Acta Cybernetica **22** (2015) 335–347.

On a Property of Non Liouville Numbers*

Jean-Marie De Koninck† and Imre Kátai‡

Dedicated to the memory of Professor Ferenc Gécseg

Abstract

Let α be a non Liouville number and let $f(x) = \alpha x^r + a_{r-1}x^{r-1} + \cdots + a_1x + a_0 \in \mathbb{R}[x]$ be a polynomial of positive degree r. We consider the sequence $(y_n)_{n\geq 1}$ defined by $y_n = f(h(n))$, where h belongs to a certain family of arithmetic functions and show that $(y_n)_{n\geq 1}$ is uniformly distributed modulo 1.

Keywords: non Liouville numbers, uniform distribution modulo 1

1 Introduction and notation

Let $t(n)$ be an arithmetic function and let $f \in \mathbb{R}[x]$ be a polynomial. Under what conditions is the sequence $(f(t(n)))_{n\geq 1}$ uniformly distributed modulo 1? In the particular case where f is of degree one, the problem is partly solved. For instance, it is known that, if α is an irrational number and if $t(n) = \omega(n)$ or $\Omega(n)$, where $\omega(n)$ stands for the number of distinct prime factors of n and $\Omega(n)$ for the number of prime factors of n counting their multiplicity, with $\omega(1) = \Omega(1) = 0$, then the sequence $(\{\alpha t(n)\})_{n\geq 1}$ is uniformly distributed modulo 1 (here $\{y\}$ stands for the fractional part of y). In 2005, we [1] proved that if α is a positive irrational number such that for each real number $\kappa > 1$ there exists a positive constant $c = c(\kappa, \alpha)$ for which the inequality $\|\alpha q\| > c/q^\kappa$ holds for every positive integer q, then the sequence $(\{\alpha\sigma(n)\})_{n\geq 1}$ is uniformly distributed modulo 1. (Here $\|x\|$ stands for the distance between x and the nearest integer and $\sigma(n)$ stands for the sum of the positive divisors of n.) Observe that one can construct an irrational number α for which the corresponding sequence $(\{\alpha\sigma(n)\})_{n\geq 1}$ is not uniformly distributed modulo 1. On the other hand, given an integer $q \geq 2$ and letting $s_q(n)$ stand for the sum of the digits of n expressed in base q, it is not hard to prove that, if α is an irrational number, the sequence $(\{\alpha s_q(n)\})_{n\geq 1}$ is uniformly distributed modulo 1. In fact, in the past 15 years, important results have been obtained concerning the

*The research of the first author was supported in part by a grant from NSERC.

†Département de mathématiques et de statistique Université Laval, Québec G1V 0A6, Canada. E-mail: jmdk@mat.ulaval.ca

‡Computer Algebra Department, Eötvös Loránd University, 1117 Budapest, Pázmány Péter Sétány I/C, Hungary. E-mail: katai@inf.elte.hu

DOI: 10.14232/actacyb.22.2.2015.6

topic of the so-called q-ary arithmetic functions. For instance, it was proved that the sequence $(\{\alpha s_q(p)\})_{p\in\wp}$ (here $\wp$ is the set of all primes) is uniformly distributed modulo 1 if and only if $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. In 2010, answering a problem raised by Gelfond [10] in 1968, Mauduit and Rivat [13] proved that the sequence $(\{\alpha s_q(n^2)\})_{n\geq 1}$ is uniformly distributed modulo 1 if and only if $\alpha \in \mathbb{R}\setminus\mathbb{Q}$.

Recall that an irrational number β is said to be a *Liouville number* if for all integers $m \geq 1$, there exist two integers t and $s > 1$ such that

$$0 < \left|\beta - \frac{t}{s}\right| < \frac{1}{s^m}.$$

Hence, Liouville numbers are those real numbers which can be approximated "quite closely" by rational numbers.

Here, if α is a non Liouville number and

$$f(x) = \alpha x^r + a_{r-1}x^{r-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x] \quad \text{is of degree } r \geq 1, \tag{1}$$

we prove that $(f(t(n)))_{n\geq 1}$ is uniformly distributed modulo 1, for those arithmetic functions $t(n)$ for which the corresponding function $a_{N,k} := \dfrac{1}{N}\#\{n \leq N : t(n) = k\}$ is "close" to the normal distribution as N becomes large.

