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Asymptotic Approximation for the Quotient Complexities of Atoms

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For Ferenc Gécseg, in memoriam

Abstract

In a series of papers, Brzozowski together with Tamm, Davies, and Szykuła studied the quotient complexities of atoms of regular languages [6, 7, 3, 4]. The authors obtained precise bounds in terms of binomial sums for the most complex situations in the following five cases: ($\mathcal{G}$): general, ($\mathcal{R}$): right ideals, ($\mathcal{L}$): left ideals, ($\mathcal{T}$): two-sided ideals and ($\mathcal{S}$): suffix-free languages. In each case let $\kappa_{\mathcal{C}}(n)$ be the maximal complexity of an atom of a regular language L, where L has complexity $n \geq 2$ and belongs to the class $\mathcal{C} \in \{\mathcal{G}, \mathcal{R}, \mathcal{L}, \mathcal{T}, \mathcal{S}\}$. It is known that $\kappa_{\mathcal{T}}(n) \leq \kappa_{\mathcal{L}}(n) = \kappa_{\mathcal{R}}(n) \leq \kappa_{\mathcal{G}}(n) < 3^n$ and $\kappa_{\mathcal{S}}(n) = \kappa_{\mathcal{L}}(n-1)$. We show that the ratio $\frac{\kappa_{\mathcal{C}}(n)}{\kappa_{\mathcal{C}}(n-1)}$ tends exponentially fast to 3 in all five cases but it remains different from 3. This behaviour was suggested by experimental results of Brzozowski and Tamm; and the result for $\mathcal{G}$ was shown independently by Luke Schaeffer and the first author soon after the paper of Brzozowski and Tamm appeared in 2012. However, proofs for the asymptotic behavior of $\frac{\kappa_{\mathcal{G}}(n)}{\kappa_{\mathcal{G}}(n-1)}$ were never published; and the results here are valid for all five classes above. Moreover, there is an interesting oscillation for all $\mathcal{C}$: for almost all n we have $\frac{\kappa_{\mathcal{C}}(n)}{\kappa_{\mathcal{C}}(n-1)} > 3$ if and only if $\frac{\kappa_{\mathcal{C}}(n+1)}{\kappa_{\mathcal{C}}(n)} < 3$.

1 Introduction and Preliminaries

Let Σ denote a finite non-empty alphabet, Σ^* the set of words over Σ and $1 \in \Sigma^*$ the empty word. A *language* L is a subset of Σ^*. A class of languages is called a *Boolean algebra* if it is closed under finite unions and complementation. By $L \subseteq \Sigma^*$ we denote a regular language with $\emptyset \neq L \neq \Sigma^*$. The set of regular languages is denoted by $\mathcal{G}$, because it is the "general" case, here. The set $\overline{L} = \Sigma^* \setminus L$ is the complement of L. The language L is a left, right or two-sided ideal if $L = \Sigma^* L$, $L = L\Sigma^*$ or $L = \Sigma^* L \Sigma^*$. A language L is suffix-free if $w \in L$ and $xw \in L$ implies $x = 1$. We denote by $\mathcal{L}, \mathcal{R}, \mathcal{T}$ and

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$\mathcal{S}$ the classes of $\mathcal{L}$eft ideals, $\mathcal{R}$ight ideals, $\mathcal{T}$wo-sided ideals and $\mathcal{S}$uffix-free languages, respectively.

For $x \in \Sigma^*$ denote by $L(x) = \{y \in \Sigma^* \mid xy \in L\}$ the (left) quotient of L by x. Frequently, a left quotient $L(x)$ is also denoted by $x^{-1}L$. We prefer the notation $L(x)$ because Σ^* acts naturally on the right; and then the formula for the action becomes $L(x) \cdot y = L(xy)$. Indeed, the classical Myhill-Nerode Theorem asserts that this action leads to the minimal deterministic finite automaton accepting L. The set of states for this DFA is $Q_L = \{L(x) \mid x \in \Sigma^*\}$, the initial state is $L = L(1)$ and the final states are those $L(x)$ with $1 \in L(x)$. The transitions are given by $L(x) \cdot a = L(xa)$ for $x \in \Sigma^*$ and $a \in \Sigma$. The size $|Q_L|$ is therefore the number of quotients of L. It is also called the *quotient complexity*, or simply the *complexity*, of L; and the complexity of L is denoted by $\kappa(L)$.

