points are grouped into four sets in the first phase of an iteration, which means that this solution lies far from the sequential thinning scheme according to Algorithm 1.

In this paper we formulate some criteria which are sufficient for order-independency. The paper is organized as follows. In Section 2, we introduce some basic notions of digital topology, Section 3 presents the mentioned sufficient conditions and the proof of their correctness. Using these conditions, in Section 4, two possible 2D template sets, $\mathcal{T}^1$ and $\mathcal{T}^2$, are proposed. In Section 5, it will be proved that if we consider $\mathcal{T}^1$- or $\mathcal{T}^2$-deletable points as "deletable" in Algorithm 1, we get order-independent and topologically correct thinning algorithms. Finally, some experimental results are shown in Section 6.

2 Basic Notions

Applying the basic concepts of digital topology as reviewed in [5], we introduce some additional notions for our purposes in this section.

First, we give an extension of the definition of a 2-dimensional $(8,4)$ binary digital picture: a 2-dimensional $(8,4)$ *labeled binary digital picture* (in the following referred to as $(8,4)$ picture or simply as picture) can be described with the 5-tuple $(\mathbb{Z}^2, 8, 4, \mathbf{B}, \mathbf{B}^+)$, where $\mathbb{Z}^2$ is the set of picture points, $\mathbf{B} \subseteq \mathbb{Z}^2$ is the set of black points, its complement, $\mathbb{Z}^2 \backslash \mathbf{B}$ is the set of white points, $\mathbf{B}^+ \subseteq \mathbf{B}$ denotes the set of *active* black points, and $\mathbf{B} \backslash \mathbf{B}^+$ is the set of *inactive* black points.

A *black component* is a maximal 8–connected set of black points, while a *white* *component* is defined as a maximal 4–connected set of white points.

A black point p in the picture $(\mathbb{Z}^2, 8, 4, \mathbf{B}, \mathbf{B}^+)$ is called as a *border point*, if it is 4–adjacent to at least one white point. A black point which is not a border point is said to be an *interior point*. The notation $N_k(p)$ will be used to refer to the set of points k–adjacent to p ($k \in \{4, 8\}$), and let $N_k^*(p) = N_k(p)\setminus\{p\}$. Further on, let us denote by $C(p)$ the number of 8-connected black components in the picture $(\mathbb{Z}^2, 8, 4, \mathbf{B} \cap N_8^*(p), \mathbf{B}^+)$.

A black point p is said to be a *simple point* if its deletion (i.e., changing it to white) preserves the topolopgy of the picture [5]. We make use the following characterization of simple points.

Theorem 1. *[1] Black point p is simple if and only if p is a border point and $C(p) = 1$.*

In order to retain some relevant information about the shape of objects, thinning algorithms preserve endpoints. Hence, thinning algorithms are coupled with endpoint-characterizations. We define endpoints as follows.

Definition 1. *The black point $p \in \mathbf{B}$ in $(\mathbb{Z}^2, 8, 4, \mathbf{B}, \mathbf{B}^+)$ is an e_j-endpoint if and only if there is not any inactive point in $N_j(p)$ ($j \in \{4, 8\}$).*

Here we note that if we consider the interior points as inactive black points, then similar criteria can be discovered as "hidden" endpoint-characterizations in the thinning scheme used by Manzanera et al. [7]. The importance of the notion of inactive and active black points rests on the fact that the thinning algorithm presented in [7] is parallel, which does not alter the state of interior points during an iteration before the simultaneous removal of deletable points. However, in sequential algorithms an interior point may change to a border point right after removing a deletable point. Thus, the algorithm must "memorize" actual border points in the beginning of the given iteration in order to be able to use such endpoint-characterizations like in [7].

A background point $p \in \overline{\mathbf{B}}$ is called an *isolated cavity point* if for any $q \in N_4(p)$, $q \in \mathbf{B}$ or $N_4(q) \cap (\mathbf{B} \setminus \mathbf{B}^+) \neq \emptyset$. An object point p is called as *single border point* if $N_4(p) \cap \overline{\mathbf{B}} = \{q\}$ where $N_4(q) \cap (\mathbf{B} \setminus \mathbf{B}^+) = \emptyset$. As an example, Fig. 1 shows a possible configuration for isolated cavity points and single border points, respectively. The symbols "$\bullet$", "$\bigstar$", "$\circ$" stand for active black points, inactive black points, and white points, respectively.

