Employing Pythagorean Hodograph Curves for Artistic Patterns

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Abstract

In this paper we present a novel design element creator tool for the digital artist. The purpose of our tool is to support the creation of vines, swirls, swooshes and floral components. To create visually pleasing and gentle curves we employ Pythagorean Hodograph quintic spiral curves to join a hierarchy of *control circles* defined by the user. The control circles are joined by spiral segments with at least G^2 continuity, ensuring smooth and seamless transitions. The control circles give the user a fast and intuitive way to define the desired curve. The resulting curves can be exported as cubic Bézier curves for further use in vector graphics applications.

1 Introduction

Floral elements, vines, tangled spirals and similar features are among the most popular design components. These components could be found in traditional ornamental and contemporary abstract designs as well. Whatever the actual media or purpose is, the common property of these patterns is they consist of an intricate network of gentle curves. Creating such curves by hand is a non-trivial task. Users of vector graphic applications often employ spirals as a starting shape, then apply transformations to achieve the desired curve. This method of curve creation heavily restricts the set of producible shapes. The motivation for our work was to enlarge this set of curves available for the artist, while guaranteeing the *pleasingness* of the curves.

To give a formal definition of pleasing curves, we built upon the empirical observation and the same assumption as Xu and Mould, that is, a pleasing curve is a curve of monotically changing curvature. Our tool enables the digital artists to create such curve or sequence of curves with ease. Out method uses cubic and quintic splines to generate resolution independent curves, which can be used directly, serve as a skeleton of a design, or act as a path for strokes or objects. To support the widest range of third party tools possible, the generated curves can be exported as cubic Bézier splines.

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To ensure this property of the drawn curves, we employ Pythagorean Hodograph (PH) curves. PH curves introduced by Farouki and Sakkalis [5] have very favourable properties, most importantly it is possible to define spiral segments using PH quintic curves whose curvature changes monotonically with arc-length. Spiral curves have been used in highway, railway and robot trajectory design [1], [7], [9]. Walton et. al. proposed to use PH spirals for such applications [12], which was followed by further research on PH quintic spirals [4] [6]. Now we demonstrate the efficiency of these curves as design elements as well.

2   Background

There have been several efforts to automate aspects of the artistic process of creating such designs and ornaments. Since the fundamental work of Prusinkiewicz and Lindenmayer [10] L-systems has been used widely to generate flowers and flower like patterns.

Wong et al. [8] presented a system to automatically generate space-filling floral ornaments. Their system uses proxy objects during generation what could be replaced by arbitrary elements created by any means.

Xu and Mould's *Magnetic Curves* [14] are more closely related to our work. They focus on the creation of the curves themselves, but their method uses a discrete time-step approach, and they commit the problem of creating smooth curves to approximation routines.

In our tool the user defines the curves by placing circles on the canvas. Using various intuitive defining elements such as this has been subject to research [2], for example Singh proposed a method in which ellipse arcs are used for defining the curves [11].

3   Pythagorean Hodograph curves

Let $x(t)$ and $y(t)$ be the x and y components of the parametric curve $\mathbf{Q}(t) : [0,1] \to \mathbb{R}^2$. $\mathbf{Q}(t)$ is said to be a PH curve if there exists a polynomial $\sigma(t)$ such that $x'^2(t) + y'^2(t) \equiv \sigma^2(t)$, that is, its coordinate polynomials' derivatives $x'(t), y'(t)$ form a Pythagorean-triple with the parametric speed $\sigma(t)$. This holds if and only if

$$x'(t) = (u^2(t) + v^2(t))w(t)$$
$$y'(t) = 2u(t)v(t)w(t),$$

where $u(t)$, $v(t)$, and $w(t)$ are polynomials and $u(t)$ and $v(t)$ are relative prime [5]. We set $w(t) \equiv 1$, that is we use primitive Pythagorean Hodograph curves, following Walton and Meek's work as in [12].