Given a regular language L it is natural to consider the smallest Boolean algebra $\mathrm{BQ}(L)$ which contains L and is closed under quotients. A priori, it is not obvious that $\mathrm{BQ}(L)$ is finite; but it is: every set in $\mathrm{BQ}(L)$ can be written as a union of *atoms* A_S where $S \subseteq Q_L$ and

$$A_S = \bigcap_{L(x) \in S} L(x) \cap \bigcap_{L(y) \notin S} \overline{L(y)}.$$

Atoms have been introduced by Brzozowski and Tamm in [5, 2]. The complexity of atoms was studied in [6, 7].

More generally, for $X, Y \subseteq Q_L$ define

$$L(X, Y) = \bigcap_{L(x) \in X} L(x) \cap \bigcap_{L(y) \in Y} \overline{L(y)}.$$

In particular, $A_S = L(S, Q_L \setminus S)$.

The observation $L(X, Y)(w) = L(X(w), Y(w)) = L(X', Y')$ with $X' = \{L(xw) \mid L(x) \in X\}$ and $Y' = \{L(xw) \mid L(x) \in Y\}$ leads to the following remark.

Remark 1.1. Let L be regular, n its complexity and $X, Y \subseteq Q_L$. Then the following assertions hold.

- $X \cap Y \neq \emptyset$ implies $L(X, Y) = \emptyset$.
- The non-empty quotients of A_S have the form $L(X, Y)$ with $|X| \leq |S|$ and $X \cap Y = \emptyset$.
- $S \neq T$ implies $A_S \cap A_T = \emptyset$.
- Since $|\{A_S \mid S \subseteq Q_L\}| \leq 2^n$ and since every element in $\mathrm{BQ}(L)$ is a union of atoms, we have $|\mathrm{BQ}(L)| \leq 2^{2^n}$. The upper bound 2^{2^n} is optimal: It is proved in [6] that for every $n \geq 2$ there exists a language L of complexity n with 2^n atoms. As $A_S \cap A_T = \emptyset$ for $S \neq T$, the atoms form a partition of Σ^*. Hence, there are 2^{2^n} distinct unions of atoms.

A 3-*coloring* of Q is a disjoint union $Q = X \cup Y \cup W$ where X, Y, W are called colors. Thus, there are 3^n different 3-colorings. A combinatorial interpretation leads to the well-known formula

$$3^n = \sum_{x=0}^{n} \sum_{y=0}^{n-x} \binom{n}{x} \binom{n-x}{y}.$$

Indeed, each 3-coloring is uniquely described by first choosing the elements with color X out of n elements and then choosing the elements with color Y out of the remaining $n - |X|$ elements. As $X \cap Y = \emptyset$ induces a unique 3-coloring with $W = Q \setminus (X \cup Y)$, there are at most 3^n non-empty sets of the form $L(X, Y)$. We will use the concept of 3-colorings in order to give a combinatorial interpretation for the bounds of [3].

2 Upper bounds

In this section we will deduce simple upper bounds for the complexity of atoms in each case by making observations on the structure of the quotients. These upper bounds are not optimal, but straightforward and still good enough to show the asymptotic behaviour.

Lemma 2.1. *Let L be a regular language of complexity $n \geq 2$ and A_S be an atom of L. Then A_S has complexity of at most $3^n + 1$.*

Proof. There are at most 3^n quotients of the form $L(X, Y)$ and the empty set. □

Lemma 2.2. *Let L be a right ideal of complexity $n \geq 2$ and A_S be an atom of L. Then A_S has complexity of at most 3^{n-1}.*

Proof. For all x with $1 \in L(x)$ we have $1 \cdot w \in L\Sigma^*(x) = L(x)$ for all $w \in \Sigma^*$ and, thus, $L(x) = \Sigma^*$. Therefore, Σ^* is the unique final state in Q_L. Additionally, we must have $\Sigma^* \in S$, as $\Sigma^* \notin S$ implies $A_S = \emptyset$. By $\Sigma^*(x) = \Sigma^*$ for all $x \in \Sigma^*$, we see that every quotient $A_S(x) = L(X, Y)$ must contain Σ^* in X. Thus, there are at most 3^{n-1} quotients $A_S(x)$, which shows that A_S has complexity of at most 3^{n-1}. □