The reason of order-dependency in the case of sequential thinning algorithms lies mainly in the fact that there is at least one pair of simple and non-end points $\{p, q\}$ in the picture which for p is no more simple after removing q and vice versa. We call such a pair of points as decision pair. For a more formal definition, see [4]. In the case of a template-based thinning algorithm with a set $\mathcal{T}$ of matching templates, we can restrict this definition with the necessary condition that both p and q must be $\mathcal{T}$-deletable.

For the masks T, T' let us suppose that T' differs from T only in one point q, where q marks a border point in T', while it is a background point in T such that it is neither an isolated cavity point nor a 4-neighbor of a single border point in T.

a b

Figure 1: Examples where p is an isolated cavity point (a) and a single border point (b). The positions marked "•", "★", and "○" are considered to be active black points, inactive black points, and white points, respectively.

Then, q is the *difference point* of T and T', and T' is a *contour-expanded version of* T.

3 The Conditions for Order-Independency

Before we formulate our sufficient conditions for the mentioned property we must write up Algorithm 1 in a more formal way by using the introduced notions in Section 2 (see Algorithm 2). We will use the abbreviation $STA(\mathcal{T})$ (where STA stands for "Sequential Thinning Algorithm" and $\mathcal{T}$ for the input set of matching templates) to refer to this scheme.

Algorithm 2: $STA(\mathcal{T})$

Input: picture $(Z^2, 8, 4, X, \emptyset)$ and a template set $\mathcal{T}$
Output: picture $(Z^2, 8, 4, Y, Y^+)$
$Y = X$
$Y^+ = \emptyset$
repeat
  // Phase 1: contour tracking
  foreach p in Y **do**
    if p is a border point **then** $Y^+ = Y^+ \cup \{p\}$
  // Phase 2: reduction
  changed = false
  foreach $p \in Y^+$ **do**
    if p is $\mathcal{T}$-deletable **then**
      $Y = Y \setminus \{p\}$
      changed = true
until changed = false

Theorem 2. *Algorithm $STA(\mathcal{T})$ is order-independent if it fulfils both of the following conditions:*

I. No template in $\mathcal{T}$ contains any position that is coincident with a $\mathcal{T}$-deletable point (with the exception of its central element).

II. *Let q be the difference point of an arbitrary template $T \in \mathcal{T}$ and its contour-expanded version T'. Then, q is not $\mathcal{T}$-deletable in T'.*

Proof. We give an indirect proof. Let us suppose that Conditions I and II are both satisfied, but Algorithm $STA(\mathcal{T})$ is not order-independent. Hence, one of the following two situations must occur:

Case 1: There exists an object point p in the actual image, which is $\mathcal{T}$-deletable in the beginning of the iteration, but it is not $\mathcal{T}$-deletable when it is visited.

Case 2: There exists an object point p in the actual image, which is not $\mathcal{T}$-deletable in the beginning of the iteration, but it is $\mathcal{T}$-deletable when it is visited.

Let us suppose that Case I holds. Let us consider a visiting sequence $Q = \langle s_1, s_2, \ldots, s_n \rangle$ of border points, where $p = s_k$ for a given $k \in \{1, 2, \ldots n\}$. Let $S_0 = \emptyset$, $S_i = \{s_1, s_2, ...s_i\}$ $(1 \leq i \leq n)$,

$$D_i = \{x \mid x \in S_i \text{ and } STA(\mathcal{T}) \text{ removes } x \text{ when the visiting order is } Q\},$$

and let us define the picture $\mathcal{P}_i = (Z^2, 8, 4, Y \setminus D_i, Y^+)$. Note that $D_0 = \emptyset$ and $S_i \subseteq Y^+$, hence $p \in Y^+$.

Obviously, there must be a point $s_i = q \in D_i$ $(1 \leq i < k)$ such that p is $\mathcal{T}$-deletable in $\mathcal{P}_{i-1}$, but p is not $\mathcal{T}$-deletable in $\mathcal{P}_i$. As $q \in D_i$, q is also $\mathcal{T}$-deletable in $\mathcal{P}_{i-1}$. q must fall into a marked position in a template $T \in \mathcal{T}$, else the removal of q would not influence the deletabilty of p. This, however, leads to a contradiction with Condition I.