Given $\mathcal{P} \subseteq \wp$, let $\Omega_{\mathcal{P}}(n) = \sum_{\substack{p^r \| n \\ p\in\mathcal{P}}} r$. From here on, we let $q \geq 2$ stand for a fixed integer. Now, consider the sequence $(y_n)_{n\geq 1}$ defined by $y_n = f(h(n))$, where $h(n)$ is either one of the five functions

$$\omega(n), \qquad \Omega(n), \qquad \Omega_{\mathcal{P}}(n), \qquad s_q(n), \qquad s_q(n^2). \tag{2}$$

Here, we show that the sequence $(y_n)_{n\geq 1}$ is uniformly distributed modulo 1.

For the particular case $h(n) = s_q(n)$, we also examine an analogous problem, as n runs only through the primes. Finally, we consider a problem involving strongly normal numbers.

Recall that the *discrepancy* of a set of N real numbers $x_1, \ldots, x_N$ is the quantity

$$D(x_1, \ldots, x_N) := \sup_{[a,b)\subseteq[0,1)} \left| \frac{1}{N} \sum_{\{x_\nu\}\in[a,b)} 1 - (b-a) \right|.$$

For each positive integer N, let

$$M = M_N = \lfloor \delta_N \sqrt{N} \rfloor, \qquad \text{where } \delta_N \to 0 \text{ and } \delta_N \log N \to \infty \text{ as } N \to \infty. \tag{3}$$

We shall say that an infinite sequence of real numbers $(x_n)_{n\geq 1}$ is *strongly uniformly distributed* mod 1 if

$$D(x_{N+1}, \ldots, x_{N+M}) \to 0 \qquad \text{as} \qquad N \to \infty$$

for every choice of M (and corresponding δ_N) satisfying (3). Then, given a fixed integer $q \geq 2$, we say that an irrational number α is a *strongly normal number* in base q (or a strongly q-normal number) if the sequence $(x_n)_{n\geq 1}$, defined by $x_n = \{\alpha q^n\}$, is strongly uniformly distributed modulo 1. The concept of strong normality was recently introduced by De Koninck, Kátai and Phong [2].

We will at times be using the standard notation $e(x) := \exp\{2\pi i x\}$. Finally, we let φ stand for the Euler totient function.

2 Background results

The sum of digits function $s_q(n)$ in a given base $q \geq 2$ has been extensively studied over the past decades. Delange [4] was one of the first to study this function. Drmota and Rivat [7], [14] studied the function $s_q(n^2)$ and then, very recently, Drmota, Mauduit and Rivat [9] analyzed the distribution of the function $s_q(P(n))$, where $P \in \mathbb{Z}[x]$ is a polynomial of a certain type.

Here, we state as propositions some other results and recall two relevant results of Halász and Kátai.

First, given an integer $q \geq 2$, we set

$$\mu_q = \frac{q-1}{2}, \qquad \sigma_q^2 = \frac{q^2-1}{12}.$$

Proposition 1. *Let $\delta > 0$ be an arbitrary small number and let $\varepsilon > 0$. Then, uniformly for $\left|k - \mu_q \log_q N\right| < \frac{1}{\delta}\sqrt{\log_q N}$,*

$$\#\{n \leq N : s_q(n) = k\} = \frac{N}{\sqrt{2\pi\sigma_q^2 \log_q N}} \left(\exp\left\{ -\frac{(k-\mu_q \log_q N)^2}{2\sigma_q^2 \log_q N} \right\} + O\left(\frac{1}{\log^{\frac{1}{2}-\varepsilon} N} \right) \right).$$

Proof. This result is in fact a particular case of Proposition 3 below. □

Proposition 2. *Let $\varepsilon > 0$. Uniformly for all integers $k \geq 0$ such that $(k, q-1) = 1$,*

$$\#\{p \leq N : s_q(p) = k\} = \frac{q-1}{\varphi(q-1)} \frac{\pi(N)}{\sqrt{2\pi\sigma_q^2 \log_q N}} \left(\exp\left\{ -\frac{(k-\mu_q \log_q N)^2}{2\sigma_q^2 \log_q N} \right\} + O\left(\frac{1}{\log^{\frac{1}{2}-\varepsilon} N} \right) \right).$$