Lemma 2.3. *Let L be a left ideal of complexity $n \geq 2$ and A_S be an atom of L. Then A_S has complexity of at most $3^{n-1} + 2$.*

Proof. As $L = \Sigma^* L$, we have

$$L \subseteq L(x) = \{y \in \Sigma^* \mid xy \in L\} = \{y \in \Sigma^* \mid xy \in \Sigma^* L\}$$

for all $x \in \Sigma^*$. Hence, for any X with $L \in X$ we have

$$L(X, Y) = L \cap \bigcap_{L(y) \in Y} \overline{L(y)}.$$

Thus, if $Y \neq \emptyset$ then $L(X, Y) = \emptyset$. Also, $L \subseteq L(x)$ implies $\overline{L(x)} \subseteq \overline{L}$ which yields $L(X, Y) = L(X, Y \cup \{L\})$ for $L \notin X$. It follows that there are at most $3^{n-1} + 2$ quotients.

The first term counts the $L(X, Y)$ with $X = \{L\}$ which is not smaller than to count the $L(X, Y)$ with $L \in X$. By the argument above, only $L(\{L\}, \emptyset)$ and $\emptyset$ are of this type.

The second term counts those (X, Y, W) with $L \notin X$ (in which case we can assume $L \in Y$ by the argumentation above).

□

Lemma 2.4. *Let L be a two-sided ideal of complexity $n \geq 2$ and A_S be an atom of L. Then A_S has complexity of at most $3^{n-2}+2$.*

Proof. This is similar to the analysis in the case of left ideals, since there are only two cases with $L \in X$. Again, for $L \notin X$ we have $L(X,Y) = L(X, Y \cup \{L\}))$, i.e., we may assume $L \in Y$. As every two-sided ideal is in particular a right ideal, we have that Σ^* is the unique final state in Q_L. Again, only those $L(X,Y)$ with $\Sigma^* \in X$ are reachable as quotients of an atom. Thus, we can deduce that A_S has at most $3^{n-2}+2$ quotients. □

3 Lower bounds

In this section we revisit the complexity bounds of atoms for left, right and two-sided ideals obtained by Brzozowski, Tamm and Davies. The bounds are optimal. We use them to derive (weaker) lower bounds in explicit form. For $|S| \in \mathcal{O}(1)$ or $n - |S| \in \mathcal{O}(1)$ it holds $\kappa(A_S) \in \mathcal{O}(2^n)$ where A_S is an atom of a language L of complexity n. As we are only interested in the maximal complexity of atoms of some language L, we will restrict the proposition below to $0 < |S| < n-1$. This excludes special cases not needed in our analysis.

Proposition 3.1 ([7, 1, 3])**.** *Let $k, n \in \mathbb{N}$ with $0 < k < n-1$ and $\mathcal{C} \in \{\mathcal{G}, \mathcal{R}, \mathcal{L}, \mathcal{T}\}$. Then there there exists a language $L \in \mathcal{C}$ of complexity n and an atom A_S of L with $|S| = k$ such that the complexity of A_S is given by:*

$$\kappa(A_S) = \begin{cases} 1 + \sum_{x=1}^{|S|} \sum_{y=1}^{n-|S|} \binom{n}{x}\binom{n-x}{y}, & \text{for } \mathcal{C} = \mathcal{G} \\ 1 + \sum_{x=1}^{|S|} \sum_{y=1}^{n-|S|} \binom{n-1}{x-1}\binom{n-x}{y}, & \text{for } \mathcal{C} = \mathcal{R} \\ 1 + \sum_{x=1}^{|S|} \sum_{y=1}^{n-|S|} \binom{n-1}{x}\binom{n-x-1}{y-1}, & \text{for } \mathcal{C} = \mathcal{L} \\ 1 + \sum_{x=1}^{|S|} \sum_{y=1}^{n-|S|} \binom{n-2}{x-1}\binom{n-x-1}{y-1}, & \text{for } \mathcal{C} = \mathcal{T}. \end{cases}$$

Moreover, for every L of complexity n in the corresponding class $\mathcal{C}$ and every S, the right hand sides are upper bounds.