One of the appealing properties of PH curves is their curvature can be expressed

as a rational function

$$\kappa(t) = \frac{2\left(u(t)v'(t) - u'(t)v(t)\right)}{\left(u^2(t) + v^2(t)\right)^2}.$$

The polynomials $u(t)$ and $v(t)$ are defined to be quadratic, hence there are six parameters (six scalar degrees of freedom) for the final curve. Using quadratic $u(t)$ and $v(t)$ results in quintic $\mathbf{Q}(t)$ what has desirable properties for our needs. This $\mathbf{Q}(t)$ then can be expressed in the Bézier form

$$\mathbf{Q}(t) = \sum_{i=0}^{5} \binom{5}{i} \mathbf{P}_i (1-t)^{5-i} t^i \qquad t \in [0,1],$$

where $\mathbf{P}_0$ is the beginning point of the curve and $\mathbf{P}_i, i \in \{1,2,3,4\}$ are functions of the aforementioned six parameters. For the exact definition see [12].
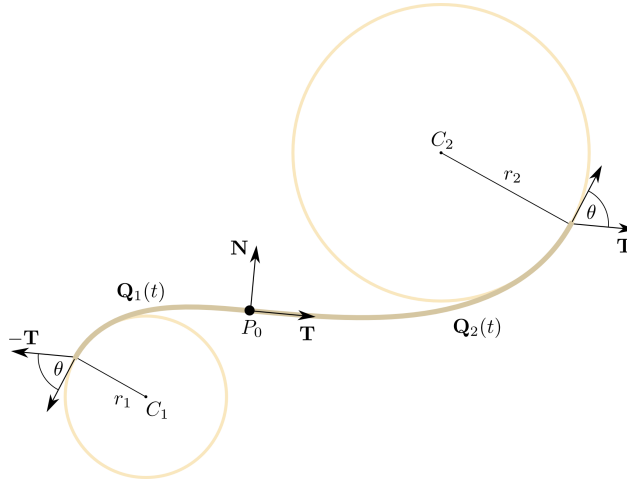

Figure 1: Definition of points and angles as used by Walton and Meek.

Without the loss of generality, we can consider the case where the curve starts from the origin and the x axis is tangential to it, as detailed in Habib and Sakai [6]. The user defined values for the actual starting point and tangent for the curve then can be translated, rotated, and if needed mirrored to align with this special case. The control points are computed in this space, then they are transformed back to the original using the inverse transformations of the applied ones.

We use these curves to define spiral segments with monotonically changing curvature. These spiral segments are used to create our circle-in-circle and circle-inside-circle transitions as detailed in the following section.
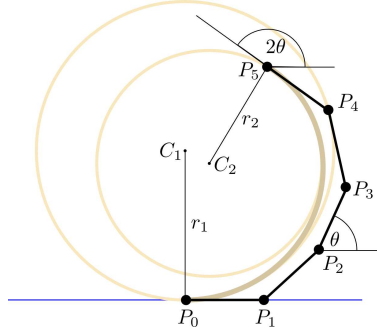
Figure 2: Definition of points and angles as used by Habib and Sakai.

4 Control circle hierarchies

To create the desired pattern, in our tool the user defines curves through a hierarchy of *control circles*. These control circles define the constraints required to derive a gentle curve. The control circles form a directed tree, where each circle is potentially connected to its ancestor. These connection are automatically created by our system as an appropriate curve.

Using the control circle hierarchy we create smooth transition curves between each child and the ancestor circle. Depending on their geometric relation we consider two cases: the *circle-to-circle transition curve* between two non-touching and non-overlapping circles, and the *circle-in-circle transition curve*, when the descendant circle is fully contained within its ancestor. For the circle-to-circle case we only support S-shaped transition curves, i.e. curves with an inflection point in the middle.

To realize these curves, the previously introduced PH curves are used. These curves have been defined to have G^2 contact to their control circle. Therefore, each incoming curve can have G^2 contact with any outgoing curve of the same circle if their endpoints coincide.