Proof. This is Theorem 1.1 in the paper of Drmota, Mauduit and Rivat [8]. □

Let $G = (G_j)_{j\geq 0}$ be a strictly increasing sequence of integers, with $G_0 = 1$. Then, each non negative integer n has a unique representation as $n = \sum_{j\geq 0} \epsilon_j(n) G_j$ with integers $\epsilon_j(n) \geq 0$ provided that $\sum_{j<k} \epsilon_j(n) G_j < G_k$ for all integers $k \geq 1$. Then, the sum of digits function $s_G(n)$ is given by

$$s_G(n) = \sum_{j\geq 0} \epsilon_j(n). \tag{4}$$

Setting $a_{N,k} := \#\{n \le N : s_G(n) = k\}$, consider the related sequence $(X_N)_{N\ge1}$ of random variables defined by

$$P(X_N = k) = \frac{a_{N,k}}{N},$$

so that the expected value of X_N and its variance are given by

$$E[X_N] = \frac{1}{N}\sum_{n\le N} s_G(n) \qquad \text{and} \qquad V[X_N] = \frac{1}{N}\sum_{n\le N}(s_G(n) - E[X_N])^2. \tag{5}$$

Let us choose the sequence $(G_j)_{j\ge0}$ as the particular sequence

$$G_0 = 1, \qquad G_j = \sum_{i=1}^{j} a_i G_{j-1} + 1 \quad (j > 0), \tag{6}$$

where the a_i's are simply the positive integers appearing in the Parry α-expansion (here $\alpha > 1$ is a real number) of 1, that is

$$1 = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \frac{a_3}{\alpha^3} + \cdots$$

It can be shown (see Theorem 2.1 of Drmota and Gajdosik [5]) that, for such a sequence $(G_j)_{j\ge0}$, setting

$$G(z,u) := \sum_{j=1}^{\infty}\left(\sum_{\ell=0}^{a_j-1} z^\ell\right) z^{a_1+\cdots+a_{j-1}} u^j$$

and letting $1/\alpha(z)$ denote the analytic solution $u = 1/\alpha(z)$ of the equation $G(z,u) = 1$ for z in a sufficiently small (complex) neighbourhood of $z_0 = 1$ such that $\alpha(1) = \alpha$, then,

$$E[X_N] = \mu\frac{\log N}{\log\alpha} + O(1)$$

and

$$V[X_N] = \sigma^2\frac{\log N}{\log\alpha} + O(1),$$

where

$$\mu = \frac{\alpha'(1)}{\alpha} \qquad \text{and} \qquad \sigma^2 = \frac{\alpha''(1)}{\alpha} + \mu - \mu^2.$$

Proposition 3. *Let $G = (G_j)_{j\ge0}$ be as in (6). If $\sigma^2 \ne 0$, then, given an arbitrary small $\varepsilon > 0$, uniformly for all integers $k \ge 0$,*

$$\#\{n \le N : s_G(n) = k\} = \frac{N}{\sqrt{2\pi V[X_N]}}\left(\exp\left\{-\frac{(k - E[X_N])^2}{2V[X_N]}\right\} + O\left(\frac{1}{\log^{\frac12-\varepsilon} N}\right)\right).$$

Proof. This is Theorem 2.2 in the paper of Drmota and Gajdosik [5]. □

Let a be a positive integer. Let $q = -a + i$ (or $q = -a - i$) and set $Q = a^2 + 1$ and $\mathcal{N} = \{0, 1, \ldots, Q - 1\}$. It is well known that every Gaussian integer z can be written uniquely as

$$z = \sum_{\ell \geq 0} \epsilon_\ell(z) q^\ell \qquad \text{with each } \epsilon_\ell \in \mathcal{N}.$$