Remark 3.1. The maximal complexity of atoms of left ideals and right ideals turns out to be same. This was also observed in [3]. Indeed, using the trinomial revision (see for example [8]) for the last equality below, we can do the following calculation:

$$\begin{aligned} \sum_{x=1}^{|S|} \sum_{y=1}^{n-|S|} \binom{n-1}{x-1}\binom{n-x}{y} &= \sum_{y=1}^{n-|S|} \sum_{x=1}^{|S|} \binom{n-1}{x-1}\binom{n-x}{y} \\ &= \sum_{x=1}^{n-|S|} \sum_{y=1}^{|S|} \binom{n-1}{y-1}\binom{n-y}{x} \\ &= \sum_{x=1}^{n-|S|} \sum_{y=1}^{|S|} \binom{n-1}{x}\binom{n-x-1}{y-1}. \end{aligned}$$

In the following we give a combinatorial interpretation of the sums in Proposition 3.1.

Lemma 3.1. *For every $n \geq 3$ there exists a regular language L of complexity n such that L has an atom A_S of complexity in $3^n - \Theta(8^{n/2})$.*

Proof. Let S be such that $|S| = n/2$ (if n is even; the proof is similar if n is odd). By Proposition 3.1 there exists a regular language L of complexity n such that the atom A_S of L has complexity $1+\sum_{x=1}^{n/2}\sum_{y=1}^{n/2}\binom{n}{x}\binom{n-x}{y}$ for some $S \subseteq Q_L$. Observe that $\sum_{x=0}^{n}\sum_{y=0}^{n-x}\binom{n}{x}\binom{n-x}{y} = 3^n$ has the combinatorial interpretation of counting all 3-colorings of $Q = X \cup Y \cup W$. We will count the 3-colorings which are missing in $\sum_{x=1}^{n/2}\sum_{y=1}^{n/2}\binom{n}{x}\binom{n-x}{y}$. As the indices start with 1 instead of 0 and end with $n/2$ instead of n, the cases for $X = \emptyset$ or $Y = \emptyset$ and for $|X| > n/2$ or $|Y| > n/2$ are missing. There are 2^n possibilities with $|X| = 0$ and 2^n many with $|Y| = 0$. There are at most 2^n possibilities for X with $|X| > n/2$. Since $|X| > n/2$, we must have $|Y| < n/2$ and, thus, there are at most $2^{n/2}$ choices remaining for Y. This leaves at most $2^n \cdot 2^{n/2} = 8^{n/2}$ missing 3-colorings with $|X| > n/2$. The case $|Y| > n/2$ is symmetrical. Combining all those cases shows that the number of missing 3-colorings is in $\Theta(8^{n/2})$. □

Lemma 3.2. *For every $n \geq 3$ there exists a right ideal L of complexity n such that L has an atom A_S of L with complexity in $3^{n-1} - \Theta(8^{n/2})$.*

Proof. By Proposition 3.1 we obtain a right ideal L of complexity n such that L has an atom A_S of complexity $1 + \sum_{x=1}^{n/2}\sum_{y=1}^{n/2}\binom{n-1}{x-1}\binom{n-x}{y}$. Observe that $\sum_{x=1}^{n}\sum_{y=0}^{n-x}\binom{n-1}{x-1}\binom{n-x}{y} = 3^{n-1}$ has the combinatorial interpretation of counting 3-colorings of $Q = X \cup Y \cup W$ with a precolored element $\Sigma^* \in X$ (see Section 2 on why Σ^* is in X). Again, we count the 3-colorings which are missing in $\sum_{x=1}^{n/2}\sum_{y=1}^{n/2}\binom{n-1}{x-1}\binom{n-x}{y}$; namely, those with $Y = \emptyset$, $|X| > n/2$ or $|Y| > n/2$. The analysis in the proof of Lemma 3.1 shows that this is in $\Theta(8^{n/2})$. □

Lemma 3.3. *For every $n \geq 3$ there exists a two-sided ideal L of complexity n such that there is an atom A_S of L with complexity in $3^{n-2} - \Theta(8^{n/2})$.*