Therefore, only Case II can occur, which means that there must be a point $s_i = q \in D_i$ $(1 \leq i < k)$ such that p is not $\mathcal{T}$-deletable in the picture $\mathcal{P}_{i-1}$, but p is $\mathcal{T}$-deletable in $\mathcal{P}_i$. Let T be the template that p matches in $\mathcal{P}_i$. As q falls into a marked position in T, $q \notin N_4(p)$ or p is not a single border point in $\mathcal{P}_i$, or else p would have been an interior point in the beginning of the given iteration of the algorithm, which is not possible, as $p \in Y^+$ in our example. Furthermore, q can not be a cavity point in $\mathcal{P}_i$, or else q would have been an interior point in $\mathcal{P}_{i-1}$, thus it would not have been visited at all. From these observations follows that q must be the difference point of T and one of its contour-expanded versions, T'. As q is $\mathcal{T}$-deletable in picture $\mathcal{P}_i$, its position yields a $\mathcal{T}$-deletable point in template T', as well. However, this contradicts Condition II. □

4 The Proposed Template Sets

Here we introduce two template sets, $\mathcal{T}^1$ and $\mathcal{T}^2$ with the help of Fig. 2, which can be used for two possible realizations of Algorithm 2.

In the 5×5 templates $T_a - T_l$ depicted in Fig. 2, each black template element "•" or "♦" coincides with an active black point, while the black template elements "★" match inactive black points. Each white template element coincides with a white point. "Don't care" elements (i.e., empty positions) stand for points which can be either active black points, or inactive black points, or white points. The points marked "." can be either active black points or white points. Let us define $T_x^1 = T_x$, and $T_y^2 = T_y$ for any $x \in \{a,b,c,d,e,f,g,h,i,j\}$, $y \in \{a,b,c,d,e,f,g,h,k,l\}$, with the supplement that points marked "$\odot$" count as "don't care" in the templates T_y^2, but in T_x^1, they must be either active black points or white points. Taking these assumptions into consideration, we introduce the following sets:

$$\mathcal{T}^1_{base} = \big\{T_x^1 \mid x \in \{a,b,c,d,e,f,g,h,i,j\}\big\},$$

$$\mathcal{T}^2_{base} = \big\{T_y^2 \mid y \in \{a,b,c,d,e,f,g,h,k,l\}\big\}.$$

Finally, the proposed template sets $\mathcal{T}^1$, $\mathcal{T}^2$ contain the templates of $\mathcal{T}^1_{base}, \mathcal{T}^2_{base}$, respectively (see Fig. 2), plus all their possible $k\times 90°$ rotated and reflected versions ($k \in \{1,2,3\}$). Later, we will also refer to the set of all such possible transformed versions of a given mask T_x^1 or T_y^2, which will be denoted by $\mathcal{T}_x^1$ or $\mathcal{T}_y^1$, respectively.

5 Discussion

Now we will show that the algorithms $STA(\mathcal{T}^1)$ and $STA(\mathcal{T}^2)$ are both topology-preserving and order-independent. For the proof of the mentioned properties, we need to introduce some further notions. For a given object point p, let us denote by $I_k(p)$ the number of elements in $N_k(p) \cap (B \setminus B^+)$ ($k \in \{4,8\}$). Further on, for any active black points that are 4–adjacent to points p and q, the number of elements in $N_4^*(p) \cap N_8^*(q) \cap B^+$ will be denoted by $B_p(q)$.

Let $T \in \mathcal{T}^1 \cup \mathcal{T}^2$. Let us also consider an additional 5×5 template T' being p its central point where T' has the following properties:

i) if the cell on the position (x,y) $(x,y \in \{0,1,2,3,4\})$ marks a black/white point in T, then the cell on the position (x,y) also marks a black/white point in T';

ii) if the cell on the position (x,y) $(x,y \in \{0,1,2,3,4\})$ marks a point denoted by '.' in T, then the cell on the position (x,y) marks an active black point or a white point in T';

iii) if the cell on the position (x,y) $(x,y \in \{0,1,2,3,4\})$ is a "don't care" point (i.e. an empty cell) in T, then the cell on the position (x,y) refers to a black or a white point in T';

iv) each active black point in a position coinciding with a member of $N_8(p)$ in T' has at least one white 4-neighbor q for which $N_4(q)$ does not contain any inactive black point in T'. (A member of $N_4(q)$ being not coincident with any position of T' can have any values.)