Then, define the sum of digits function $s_q(z)$ of $z \in \mathbb{Z}[i]$ in base q as

$$s_q(z) = \sum_{\ell \geq 0} \epsilon_\ell(z).$$

Proposition 4. *Let $\mathcal{A}$ be the set of those positive integers a for which if $p \mid q = -a \pm i$ and $|p| \neq 1$, then $|p|^2 \geq 689$. Let $\mathcal{D}_N = \{z \in \mathbb{C} : |z| \leq \sqrt{N}\} \cap \mathbb{Z}[i]$ or $\mathcal{D}_N = \{z \in \mathbb{C} : |\Re(z)| \leq \sqrt{N}, |\Im(z)| \leq \sqrt{N}\} \cap \mathbb{Z}[i]$. Then, uniformly for all integers $k \geq 0$, we have*

$$\frac{1}{\#\mathcal{D}_N} \#\{z \in \mathcal{D}_N : s_q(z^2) = k\} = \frac{Q(k, q-1)}{\sqrt{2\pi\sigma_Q^2 \log_Q(N^2)}} \left(\exp\{-\frac{\Delta_k^2}{2}\} + O\left(\frac{(\log\log N)^{11}}{\sqrt{\log N}} \right) \right),$$

where

$$\Delta_k = \frac{k - \mu_Q \log_Q(N^2)}{\sqrt{\sigma_Q^2 \log_Q(N^2)}}, \quad \mu_Q = \frac{Q-1}{2}, \quad \sigma_Q^2 = \frac{Q^2 - 1}{12}.$$

Proof. This result is a simplified version of Theorem 4 in Morgenbesser [15]. □

Let $a \in \mathbb{N}$ and $q = -a + i \in \mathbb{Z}[i]$. Set $\mathcal{N} = \{0, 1, \ldots, a^2\}$. Then, every $z \in \mathbb{Z}[i]$ can be written uniquely as

$$z = \sum_{j \geq 0} \epsilon_j(z) q^j \quad \text{with each } \epsilon_j(z) \in \mathcal{N}.$$

Let L be a non negative integer and consider a function $F : \mathcal{N}^{L+1} \to \mathbb{Z}$ satisfying $F(0, 0, \ldots, 0) = 0$ and set

$$s_F(z) = \sum_{j=-L}^{\infty} F(\epsilon_j(z), \epsilon_{j+1}(z), \ldots, \epsilon_{j+L}(z)).$$

The following is due to Drmota, Grabner and Liardet [6].

Proposition 5. *Under certain conditions on F stated in Corollary 3 in Drmota, Grabner and Liardet [6],*

$$\#\{z \in \mathbb{Z}[i] : |z|^2 < N,\ s_F(z) = k\} = \frac{\pi N}{\sqrt{2\pi\sigma^2 \log_{|q|^2} N}} \exp\left\{-\frac{(k-\mu\log_{|q|^2} N)^2}{2\sigma^2 \log_{|q|^2} N}\right\}\left(1 + O\left(\frac{1}{\sqrt{\log N}}\right)\right)$$

uniformly for $|k - \mu\log_{|q|^2} N| \le c\sqrt{\log_{|q|^2} N}$, where c can be taken arbitrarily large.

For any particular set of primes $\mathcal{P}$, let $E(x) = E_{\mathcal{P}}(x) := \sum_{\substack{p\le x\\ p\in\mathcal{P}}} \frac{1}{p}$.

The following two results, which we state as propositions, are due respectively to Halász [11] and Kátai [12].

Proposition 6. (Halász) *Let $0 < \delta \le 1$ and let $\mathcal{P}$ be a set of primes with corresponding functions $\Omega_{\mathcal{P}}(n)$ and $E(x) = E_{\mathcal{P}}(x)$. Then, assuming that $E(x) \to \infty$ as $x \to \infty$, the estimate*

$$\sum_{\substack{n\le x\\ \Omega_{\mathcal{P}}(n)=k}} 1 = \frac{xE(x)^k}{k!} e^{-E(x)} \left\{1 + O\left(\frac{|k - E(x)|}{E(x)}\right) + O\left(\frac{1}{\sqrt{E(x)}}\right)\right\}$$

holds uniformly for all positive integers k and real numbers $x \ge 3$ satisfying

$$E(x) \ge \frac{8}{\delta^3} \qquad \text{and} \qquad \delta \le \frac{k}{E(x)} \le 2 - \delta.$$

Proposition 7. (Kátai) *For $1 \le h \le x$, let*

$$A_k(x,h) := \sum_{\substack{x\le n\le x+h\\ \omega(n)=k}} 1, \qquad B_k(x) := \sum_{\substack{n\le x\\ \omega(n)=k}} 1,$$

$$\delta_k(x,h) := \frac{A_k(x,h)}{h} - \frac{B_k(x)}{x}, \qquad E(x,h) := \sum_{k=1}^{\infty} \delta_k^2(x,h).$$

