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Realizing Small Tournaments Through Few Permutations*

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Abstract

Every tournament on 7 vertices is the majority relation of a 3-permutation profile, and there exist tournaments on 8 vertices that do not have this property. Furthermore every tournament on 8 or 9 vertices is the majority relation of a 5-permutation profile.

Keywords: voting systems; digraph realization; extremal combinatorics

1 Introduction

A *tournament* $T = (V, A)$ is a directed graph on a vertex set V whose arc set A contains exactly one arc between any pair of distinct vertices. A finite family of (not necessarily distinct) permutations of V forms a *realization* of the tournament, if for every arc $uv \in A$ vertex u precedes vertex v in more than half of the permutations. A realization of the tournament by k permutations is called a *k-permutation profile.* McGarvey [2] proved that every tournament has a realization by a finite number of permutations. Subsequent results by Stearns [6] and Erdős & Moser [1] yield that every tournament on n vertices can be realized by $O(n/\log n)$ permutations, and that some tournaments on n vertices cannot be realized by fewer than $\Omega(n/\log n)$ permutations. We define the *McGarvey number* $\mathrm{McG}(T)$ of a tournament T as the size of the smallest possible permutation family that realizes the tournament; note that $\mathrm{McG}(T)$ always is an odd integer.

Shepardson & Tovey [5] analyzed several combinatorial questions on the so-called *predictability number* of tournaments, a parameter closely related to realizations of tournaments. Page 502 of [5] formulates the conjecture that every 7-vertex tournament T has $\mathrm{McG}(T) \leq 3$. In this technical note we confirm this conjecture, and we also discuss a number of related questions. Our results confirm this conjecture:

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- Every 7-vertex tournament T satisfies $\text{McG}(T) \leq 3$.
- Every 8-vertex and every 9-vertex tournament T satisfies $\text{McG}(T) \leq 5$.
- There exist 96 non-isomorphic 8-vertex tournaments T with $\text{McG}(T) = 5$.
- There exist 17.674 non-isomorphic 9-vertex tournaments with $\text{McG}(T) = 5$.

All our results have been derived with the help of computer programs, and in particular with the help of the software packages AIMMS and CPLEX.

2 Mathematical model and computational results

We express every permutation of the vertex set $V = \{1, 2, \ldots, n\}$ by a a transitive tournament, which can be considered as a permutation (total order) of V. It is well-known (see for instance Moon [4]) that a tournament without directed triangles is transitive. We use n^2 integer variables $x_{uv} \in \{0, 1\}$ with $u, v = 1, \ldots, n$ to encode the arcs of the tournament, and we impose the following two families of linear inequalities.

$$\begin{aligned} x_{uv} + x_{vu} &= 1 \qquad \text{for all } u, v \in V \\ x_{uv} + x_{vw} + x_{wu} &\leq 2 \qquad \text{for all } u, v, w \in V \end{aligned}$$

The first constraint family enforces that for every two vertices u, v there is either an arc uv or an arc vu but not both. The second constraint family forbids the occurrence of directed triangles (and thus makes the tournament transitive).

In order to decide whether a given tournament $T = (V, A)$ can be realized by three permutations, we introduce three such sets of integer variables x_{uv}, x'_{uv}, x''_{uv} together with the corresponding families of constraints. Furthermore we add the constraints

$$x_{uv} + x'_{uv} + x''_{uv} \geq 2 \qquad \text{for all } uv \in A.$$

These constraints ensure that vertex u precedes vertex v in more than half of the three permutations. McKay [3] gives a list that enumerates all 456 non-isomorphic tournaments on seven vertices. We worked through the tournaments on this list one by one, and for each of them the software package AIMMS managed to find a feasible solution to the corresponding linear integer program. We also worked through the list of $6,880$ non-isomorphic tournaments on eight vertices and through the list of $191,536$ non-isomorphic tournaments on nine vertices; for all these tournaments AIMMS found a realization by five permutations.