T' is called the *properly composed version of* T. For a better understanding of this definition, Fig. 3 shows two templates where the first of them in Fig. 3a is a properly composed version of template T_k. However, in the template of Fig. 3b, both the

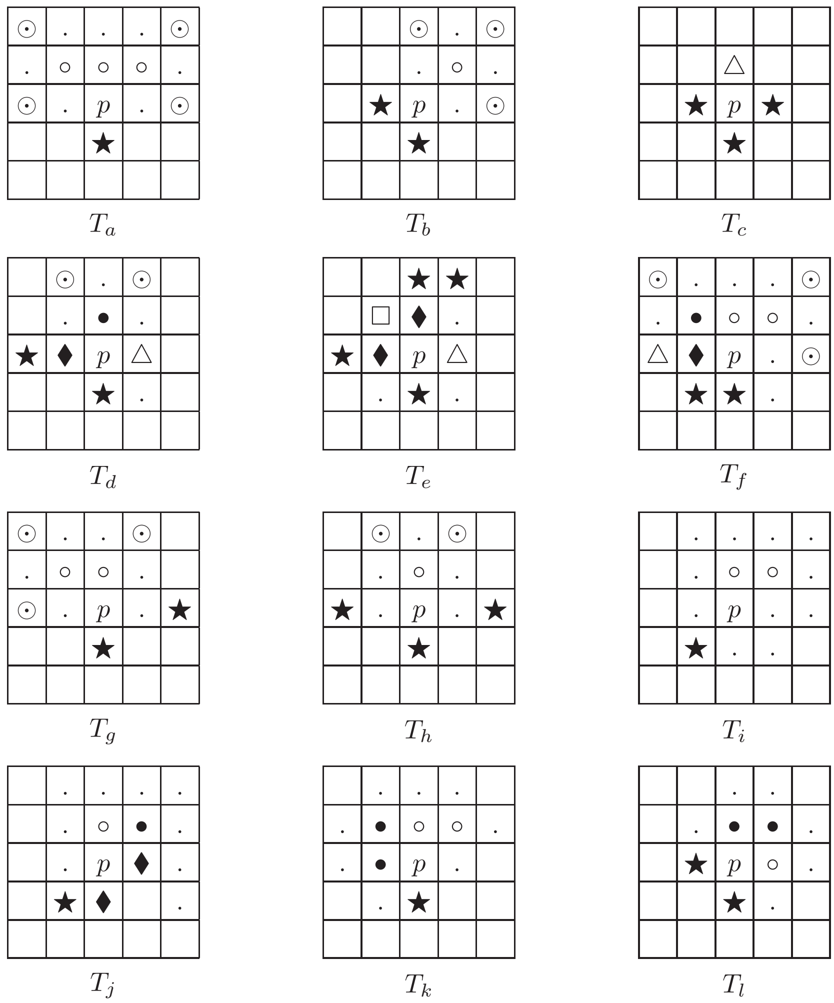


Figure 2: Sets of templates $\mathcal{T}^1_{base}$ and $\mathcal{T}^2_{base}$. Notations: each position marked "$\bullet$" and "$\blacklozenge$" matches an active black point; positions denoted by "$\bigstar$" match inactive black points; each position marked "$\circ$", "$\square$", and "$\triangle$" matches a white point; each "." can yield either an active black point or a white point; each empty cell matches a "don't care" point which can be either an (active or inactive) black point or a white point; the symbols "$\odot$" are to be considered as "." for the members of $\mathcal{T}^1$ and "don't care" for the members of $\mathcal{T}^2$.

a b

Figure 3: Examples for being (a) and not being (b) a properly composed version of the template T_k.

left and the right 4-neighbor of p are active black points, which for Condition iv) is not fulfiled.