Letting $\varepsilon > 0$ be an arbitrarily small number and $x^{7/12+\varepsilon} \le h \le x$, then

$$E(x,h) \ll \frac{1}{\log^2 x \cdot \sqrt{\log\log x}}.$$

3 Main results

Theorem 1. *Let $f(x)$ be as in (1), $h(n)$ be one of the five functions listed in (2) and $y_n := f(h(n))$. Then, the sequence $(y_n)_{n\ge1}$ is uniformly distributed modulo 1.*

Theorem 2. *Let $f(x)$ be as in (1). Then, the sequence $(z_p)_{p\in\wp}$, where $z_p := f(s_q(p))$, is uniformly distributed modulo 1.*

Theorem 3. *Let $Q \geq 2$ and $q \geq 2$ be fixed integers. Let α be a strongly Q-normal number. Let g be a real valued continuous function defined on $[0,1]$ such that $\int_0^1 g(x)\,dx = 0$. Then,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(\alpha Q^{h(n)}) = 0, \tag{7}$$

where $h(n) = s_q(n)$ or $s_q(n^2)$. Moreover, letting $\pi(N)$ stand for the number of prime numbers not exceeding N, we have

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p\leq N} g(\alpha Q^{s_q(p)}) = 0. \tag{8}$$

The following corollary follows from estimate (7) of Theorem 3.

Corollary 1. *With α and $h(n)$ as in Theorem 3, the sequence $(\alpha Q^{h(p)})_{p\in\wp}$ is uniformly distributed modulo 1.*

In light of Proposition 3, we have the following two corollaries.

Corollary 2. *Let G be as in (4). Then, letting f be as in (1), the sequence $(\{f(s_G(n))\})_{n\geq 0}$ is uniformly distributed modulo 1.*

Corollary 3. *Let G be as in (4). Then, if α is a strongly normal number in base Q, the sequence $(\{\alpha \cdot Q^{s_G(n)}\})_{n\geq 0}$ is uniformly distributed modulo 1.*

As a direct consequence of the Main Lemma and of Proposition 4, we have the following result.

Theorem 4. *Let $\mathcal{D}_N$ be as in Proposition 4. Let f be as in (1). For each $z \in \mathcal{D}_N$, set $y_z := f(s_q(z^2))$. Then, the discrepancy of the sequence y_z tends to 0 as $N \to \infty$, that is*

$$D(y_z : z \in \mathcal{D}_N) \to 0 \qquad \text{as } N \to \infty.$$

Theorem 5. *Let $\mathcal{D}_N$ be as in Proposition 4. Let α be a strongly normal number in base Q and consider the sequence $(y_z)_{z\in\mathcal{D}_N}$. Then*

$$D(y_z : z \in \mathcal{D}_N) \to 0 \qquad \text{as } N \to \infty.$$

In line with Proposition 7, we have the following.

Theorem 6. *Let $\varepsilon > 0$ be a fixed number. Let $H = \lfloor x^{7/12+\varepsilon} \rfloor$ and set*

$$\pi_k([x, x+H]) := \#\{n \in [x, x+H] : \omega(n) = k\}.$$

Let f be as in (1) and set

$$S(x) = \sum_{x\leq n\leq x+H} e(f(\omega(n))).$$

Then

$$\frac{S(x)}{H} \to 0 \qquad \text{as } x \to \infty.$$

4 Preliminary lemmas

Lemma 1. *Let α be a non Liouville number and let $f(x)$ be as in (1). Then,*

$$\sup_{U\geq 1}\frac{1}{N}\left|\sum_{n=U+1}^{U+N} e(f(n))\right|\to 0 \qquad \text{as } N\to\infty.$$

Proof. Since α is a non Liouville number, there exists a positive integer ℓ such that if τ is a fixed positive number and

$$\left|\alpha-\frac{t}{s}\right|\leq\frac{1}{s\tau},\quad (t,s)=1,\quad s\leq\tau,$$

then $\tau^{1/\ell}<s$.