The typical workflow of curve creation in our prototype application using control circles is as follows:

1. The user places an initial circle on the drawing canvas or selects and existing one.

2. Adjusts the position and radius of the circle as needed.

3. Creates a new control circle anywhere on the canvas, and sets its radius.

4. The new circle is created as a child of the previously selected one.

5. The circles define the starting and ending curvature of the curve.

6. If it is possible, a new transition curve is created by our tool, the curve will *coil* onto the control circles, touching them only at the endpoints with G^2 contact.

4.1   *Circle-to-circle* transitions

Two non-touching, non-overlapping control circles are connected by an S-shape, if distance centres are within a certain threshold. The derivation of the control points of the S-curves is following the work of Walton and Meek [13].

According to their work, if the radii and centres of the circles are $r_1, r_2, \mathbf{C}_1, \mathbf{C}_2$ respectively, then the condition for the distance threshold is

$$r_1 + r_2 < \|\mathbf{C}_2 - \mathbf{C}_1\| < \sqrt{\left(\frac{321}{120}\right)^2 + \left(\frac{91}{60}\right)^2}\,(r_1 + r_2) < 3.075(r_1 + r_2).$$

If this criteria is met, the transition curve is automatically created.

The positions and radii of the circles define the shape of the S, save for reflectional symmetry. The choice between the two possible curves is the only required additional user input.

The S-shape is produced using two distinct PH spiral segments. Let $\mathbf{Q}_1(t), \mathbf{Q}_2(t)$ denote these two curves. We derive both $\mathbf{Q}_1(t)$ and $\mathbf{Q}_2(t)$ so $\mathbf{Q}_1(0) = \mathbf{Q}_2(0)$, $\mathbf{Q}_1(1), \mathbf{Q}_2(1)$ lie on the first and second circle respectively, and $\kappa_1(0) = \kappa_2(0)$, $\kappa_1(1) = \frac{1}{r_1}$, $\kappa_2(1) = \frac{1}{r_2}$, where $\kappa_1(t), \kappa_2(t)$ denotes the curvature of $\mathbf{Q}_1(t), \mathbf{Q}_2(t)$ respectively.

4.2   *Circle-in-circle* transitions

A fully contained circle is joined to its ancestor with a spiral segment, if such transition is possible. The conditions and the derivation of the control points are given in Habib and Sakai's [6] work.

For a given containing circle with radius and centre $r_1, \mathbf{C}_1$, only the radius r_2 of the contained circle is required to compute the control points of the new spiral segment. Using the radii of the circles the algorithm first computes the range of allowed distances between the centres. From this range our software chooses the smallest possible distance. While it would be possible to introduce this as a user parameter, practice show that this does not expand the range of possible curves significantly.

As a consequence of the derivation, from the parameters that are defined by the user the ones used are only $r_1, r_2, \mathbf{C}_1$, but not $\mathbf{C}_2$. The actual value of $\|\mathbf{C}_1 - \mathbf{C}_2\|$ is chosen by the software, and the final $\mathbf{C}_2$ centre is computed at a later point. For a given pair of radii and a distance, the spiral segment is uniquely defined, thus it is the radii and centre-to-centre distance that dictates the positioning of the smaller circle. If the creation of the transition curve is possible, only the radius of the smaller circle is used from the user input, and its centre is repositioned as defined by the algorithm. Similarly to the S-curves, a mirroring property can be defined.

An additional parameter of a spiral segment is its starting point, defined in degrees on the ancestor control circle. For new segments, this is automatically calculated, so the new curve continues the ancestor's incoming curve, if it exists.

5   Implementation

The computations of the control points in both cases are quite involved, especially for the case of *circle-in-circle* transitions, and require numerical algorithms in the implementation. Since the solution of the emergent equations differ greatly the two kinds of transition curves, we discuss them separately.