For the proof of order-independency we will make use of an earlier result which states an important property of decision pairs.

Proposition 1. *[4] If $\{p, q\}$ is a decision pair, then $N_8(p, q)$ matches at least one of the configurations in Fig. 4 or their rotations by* $90°$, $180°$, *or* $270°$.

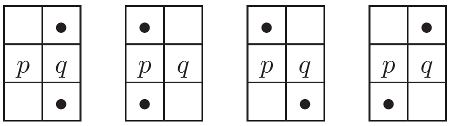


Figure 4: Possible configurations of the horizontal decision pair $\{p, q\}$. Points marked by empty cells may be either black or white.

By careful examination of the templates in $\mathcal{T}^1$ and $\mathcal{T}^2$ we can observe the following two facts.

Proposition 2. *In any $T \in \mathcal{T}^1$ there is an inactive black point (marked "★") in $N_8(p)$.*

Proposition 3. *In any $T \in \mathcal{T}^2$ there is an inactive black point (marked "★") in $N_4(p)$.*

Remark 1. From these propositions and from our endpoint-criterion introduced in Section 1 follows that algorithm $STA(\mathcal{T}^1)$ preserves e_8-endpoints while $STA(\mathcal{T}^2)$ preserves e_4-endpoints.

An important question in the view of order-independency is how the decision pairs are handled by algorithms $STA(\mathcal{T}^1)$ and $STA(\mathcal{T}^2)$. The lemma below serves as an answer.

Lemma 1. *Let $T \in \mathcal{T}^1 \cup \mathcal{T}^2$ and let T' be a properly composed version of T. If p is a member of a decision pair $\{p, q\}$ in T', then one of the following conditions is satisfied:*

- $I_4(p) < I_4(q)$, or
- $I_4(p) = I_4(q)$ and $I_8(p) < I_8(q)$, or
- $I_4(p) = I_4(q)$, $I_8(p) = I_8(q)$, and $B_p(q) < B_q(p)$.

Remark 2. As at most one point can be removed from the decision pair $\{p, q\}$ without altering the topology, we must somehow set rules to be able to decide whether we may safely remove any point from the pair at all, and if so, then which one should be preferred. This lemma gives a possible preference for this decision, which prevails for the template sets $\mathcal{T}^1$ and $\mathcal{T}^2$. The number of interior 4- and 8-neighbors is a useful property for the comparison of p and q, as the interior points do not change during an iteration. The values $B_p(q)$ and $B_q(p)$ can be also applied for this purpose, if it is ensured that the sets $N_4^*(p) \cap N_8^*(q) \cap B^+$ and $N_4^*(q) \cap N_8^*(p) \cap B^+$ contain only non-deletable border points. Note however that if the neighborhood of the decision pair is symmetric, then these rules do not determine any preferred point in $\{p, q\}$.

Proof. First we will show that at least one of the mentioned conditions holds for each $T \in \mathcal{T}^1_{base} \cup \mathcal{T}^2_{base}$.

- If $T \in \{T_a^1, T_a^2, T_c^1, T_c^2\}$, then it is obvious that p is not a member of any decision pair in T'.
- If $T \in \{T_b^1, T_b^2\}$, then at least one of the positions marked "." in $N_4(p)$ must be a white point, else p would be an interior point. Therefore, by Proposition 1, p is not a member of any decision pair in T'.
- If $T \in \{T_d^1, T_d^2\}$, then $I_4(p) = 1$ and $I_8(p) \geq 1$ in T'.
  - Let q be the left 4-neighbor of p in T'. Then, $I_4(q) \geq 1 = I_4(p)$. If $I_4(q) = 1$, then $I_8(q) = 2$ and $I_8(p) = 1$, which implies $I_8(q) > I_8(p)$.
  - Let q be the upper 4-neighbor of p in T. If $T = T_d^2$, then q is an e_8-endpoint. Let $T = T_d^1$. q does not match the templates in $\mathcal{T}_x^1$ ($x \in \{a, b, c, d, e, f, g, h\}$), because $I_4(q) = 0$, and in the mentioned templates the central point has at least one inactive black 4-neighbor. By careful examination of the possible remaining templates it is also easy to see that q does not match any member of $\mathcal{T}_i^1$ and $\mathcal{T}_j^1$. Thus, q is neither $\mathcal{T}^1$-deletable nor $\mathcal{T}^2$-deletable, hence $\{p, q\}$ cannot be a decision pair.
- If $T \in \{T_e^1, T_e^2\}$, then $I_4(p) = I_8(p) = 1$ in T'. Let q be the left or upper 4-neighbor of p in T'. Then, in both possible cases we get $I_4(q) = 1 = I_4(p)$, $I_8(q) \geq 2 > I_8(p)$.
- If $T \in \{T_f^1, T_f^2\}$, then $I_4(p) = 1, I_8(p) = 2$ in T'. Let q be the left 4-neighbor of p. By Proposition 1, only the set $\{p, q\}$ may be a decision pair in T', and $I_4(q) = I_4(p) = 1$, $I_8(q) \geq 2 = I_8(p)$, further on, $B_p(q) = 0, B_q(p) = 1$, hence $B_p(q) < B_q(p)$.