Proof. By Proposition 3.1 we obtain a two-sided ideal L of complexity n such that L has an atom A_S of complexity $1 + \sum_{x=1}^{n/2}\sum_{y=1}^{n/2}\binom{n-2}{x-1}\binom{n-x-1}{y-1}$. We count the number of 3-colorings of $Q = X \cup Y \cup W$ with precolored elements $L \in Y$ and $\Sigma^* \in X$. There are $3^{n-2} = \sum_{x=1}^{n}\sum_{y=1}^{n-x}\binom{n-2}{x-1}\binom{n-x-1}{y-1}$ such 3-colorings. Thus, in $\sum_{x=1}^{n/2}\sum_{y=1}^{n/2}\binom{n-2}{x-1}\binom{n-x-1}{y-1}$ the 3-colorings with $|X| > n/2$ or $|Y| > n/2$ are not counted. The analysis in the proof of Lemma 3.1 shows that this is in $\Theta(8^{n/2})$. □

4 Asymptotic behaviour

As above, let $\mathcal{C}$ be one of the classes: ($\mathcal{G}$) general regular languages, ($\mathcal{R}$) right ideals, ($\mathcal{L}$) left ideals, ($\mathcal{T}$) two-sided ideals or ($\mathcal{S}$) suffix-free languages. Define

$$\kappa_{\mathcal{C}}(n) = \max\{\kappa(A_S) \mid A_S \text{ is an atom of } L \in \mathcal{C} \text{ of complexity } n\}.$$

This section studies the behaviour of $\kappa_{\mathcal{C}}(n)/\kappa_{\mathcal{C}}(n-1)$ as a function in n.

n	8	9	10	11	12	13	14	15
$\kappa_{\mathcal{G}}(n)$	5083	15361	48733	146169	455797	1364091	4212001	12601332
ratio	3.284	3.022	3.173	2.999	3.118	2.992	3.088	2.992

Table 1: $\kappa_{\mathcal{G}}(n)$ and the ratio $\kappa_{\mathcal{G}}(n)/\kappa_{\mathcal{G}}(n-1)$ for some small n

4.1 Asymptotic Approximation

Combining the explicit lower and upper bounds we obtain the following result which was announced in [3].

Theorem 4.1. *Let $\mathcal{C} \in \{\mathcal{G}, \mathcal{L}, \mathcal{R}, \mathcal{T}, \mathcal{S}\}$. Then the ratio $\kappa_{\mathcal{C}}(n)/\kappa_{\mathcal{C}}(n-1)$ converges exponentially fast to* 3.

Proof. First, we will prove this for the class of right ideals. By Lemma 3.2 and Lemma 2.2 we have

$$3^{n-1} - f(n) \leq \kappa_{\mathcal{R}}(n) \leq 3^{n-1}$$

for some $f \in \Theta(8^{n/2})$. We conclude

$$\frac{3^{n-1} - f(n)}{3^{n-2}} \leq \frac{\kappa_{\mathcal{R}}(n)}{\kappa_{\mathcal{R}}(n-1)} \leq \frac{3^{n-1}}{3^{n-2} - f(n-1)},$$

which implies the assertion. The cases of general regular languages and two-sided ideals are analogous using the respective lemmas. The case of left ideals follows as $\kappa_{\mathcal{L}}(n) = \kappa_{\mathcal{R}}(n)$ for $n \geq 3$ by Remark 3.1. The case of suffix-free languages is clear because $\kappa_{\mathcal{S}}(n) = \kappa_{\mathcal{L}}(n-1)$ as is shown in [4]. □

4.2 Oscillation

In [6] it is shown that

$$\kappa_{\mathcal{G}}(n) = 1 + \sum_{x=1}^{\lfloor n/2 \rfloor} \sum_{y=1}^{n-\lfloor n/2 \rfloor} \binom{n}{x}\binom{n-x}{y}. \tag{1}$$

This means that $\kappa(A_S)$ is maximal for $|S| = \lfloor n/2 \rfloor$. In this section we will prove that the quotient $\kappa_{\mathcal{C}}(n)/\kappa_{\mathcal{C}}(n-1)$ does not only converge to 3, but also does so oscillating. Oscillation was observed first by calculating $\kappa_{\mathcal{G}}(n)$ in the range $1 \leq n \leq 20$. It came as a little surprise as the first ten values do not reveal this, [6]. In Table 1 we display the values $\kappa_{\mathcal{G}}(n)$ and the ratios $\kappa_{\mathcal{G}}(n)/\kappa_{\mathcal{G}}(n-1)$ for $8 \leq n \leq 15$.