5.1   *Circle-to-circle* transitions

For this special case of *circle-to-circle* transitions, all the six parameter of $u(t)$ and $v(t)$ can be reduced either to zero or to the function of a single parameter θ for each PH spiral segment. This θ measures the angle between the beginning and ending tangent vectors of a spiral segment, as show on Figure 1. Because of this, the problem of finding the appropriate curve is reduced to finding an appropriate value of θ.

Unfortunately, there is no known explicit equation for θ, but it defined as the root of a rational trigonometric equation $f(\theta) = 0$. Citing [12] $f(\theta)$ is defined as

$$f(\theta) = \|\mathbf{C}_2 - \mathbf{C}_1\|^2 - \left(\frac{r_1 + r_2}{120}\right)^2 g_1^2(\theta) - \left(\frac{r_1 + r_2}{60}\right)^2 g_2^2(\theta),$$

where

$$g_1\left(\theta\right) = \frac{\sin\left(\theta\right)}{\left(1 + \cos\left(\theta\right)\right)^2} \left(321 - 58\cos\left(\theta\right) + \left(-36\right)\cos\left(\theta\right)^2\right),$$

$$g_2\left(\theta\right) = \frac{1}{1 + \cos\left(\theta\right)} \left(91 + 11\cos\left(\theta\right) + 18\cos\left(\theta\right)^2\right).$$

The first and second derivatives of g_1 and g_2:

$$g_1'\left(\theta\right) = -\frac{36\cos\left(\theta\right)^3 + 72\cos\left(\theta\right)^2 + 365\cos\left(\theta\right) - 700}{\cos\left(\theta\right)^2 + 2\cos\left(\theta\right) + 1}$$

$$g_1''\left(\theta\right) = \frac{\left(36\cos\left(\theta\right)^3 + 108\cos\left(\theta\right)^2 - 221\cos\left(\theta\right) + 1765\right)\sin\left(\theta\right)}{\cos\left(\theta\right)^3 + 3\cos\left(\theta\right)^2 + 3\cos\left(\theta\right) + 1}$$

$$g_2'\left(\theta\right) = -\frac{\left(18\cos\left(\theta\right)^2 + 36\cos\left(\theta\right) - 80\right)\sin\left(\theta\right)}{cos\left(\theta\right)^2 + 2\cos\left(\theta\right) + 1}$$

$$g_2''\left(\theta\right) = -\frac{18\cos\left(\theta\right)^3 + 36\cos\left(\theta\right)^2 + 116\cos\left(\theta\right) - 196}{\cos\left(\theta\right)^2 + 2\cos\left(\theta\right) + 1}$$

Using these we numerically find the root of $f(\theta) = 0$ with bisection. It is sufficient, since these computations are only required when parameters of the defining circles are modified.

5.2   *Circle-in-circle* transitions

When computing the control points for a *circle-in-circle* transition we built upon the work of Habib and Sakai [6]. The problem boils down to finding a specific θ value like in the previous case, but here it has a slightly different meaning. This time it is 2θ that measures the angle between the beginning and ending tangent vectors of a spiral segment. Figure 2 demonstrates θ and 2θ.

Before computing θ, a valid range for $\|\mathbf{C}_1 - \mathbf{C}_2\|$ has to be determined. Again the solution required a root solving technique. (The upper bound is simply $|\mathbf{C}_1 - \mathbf{C}_2|$.)