- If $T \in \{T_g^1, T_g^2\}$, then let q be the right 4-neighbor of p and r be the left 4-neighbor of p. By Proposition 1, the set $\{p, r\}$ cannot be a decision pair in T'. Let us suppose that $\{p, q\}$ is a decision pair. In this case, q is a border point, and by Proposition 1, the upper 4-neighbor of q must be also a border point. Therefore, there is not any white point s in $N_4(q)$ which for $N_4(s)$ would contain any inactive black point. But this leads to a contradiction with the definition of the properly composed version of T. Hence, $\{p, q\}$ cannot be a decision pair in T'.

- If $T \in \{T_h^1, T_h^2\}$, we can show the same way as in the previous case that p can not be a member of a decision pair in T'.

- If $T = T_i^1$, then $I_4(p) = 0$ and by Proposition 1, p can only be a member of a decision pair $\{p, q\}$ in T' where q is the left or bottom 4-neighbor of p. Then, $I_4(q) \geq 1 > I_4(p)$.

- If $T = T_j^1$, then $I_4(p) = 0$ and the following two cases are to be examined.
  - Let q be the right 4-neighbor of p in T'. If the bottom 4-neighbor of q, say r, is not an inactive black point, then q is an e_8-endpoint, which means, $\{p, q\}$ cannot be a decision pair. If r is an inactive black point, then $I_4(q) = 1 > I_4(p)$.
  - Let q be the bottom 4-neighbor of p in T'. Then, $I_4(q) \geq 1 > I_4(p)$.

- If $T \in \{T_k^2, T_l^2\}$, then it is easy to see by Propositions 1 and 3 that p cannot be a member of a decision pair in T'.

It is obvious that the amounts $I_4(p)$, $I_8(p)$, $B_p(q)$, $B_q(p)$ will not change for any $q \in N_4(p)$ after rotating or reflecting a template. Therefore, the lemma also holds in the case $T \in \mathcal{T}_x^1 \cup \mathcal{T}_y^2$, for any $x \in \{a, b, c, d, e, f, g, h, i, j\}$ and $y \in \{a, b, c, d, e, f, g, h, k, l\}$. $\square$

Finally, using Lemma 1 and Propositions 2-3, we give a proof for the mentioned properties of the proposed algorithms.

Theorem 3. *Algorithms $STA(\mathcal{T}^1)$ and $STA(\mathcal{T}^2)$ are both topology-preserving.*

Proof. It is easy to see that if an object point p matches in any iteration of $STA(\mathcal{T}^1)$ or $STA(\mathcal{T}^2)$ a $T \in \mathcal{T}^1$ or a $T \in \mathcal{T}^2$, then there must be a properly composed version T' of T which for p matches T' as well. Therefore, by Theorem 1 we have only to show that $C(p) = 1$ holds in such a T'. It is sufficient to prove this only for the case $T \in \mathcal{T}_{base}^1 \cup \mathcal{T}_{base}^2$, because $C(p)$ does not change after rotating or reflecting T.

- Let $T \in \{\mathcal{T}_x^1, \mathcal{T}_y^2\}$ where $x \in \{a, b, c, d, e, f, j\}$ and $y \in \{a, b, c, d, e, f, k, l\}$. By careful examination of these templates one can notice that $C(p) = 1$ in T', hence p is simple.