Theorem 4.2. *For every $\mathcal{C} \in \{\mathcal{G}, \mathcal{R}, \mathcal{L}, \mathcal{T}, \mathcal{S}\}$ there exists some $n_0 \in \mathbb{N}$ such that*

$$\kappa_{\mathcal{C}}(n)/\kappa_{\mathcal{C}}(n-1) > 3 \iff \kappa_{\mathcal{C}}(n+1)/\kappa_{\mathcal{C}}(n) < 3$$

for all $n \geq n_0$. Moreover, for almost all n we have $\kappa_{\mathcal{C}}(n)/\kappa_{\mathcal{C}}(n-1) \neq 3$.

Proof. We give the proof for the general class $\mathcal{C} = \mathcal{G}$, only. Similar calculations show the result in the other cases. This is not done here and left to the reader.

We apply the interpretation of the sums as the number of 3-colorings from above. Let HC_n be the set of all 3-colorings of $\{1, \ldots, n\}$ in which the color X appears at most $\lfloor n/2 \rfloor$ times and the color Y appears at most $n - \lfloor n/2 \rfloor = \lceil n/2 \rceil$ times. We also let $\mathrm{hc}(n) = |\mathrm{HC_n}|$.

Besides the term $+1$ and starting at $x = 1$ and $y = 1$, instead of $x = 0$ and $y = 0$ for $\mathrm{hc}(n)$, the right-hand side in Equation (1) is identical to $\mathrm{hc}(n)$. More precisely, we have the following estimation.

$$\kappa_{\mathcal{G}}(n) < \mathrm{hc}(n) = \sum_{x=0}^{\lfloor n/2 \rfloor} \sum_{y=0}^{n-\lfloor n/2 \rfloor} \binom{n}{x}\binom{n-x}{y} \leq \kappa_{\mathcal{G}}(n) + 2^{n+1}. \tag{2}$$

Thus, apart from an error term bounded by $2^{n+1} \in \mathcal{O}(2^n)$ the numbers $\kappa_{\mathcal{G}}(n)$ and $\mathrm{hc}(n)$ are equal. We show two statements.

1. If n is large enough and even, then $\kappa_{\mathcal{G}}(n+1) < 3 \cdot \kappa_{\mathcal{G}}(n)$.
2. If n is large enough and odd, then $\kappa_{\mathcal{G}}(n+1) > 3 \cdot \kappa_{\mathcal{G}}(n)$.

1.) Let n be even, i.e., $n/2 = \lceil n/2 \rceil = \lfloor n/2 \rfloor = \lfloor (n+1)/2 \rfloor$ and $\lceil n/2 \rceil + 1 = \lceil (n+1)/2 \rceil$. We calculate $\mathrm{hc}(n+1)$ by considering 3-colorings of $\{1, \dots, n\}$ and extending them by choosing a color for $n+1$. Consider first any 3-coloring of $\{1, \dots, n\}$ in $\mathrm{HC_n}$. There are 3 possible extensions of this 3-coloring by choosing the color of $n+1$, i.e., there are at most $3\mathrm{hc}(n)$ possible extensions of $\mathrm{HC_n}$. Not all of those extensions are in $\mathrm{HC_{n+1}}$. We cannot extend those 3-colorings of $\{1, \dots, n\}$, which already had $n/2$ elements in X by choosing $n+1 \in X$. Let us count how many such 3-colorings in $\mathrm{HC}_(n)$ exist: there are $\binom{n}{n/2}$ choices for X and, for each fixed X, there are $\sum_{y=0}^{n/2} \binom{n/2}{y} = 2^{n/2}$ choices for Y. In total, we see that there are $3\mathrm{hc}(n) - \binom{n}{n/2} 2^{n/2}$ extensions of $\mathrm{HC_n}$ in $\mathrm{HC_{n+1}}$.