The equation what's root is to be found is

$$ m\left(\lambda\right)\left(r_1 - r_2\right) - \|\mathbf{C}_1 - \mathbf{C}_2\| = 0, $$

where

$$ m(\lambda) = \frac{\sqrt{g_1\left(p_0\right)^2 + g_2\left(p_0\right)^2}}{\lambda^3 - 1}, $$

$$ g_1(p) = \frac{1}{120\,\lambda^6\,sin\left(\frac{\theta}{2}\right)^2}\left\{\lambda^2 + p\,\left(p^2 - 2\,\lambda^4\,\left(3 - 2\,\lambda + 6\,\lambda^2\right)\right)\,\cos\left(\theta\right)\right\}, $$

$$ g_2(p) = \frac{1}{60\,\lambda^6\,\sin\left(\frac{\theta}{2}\right)}\left\{3\,\lambda^2\,\left(p^2\,\left(1 + \lambda\right) - 20\,\lambda^7\right.\right. $$

$$ \left. + 2\,p^2\,\cos\left(\theta\right) + 20\,\lambda^4\,\cos\left(2\,\theta\right)\right)\,\sin\left(\frac{\theta}{2}\right) $$

$$ + p\,\left(\left(p^2 + 2\,\lambda^5\right)\,\cos\left(\frac{\theta}{2}\right) + 2\,\lambda^4\,\left(3 - \lambda\right)\,\cos\left(\frac{3\,\theta}{2}\right)\right. $$

$$ \left.\left. - 6\,\lambda^4\,\cos\left(\frac{5\,\theta}{2}\right)\right)\right\}, $$

$$ p_0 = \frac{3}{2}\,\lambda^2\,\left(1 + \sqrt{1 + \frac{16\,\lambda\cos\left(\frac{\theta}{2}\right)^2}{9}}\right)\,\tan\left(\frac{\theta}{2}\right), $$

$$ \lambda = \sqrt[3]{\frac{r_1}{r_2}}. $$

These definitions were given by Habib and Sakai. As it is clear from the equations above, the derivative of the concerned functions are too complex, and it would be impossible to work with it efficiently. Nevertheless, because the domain of the possible solutions is known, a simple bisection method has proven adequate.

6   Export and refinement

The patterns created in the tool can be exported as sequences of cubic Bézier curves in SVG format (the highest degree Bézier curves that SVG supports). In order to reduce the difference between the original PH curves and the Bézier curves as much as possible, each PH curve is exported as two Bézier curves. Since a quintic PH curve is a special Bézier curve, standard operations can be carried out on it. At first we split the PH curve to two quintic Bézier curve at $t = 0.5$, then reduce the degree of these curves to cubic. The resulting curve undergoes just small visual distortion.

The SVG format is widely recognized by application, thus the user can use the whole toolset of her software to apply colours, strokes, brushes or textures the curves. Figure 3 demostrates such a process.

Figure 3: Refinement process example, from left to right: SVG file opened with Inkscape, Bézier segments connected, brushes and colours applied according to curve parameter (brushes shown below).

7   Results

In this paper we demonstrated a novel use of Pythagorean Hodograph curves. We presented a tool in which the user can create smooth patterns using out *control circle hierarchies*. Using this tool the user can effortlessly create curves with pleasing slopes and seamless connection thanks to the monotonically changing curvature of the elements and to the G^2 contacts. We aim this tool to the digital artist for creation of floral ornaments, swrils, and swooshes.

Along with these, we augmented the works of principal researchers of PH curves with implementation details.

Finally we showed how our tool integrates with other vector graphics applications.

8 Future work

While control circles give an intuitive way to define the curves, they lack the precision that is sometimes required even by the artist. It it not possible to explicitly define the start or endpoint of a curve, and they can be fitted only using experimenting. This behaviour of the control circles arises from the formulation of the PH curves, where only the starting point could be defined, and the position of the endpoint is dictated by the value of θ. Being able to directly define the endpoints and curvatures, much like the cubic Bézier curves being defined by endpoints and tangents in drawing applications, would give the user much better control.

An other aspest of our future work is fully integrating our tool to a vector graphics application, such as Inkscape. This way our technique would be able to reach a wider audience, and we could receive feedback from a real user base.

Further appealing properties of the PH curves can be taken advantage of. For example the arc length of PH curves can be expressed as a polynomial, allowing the application of textures and brushes depending on arc length, not just on curve parameter.

9 Acknowledgements

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