- Let $T \in \{\mathcal{T}_g^1, \mathcal{T}_g^2\}$. It is easy to see that in the beginning of the actual iteration of $STA(\mathcal{T}^1)$ or $STA(\mathcal{T}^2)$, $C(p) = 1$ in T', and the only way that this could change is when the right 4-neighbor of p, say q, and the upper 4-neighbor of q are both black points and q gets deleted before visiting p in Phase 2. But in this case, we would come into a contradiction with the definition of T', as q would be an active black point which for $N_4(q)$ does not contain any white point without an inactive black 4-neighbor. Therefore, $C(p) = 1$ still holds in T' when p becomes the actual point in Phase 2, which means that p is simple.

- The proof for the situation $T \in \{\mathcal{T}_h^1, \mathcal{T}_h^2\}$ can be similar to the previous case.

- Let $T = \mathcal{T}_i^1$. It is easy to see that in the beginning of the actual iteration of $STA(\mathcal{T}^1)$, $C(p) = 1$ holds in T', and there are two possible situations when this could change: either both the left 4-neighbor of p, say q, and the upper-left 4-neighbor of p are black points and q will be removed before visiting p, or the bottom 4-neighbor, say r, and the bottom-right 4-neighbor of p are black points and r will be removed before visiting p. We only give the proof for the first case as the other situation can be similarly examined. Let us suppose that q gets removed before visiting p. Thus, when algorithm $STA(\mathcal{T}^1)$ visits q, it matches a properly composed version U' of a $U \in \mathcal{T}^1$. It is obvious that $U \neq T_i^1$ and $U \notin \mathcal{T}_i^1$, hence $U = T_x^1$ or $U \in \mathcal{T}_x^1$ must hold for at least one $x \in \{a, b, c, d, e, f, j\}$. Above, we have already seen that in such a case, the central point of U' is a simple point. It can be also easily seen that by removing p before q, q would not be simple any more. This means that $\{p, q\}$ is a decision pair, and from Lemma 1 follows that in this situation, q will not be removed. Hence, p remains simple until it gets removed.

□

Theorem 4. *Algorithms $STA(\mathcal{T}^1)$ and $STA(\mathcal{T}^2)$ are both order-independent.*

Proof. For the positions represented by white or black symbols in any template $T \in \mathcal{T}^1 \cup \mathcal{T}^2$ the following observations can be made.

- Points marked by "○" do not have any inactive black 8-neighbor/4-neighbor, therefore if we consider such a point q and a contour-expanded version T' of $T \in \mathcal{T}^1/T \in \mathcal{T}^2$ with difference point q, then by Proposition 2 / Proposition 3, q is not $\mathcal{T}^1$-deletable/$\mathcal{T}^2$-deletable in T'.

- Points marked by "•" do not have any inactive black 8-neighbor/4-neighbor, therefore, by Proposition 2 / Proposition 3, such a point is not $\mathcal{T}^1$-deletable/ $\mathcal{T}^2$-deletable in T.

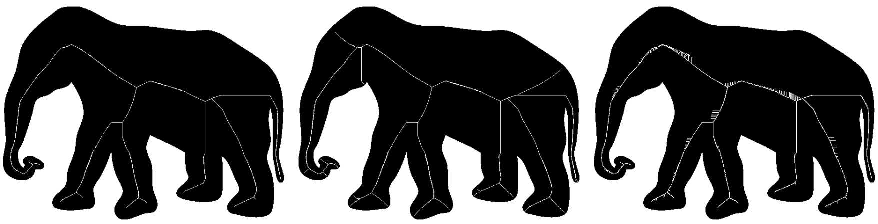
- It is easy to see that if a position denoted by "♦" marks a simple point q and $N_8(q)$ contains an inactive black point, then by Lemma 1, q does not match any $T' \in \mathcal{T}^1 \cup \mathcal{T}^2$.

- Points marked by "△" are 4-neighbors of a simple point, therefore we can not construct for such a point q any contour-expanded version T' of T with difference point q.
- The point represented by the symbol "□" is an isolated cavity point, therefore we can not construct for such a point q any contour-expanded version T' of T with difference point q.