Vaughan ([16], Lemma 2.4) proved that if $\left|\alpha-\frac{t}{s}\right|<\frac{1}{s^2}$ and $K=2^{t-1}$, then, given any small number $\varepsilon>0$,

$$\sum_{n=U+1}^{U+N} e(f(n))\ll_\varepsilon N^{1+\varepsilon}\left(\frac{1}{s}+\frac{1}{N}+\frac{s}{N^t}\right)^{1/K}. \tag{9}$$

Now, choose $\tau=N^{t/2}$ so that $N^{t/2^\ell}<s<\tau$. It then follows from (9) that

$$\sum_{n=U+1}^{U+N} e(f(n))\ll N^{1-\delta},$$

for some $\delta>0$ which depends only on ε and ℓ, thus completing the proof of Lemma 1. □

Using this result, we can establish our Main Lemma.

Lemma 2. *(***Main Lemma***) For each positive integer N, let $(E_N(k))_{k\geq 1}$ be a sequence of non negative integers called weights which, given any $\delta>0$, satisfies the following three conditions:*

(a) $\displaystyle\sum_{k=1}^{\infty}E_N(k)=1$;

(b) there exists a sequence $(L_N)_{N\geq 1}$ which tends to infinity as $N\to\infty$ such that

$$\limsup_{N\to\infty}\sum_{\substack{k=1\\ \frac{|k-L_N|}{\sqrt{L_N}}>\frac{1}{\delta}}}^{\infty}E_N(k)\to 0 \qquad \text{as } \delta\to 0;$$

(c) $\displaystyle\lim_{N\to\infty}\max_{\frac{|k-L_N|}{\sqrt{L_N}}\leq\frac{1}{\delta}}\max_{1\leq\ell\leq\delta^{3/2}}\left|\frac{E_N(k+\ell)}{E_N(k)}-1\right|=0.$

Moreover, let α and f be as in (1) and let

$$T_N(f) := \sum_{k=1}^{\infty} e(f(k)) E_N(k).$$

Then,

$$T_N(f) \to 0 \qquad \textit{as } N \to \infty. \tag{10}$$

Proof. Let $\delta > 0$ be fixed and set

$$S := \lfloor \delta^{3/2} \sqrt{L_N} \rfloor, \quad t_m = \lfloor L_N \rfloor + mS \ \ (m = 1, 2, \ldots),$$
$$U_m = [t_m, t_{m+1} - 1] \ \ (m = 1, 2, \ldots).$$

Let us now write

$$T_N(f) = S_1(N) + S_2(N), \tag{11}$$

where

$$\begin{aligned} S_2(N) &= \sum_{|k-L_N| > \frac{1}{\delta}\sqrt{L_N}} E_k(N) e(f(k)), \\ S_1(N) &= \sum_{|m| \le 1/\delta^{5/2}} \sum_{k \in U_m} E_k(N) e(f(k)) = \sum_{|m| \le 1/\delta^{5/2}} S_1^{(m)}(N), \end{aligned}$$

say.

First observe that, by condition (b) above,

$$|S_2(N)| \le \sum_{\frac{|k-L_N|}{\sqrt{L_N}} > \frac{1}{\delta}} E_N(k) = o(1) \qquad \text{as } N \to \infty. \tag{12}$$

On the other hand, it follows from condition (c) above and Lemma 1 that, as $N \to \infty$,

$$\begin{aligned} \left| S_1^{(m)}(N) \right| &\le E_{t_m}(N) \left| \sum_{k \in U_m} e(f(k)) \right| + o(1) \sum_{k \in U_m} E_k(N) \\ &= o(1) S E_{t_m}(N) + o(1) \sum_{k \in U_m} E_k(N), \end{aligned}$$

while

$$\left| S E_{t_m}(N) - \sum_{k \in U_m} E_k(N) \right| = o(1) \sum_{k \in U_m} E_k(N).$$

Gathering these two estimates, we obtain that

$$S_1(N) \to 0 \quad \text{as } N \to \infty. \tag{13}$$

Using (12) and (13) in (11), conclusion (10) follows. □

Lemma 3. *For each integer $k \geq 1$, let*

$$\begin{aligned} \pi_k(x) &:= \#\{n \leq x : \omega(n) = k\}, \\ \pi_k^*(x) &:= \#\{n \leq x : \Omega(n) = k\} \end{aligned}$$

Then, the relations

$$\begin{aligned} \pi_k(x) &= (1 + o(1)) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}, \\ \pi_k^*(x) &= (1 + o(1)) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \end{aligned}$$

hold uniformly for

$$|k - \log\log x| \leq \frac{1}{\delta_x} \sqrt{\log\log x}, \tag{14}$$

where δ_x is some function of x chosen appropriately and which tends to 0 as $x \to \infty$.