Theorem 1. *Every tournament on $n \leq 7$ vertices has a realization by three permutations. Every tournament on $n \leq 9$ vertices has a realization by five permutations.* □

0000110.011000.00010.0001.100.10.1	1001010.110010.11000.0101.001.00.0
0000110.101000.10010.0001.100.10.1	1001100.110010.10010.1010.001.00.1
0001010.011000.00100.0001.110.10.1	1001100.111000.01010.0010.101.10.1
0001100.101000.10010.0001.100.10.1	1010000.000111.01000.1100.101.11.0
0001100.101000.10100.0010.110.11.1	1010000.001101.00010.1100.101.11.0
0010001.010100.00010.0110.000.10.1	1010000.100011.00101.1100.010.01.0
0010001.110000.01010.0110.100.10.1	1010000.100011.10100.1110.100.11.1
0010010.101000.00110.0001.100.10.1	1010000.100101.00011.1100.010.01.0
0010010.101000.10100.1001.110.10.1	1010000.100101.01001.1100.110.11.0
0010100.101000.00011.0001.100.10.0	1010000.100101.10010.1110.100.11.1
0010100.101000.10010.1001.110.10.1	1010000.101001.00101.1100.110.11.0
0011000.000011.00100.1010.101.10.1	1010000.101001.00110.1100.101.11.0
0011000.000110.10000.0111.100.11.1	1010000.101100.10001.1110.110.11.1
0011000.000110.10000.1101.011.10.1	1010000.111000.01100.1110.111.11.1
0011000.000110.10000.1101.110.11.1	1010001.110001.00110.0011.000.00.0
0011000.010010.00100.1110.011.01.1	1010001.110001.00110.1010.010.00.1
0011000.010010.00100.1110.101.11.1	1010001.110001.00110.1010.100.10.1
0011000.100100.01010.0001.110.10.1	1010001.110010.00011.1100.100.01.0
0011000.100100.10100.0011.010.10.1	1010001.110010.00101.0110.000.01.0
0011000.101000.10100.0011.110.10.1	1010001.110010.10100.1011.001.00.0
0011100.110000.00110.1010.001.10.1	1010001.110010.10100.1011.010.00.1
0011100.110000.01010.0110.001.10.1	1010001.110100.00011.1010.001.00.0
0100010.001100.10010.0001.100.10.1	1010001.110100.00011.1010.100.10.1
0100010.011000.10010.0101.100.10.1	1010001.110100.01010.0011.010.00.1
0100010.101000.10101.0001.100.10.0	1010001.110100.01010.0011.100.01.0
0100100.010000.01010.0110.100.11.1	1010010.110001.00011.1100.100.01.0
0100100.010010.00110.0001.000.10.1	1010010.110001.00101.0110.000.01.0
0100100.101000.10011.0001.100.10.0	1010100.101010.01100.0001.001.10.0
0101000.110000.10110.0011.100.10.1	1010100.110001.00011.1010.010.00.1
0101100.010100.01010.0010.001.10.1	1010100.110001.01010.0011.001.00.0
0101100.011000.01010.0010.101.10.1	1010100.111000.01010.0110.001.10.1
0110000.001100.10010.1001.110.10.1	1010100.111000.01010.1010.101.10.1
0110000.101000.10110.0011.100.10.1	1100000.010110.11000.0111.100.11.1
0110010.100001.10101.1001.100.10.0	1100000.011010.00110.0001.100.10.1
0110100.100001.10011.1001.100.10.0	1100000.101010.10110.1000.110.01.1
0110100.101000.11010.0101.001.10.0	1100000.110010.10110.1100.010.01.1
0111000.100100.11010.1010.110.01.1	1100000.110100.10110.0011.000.10.1
1000010.110100.11000.0101.010.10.1	1100000.110100.11010.1001.110.10.1
1000010.111000.00110.0001.100.10.1	1100000.111000.10101.0011.100.10.0
1000100.010110.01000.0010.100.11.1	1100100.110010.11010.1010.001.00.1
1000100.110010.11000.1100.110.11.1	1101000.011100.00110.0010.001.10.1
1000100.110100.10010.1001.010.10.1	1101000.101010.11010.0001.101.00.0
1001000.100100.00011.1000.110.10.1	1101000.110010.11100.0101.001.10.0
1001000.100110.10010.0100.100.11.1	1110000.100110.11001.0011.100.01.0
1001000.101100.00110.0000.110.11.1	1110000.101010.10101.0011.100.01.0
1001000.110010.10001.1100.100.10.1	1110000.101010.11010.0101.001.00.0
1001000.110100.01100.0010.110.11.1	1110000.110001.10011.1110.100.01.0
1001000.110100.11000.0011.110.10.1	1110000.111000.11100.1110.011.10.1