From these observations and from Theorem 2 follows that algorithms $STA(\mathcal{T}^1)$ and $STA(\mathcal{T}^2)$ are order-independent. □

6 Results

In experiments our algorithms were tested on some test pictures. Our results were compared to the ones produced by the existing algorithm introduced by Ranwez and Soille [9]. Figs. 5-9 show some illustrative examples where "skeletons" extracted by the mentioned 2D algorithms are superimposed on the original objects. Numbers in parentheses indicate the counts of skeletal points. One can easily recognize that the method by Ranwez and Soille has extracted much more unwanted line segments for the selected pictures than the proposed algorithms. However, it is important to note that the aim of the paper is not to carry out a detailed comparison of order-independent sequential thinning methods, hence the shown results serve only demonstrational purposes.



$STA(\mathcal{T}^1)$ (2 243) $STA(\mathcal{T}^2)$ (2 780) Ranwez-Soille (3 571)

Figure 5: A 612×467 image with 179 293 object points of an elephant and its "skeletons".

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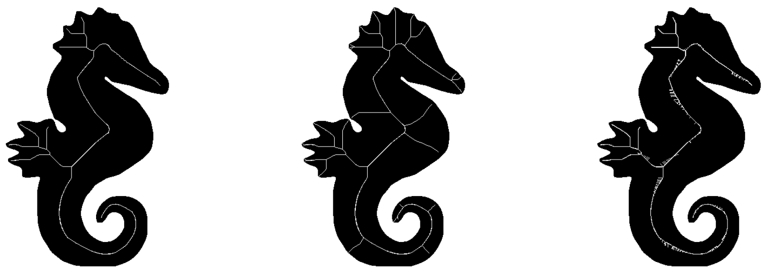
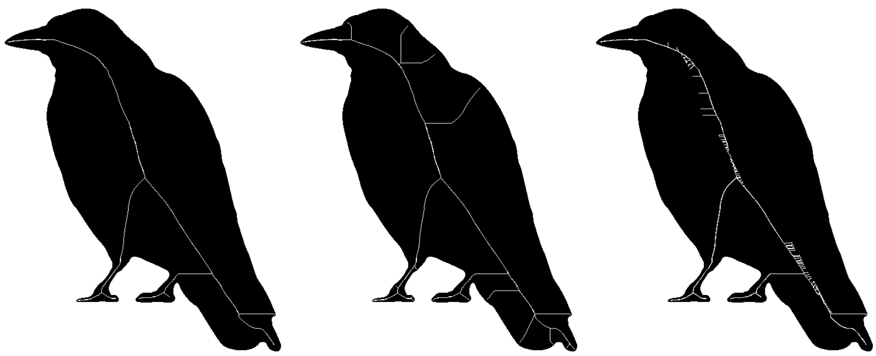


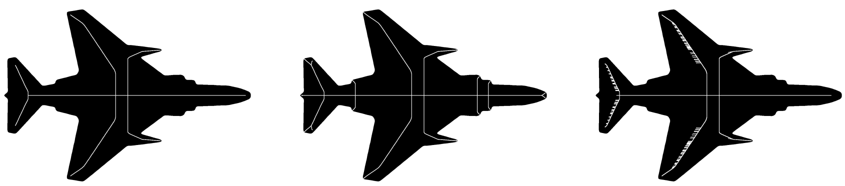
$STA(\mathcal{T}^1)$ (1 318) $STA(\mathcal{T}^2)$ (1 931) Ranwez-Soille (2 027)

Figure 6: A 530×530 image of a seahorse with 79 293 object points and its "skeletons".



$STA(\mathcal{T}^1)$ (1 300) $STA(\mathcal{T}^2)$ (1 786) Ranwez-Soille (2 194)

Figure 7: A 492×606 image of a crow with 126 538 object points and its "skeletons".



$STA(\mathcal{T}^1)$ (3 389) $STA(\mathcal{T}^2)$ (4 329) Ranwez-Soille (6 081)

Figure 8: A 1600×1600 image of a plane with 487 620 object points and its "skeletons".

Figure 9: A 745×773 image with 152 611 object points of a leaf and its "skeletons".

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