Proof. This follows from Theorem 10.4 stated in the book of De Koninck and Luca [3]. □

5 Proof of Theorem 1

We first consider the case when $h(n)$ is one of the three functions $\omega(n)$, $\Omega(n)$ and $\Omega_E(n)$. Set

$$\begin{aligned} \pi_k(N) &= \#\{n \leq N : \omega(n) = k\}, \\ \pi_k^*(N) &= \#\{n \leq N : \Omega(n) = k\}, \\ T_k(N) &= \#\{n \leq N : \Omega_E(n) = k\}. \end{aligned}$$

In light of Lemma 3 and Proposition 6, the corresponding weights of the sequences $(\pi_k(N))_{k\geq 1}$, $(\pi_k^*(N))_{k\geq 1}$ and $(T_k(N))_{k\geq 1}$ are $\pi_k(N)/N$, $\pi_k^*(N)/N$ and $T_k(N)/N$, respectively.

Now, in order to obtain the conclusion of the Theorem, we only need to prove that, for each non zero integer m,

$$\frac{1}{N} \sum_{n \leq N} e(mf(h(n))) \to 0 \qquad \text{as } N \to \infty.$$

But this is guaranteed by Lemma 1 if we take into account the fact that since α is a non Liouville number, the number $m\alpha$ is also non Liouville for each $m \in \mathbb{Z} \setminus \{0\}$. Hence, the theorem is proved.

6 Proof of Theorem 2

We cannot make a direct use of Lemma 2 because the estimate in that lemma only holds for those positive integers k such that $(k, q-1) = 1$. To avoid this obstacle, we shall subdivide the positive integers k according to their residue class modulo $q-1$. Observe that there are $\varphi(q-1)$ such classes. Hence, we write each k as

$$k = t(q-1) + \ell, \qquad (\ell, q-1) = 1.$$

Hence, for each positive integer ℓ such that $(\ell, q-1) = 1$, we set

$$\wp_\ell := \{p \in \wp : s_q(p) \equiv \ell \pmod{q-1}\}, \qquad \Pi_\ell(N) := \#\{p \le N : p \in \wp_\ell\}. \quad (15)$$

It is easy to verify that

$$\frac{\Pi_\ell(N)}{\pi(N)} = (1 + o(1))\frac{1}{\varphi(q-1)} \qquad (N \to \infty). \quad (16)$$

Thus, in order to prove Theorem 2, we need to show that the sum

$$U_\ell(N) := \sum_{\substack{p \le N \\ s_q(p) \equiv \ell \pmod{q-1}}} e(mf(s_q(p))),$$

where m is any fixed non zero integer, satisfies

$$U_\ell(N) = o(1) \quad \text{as } N \to \infty. \quad (17)$$

Setting

$$\sigma_N(k) := \#\{p \le N : s_q(p) = k\},$$

we have

$$\begin{aligned} U_\ell(N) &= \sum_{k \equiv \ell \pmod{q-1}} e(mf(k))\sigma_N(k) \\ &= \sum_{t \ge 0} e(mf(t(q-1) + \ell))\sigma_N(t(q-1) + \ell). \end{aligned} \quad (18)$$

Observe that the leading coefficient of the above polynomial $f(t(q-1) + \ell)$ is $\alpha(q-1)^k$, which is a non Liouville number as well (as we mentioned in the proof of Theorem 1), and also that the functions

$$w_N(t) := \frac{1}{\Pi_\ell(N)}\sigma_N(t(q-1) + \ell)$$

may be considered as weights (since $\sum_{k=1}^{\infty} w_N(t) = 1$). Thus, applying Lemma 2, we obtain (17), thereby completing the proof of Theorem 2.

7 Proof of Theorem 3

We shall skip the proof of estimate (7), since it can be obtained along the same lines as that of the main theorem in De Koninck, Kátai and Phong [2].

In order to obtain (8), we separate the set $\wp$ into $\varphi(q-1)$ distinct sets $\wp_\ell$, with corresponding counting function $\Pi_N(\ell)$ defined in (15).