Table 1: The 96 non-isomorphic 8-vertex tournaments T with $\mathrm{McG}(T) = 5$.

	0	1	2	3	4	5	6	7
0	-	$*$	1	1	0	0	$*$	$*$
1	$*$	-	$*$	1	1	0	0	$*$
2	0	$*$	-	$*$	1	1	0	$*$
3	0	0	$*$	-	1	1	1	0
4	1	0	0	0	-	$*$	1	1
5	1	1	0	0	$*$	-	1	0
6	$*$	1	1	0	0	0	-	1
7	$*$	$*$	$*$	1	0	1	0	-

Table 2: The adjacency matrix of the directed graph G_8.

In our computational experiments, we detected that 96 of the 6.880 non-isomorphic 8-vertex tournaments cannot be realized by three permutations. These tournaments are listed in Table 1. Each tournament is represented as the upper triangle of the adjacency matrix in row order, and consecutive rows are always separated by dots (this is the representation used in McKay's list [3]).

We also took a closer look at these 96 exceptional tournaments, and tried to understand their common properties. We used CPLEX to analyze their structure, and to identify minimal infeasible subsystems of the underlying linear integer programs. It turned out that all 96 tournaments contain the directed subgraph G_8 whose adjacency matrix is depicted in Table 2. The arcs marked by '$*$' are unspecified, and their orientation can be set arbitrarily in the tournaments. (Note: Since there are eight vertex pairs with unspecified arcs, this would yield 256 corresponding 8-vertex tournaments; however symmetries and isomorphisms reduce this number to 96.)

Observation 2. *If a tournament T contains the graph G_8 as a subgraph on eight vertices, then T has no realization by three permutations.* □

We stress that the copyright on this graph G_8 belongs to Shepardson & Tovey [5] who established that any tournament containing a subgraph G_8 has a predictability number of at most 13/20.

Finally, our programs detected that $17,674$ out of $191,536$ non-isomorphic 9-vertex tournaments cannot be realized by three permutations.

3 Conclusions

The computational approach described in this note is strong enough to handle all tournaments with $n \leq 9$ vertices. For $n = 10$ vertices the running times would still be manageable, but we did not spend much time on McKay's list [3] with $9,733,056$ non-isomorphic tournaments on ten vertices: we do not expect any surprises from them, and we firmly believe that all of them will be realizable by five permutations.

It might be interesting to determine the smallest tournament that has no realization by five permutations. We randomly explored a (tiny) fraction of the set of 20-vertex tournaments, but we did not succeed in finding anything (for $n > 20$ the computation times become prohibitively large). The counting argument of Stearns [6] yields the existence of a 41-vertex tournament T_{41} with $\mathrm{McG}(T_{41}) \geq 7$. However, for small tournaments the asymptotic bounds implied by [6] seem to be rather loose: The same counting argument only yields the existence of a 19-vertex tournament T_{19} with $\mathrm{McG}(T_{19}) \geq 5$, whereas we know that there exist 8-vertex tournaments with that property.

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