Observe that

$$g(\alpha Q^{t(q-1)+\ell})\sigma_N(t(q-1)+\ell) = g((\alpha Q^\ell)\cdot Q^{t(q-1)})\sigma_N(t(q-1)+\ell)$$

Now, since α is a strongly Q-normal number, then so is αQ^ℓ, a number which is strongly Q^{q-1}-normal.

We then have

$$\begin{aligned}\sum_{p\le N} g(\alpha Q^{s_q(p)}) &= \sum_{k\ge 1}\sum_{\substack{p\le N\\ s_q(p)=k}} g(\alpha Q^k)\\ &= \sum_{\substack{\ell=1\\ (\ell,q-1)=1}}^{q-1}\sum_{\substack{p\le N\\ p\in\wp_\ell}} g(\alpha Q^{t(q-1)+\ell})\sigma_N(t(q-1)+\ell)\\ &= \sum_{\substack{\ell=1\\ (\ell,q-1)=1}}^{q-1}\sum_{\substack{p\le N\\ p\in\wp_\ell}} g((\alpha Q^\ell)\cdot Q^{t(q-1)})\sigma_N(t(q-1)+\ell).\end{aligned}$$

Since we then have

$$\lim_{N\to\infty}\frac{1}{\Pi_\ell(N)}\sum_{\substack{p\le N\\ p\in\wp_\ell}} g(\alpha Q^{s_q(p)}) = 0 \quad \text{for each } \ell \text{ with } (\ell,q-1)=1,$$

summing up over all ℓ's such that $(\ell,q-1)=1$, estimate (8) follows immediately.

References

[1] J.M. De Koninck and I. Kátai, *On the distribution modulo 1 of the values of* $F(n)+\alpha\sigma(n)$, Publicationes Mathematicae Debrecen **66** (2005), 121–128.

[2] J.M. De Koninck, I. Kátai and B.M. Phong, *On strong normality*, preprint.

[3] J.M. De Koninck and F. Luca, *Analytic Number Theory: Exploring the Anatomy of Integers*, Graduate Studies in Mathematics, Vol. 134, American Mathematical Society, Providence, Rhode Island, 2012.

[4] H. Delange, *Sur la fonction sommatoire de la fonction "somme des chiffres"*, Enseign. Math. (2) **21**.1 (1975), 31–47.

[5] M. Drmota and J. Gajdosik, *The distribution of the sum-of-digits function*, J. Théor. Nombres Bordeaux **10** (1998), No. 1, 17–32.

[6] M. Drmota, P.J. Grabner and P. Liardet, *Block additive functions on the Gaussian integers*, Acta Arith. **135** (2008), No. 4, 299–332.

[7] M. Drmota and J. Rivat, *The sum of digits function of squares*, J. London Math. Soc. **72** (2005), 273–292.

[8] M. Drmota, C. Mauduit and J. Rivat, *Primes with an average sum of digits*, Compos. Math. **145** (2009), No. 2, 271–292.

[9] M. Drmota, C. Mauduit and J. Rivat, *The sum of digits function of polynomial sequences*, J. Lond. Math. Soc. (2) **84** (2011), No. 1, 81–102.

[10] A. O. Gelfond, *Sur des nombres qui ont des propriétés additives et multiplicatives données*, Acta Arith. **13** (1968), 259–265.

[11] G. Halász, *On the distribution of additive and the mean values of multiplicative arithmetic functions*, Studia Sci. Math. Hungar. **6** (1971), 211–233.

[12] I. Kátai, *A remark on a paper of K. Ramachandra* in Number theory (Ootacamund, 1984), 147–152, Lecture Notes in Math., 1122, Springer, Berlin, 1985.

[13] C. Mauduit and J. Rivat, *Sur un problème de Gelfond: la somme des chiffres des nombres premiers*, Ann. of Math. (2) **171** (2010), no. 3, 1591-1646.

[14] C. Mauduit and J. Rivat, *La somme des chiffres des carrés*, Acta Math. **203** (2009), No. 1, 107–148.

[15] J. F. Morgenbesser, *The sum of digits of squares in* $\mathbb{Z}[i]$, J. Number Theory **130** (2010), 1433–1469.

[16] R. C. Vaughan, *The Hardy-Littlewood method*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 2, 433–437.

Received 25th February 2015