It remains to count the number of 3-colorings in $\mathrm{HC_{n+1}}$ which are not extensions of any 3-coloring in $\mathrm{HC_n}$. These are exactly the extensions of those 3-colorings of $\{1, \dots, n\}$ in which we have $|Y| = n/2 + 1$. As $n - (n/2 + 1) = n/2 - 1$, X may contain at most $n/2 - 1$ elements, i.e., $|X| \leq n/2 - 1$. Consequently, $n + 1$ may be either colored X or W. Thus, there are $2 \cdot \binom{n}{n/2+1} 2^{n/2-1} = \binom{n}{n/2+1} 2^{n/2}$ extensions of this type. The binomial coefficient $\binom{n}{n/2}$ is the largest one among all $\binom{n}{k}$ where $k \in \mathbb{Z}$. In particular, $\binom{n}{n/2} \geq \frac{2^n}{n+1}$ for all $n \in \mathbb{N}$ and $\binom{n}{n/2} \geq \frac{2^n}{n}$ for $n \geq 2$. We conclude

$$\begin{aligned}
3\mathrm{hc}(n) - \mathrm{hc}(n+1) &= 2^{n/2} \left(\binom{n}{n/2} - \binom{n}{n/2+1} \right) \\
&= 2^{n/2} \binom{n}{n/2} \cdot \frac{1}{n/2+1} \\
&\geq 2^{n/2} \cdot 2^n \cdot \frac{1}{n \cdot (n/2+1)} = \frac{\sqrt{8}^n}{n \cdot (n/2+1)}.
\end{aligned}$$

Note that the term $\binom{n}{n/2} - \binom{n}{n/2+1} = \binom{n}{n/2} \cdot \frac{1}{n/2+1}$ is equal to the Catalan number $C_{n/2}$; and better estimations for the difference $3\mathrm{hc}(n) - \mathrm{hc}(n+1)$ are possible. The fraction $\frac{\sqrt{8}^n}{n \cdot (n/2+1)}$ is greater than three times the error term 2^{n+1} for almost all n.

Class $\mathcal{C}$	n_0
regular languages ($\mathcal{G}$)	10
left ideals ($\mathcal{L}$)	11
right ideals ($\mathcal{R}$)	11
two-sided ideals ($\mathcal{T}$)	5
suffix-free languages ($\mathcal{S}$)	12

Table 2: Smallest n_0 where oscillation starts.

Thus, there exists a (small) number n_0 such that for all even $n \geq n_0$ we obtain $\kappa_{\mathcal{G}}(n+1) < 3\kappa_{\mathcal{G}}(n)$. According to Table 1 we have $n_0 = 10$.

2.) Let n be odd and $n \geq 3$. We have $(n+1)/2 = \lfloor n/2 \rfloor + 1 = \lceil n/2 \rceil$. Again, consider the extensions of 3-colorings of $\{1, \ldots, n\}$. First, consider the extensions of $\mathrm{HC_n}$. They are not in $\mathrm{HC_{n+1}}$ if and only if $|Y| = \lceil n/2 \rceil$ and the color of $n+1$ is the color Y. For fixed Y, there are $2^{\lfloor n/2 \rfloor}$ choices for X. In total, there are $3\mathrm{hc}(n) - \binom{n}{\lceil n/2 \rceil} 2^{\lfloor n/2 \rfloor}$ extensions of colorings in $\mathrm{HC_n}$ which are in $\mathrm{HC_{n+1}}$.

It remains to count the number of colorings in $\mathrm{HC_{n+1}}$ which are not extensions of colorings in $\mathrm{HC_n}$.

These are exactly the extensions of those 3-colorings of $\{1, \ldots, n\}$ in which we have $|X| = \lfloor n/2 \rfloor + 1$. As $n - (\lfloor n/2 \rfloor + 1) = \lceil n/2 \rceil - 1$, the color Y may contain at most $\lceil n/2 \rceil - 1$ elements, i.e., $|Y| \leq \lceil n/2 \rceil - 1$. Consequently, $n+1$ may be either colored Y or W. Thus, there are $2 \cdot \binom{n}{\lfloor n/2 \rfloor + 1} 2^{\lceil n/2 \rceil - 1} = 2 \cdot \binom{n}{\lfloor n/2 \rfloor + 1} 2^{\lfloor n/2 \rfloor}$ extensions of this type. Consequently, we obtain

$$\begin{aligned} \mathrm{hc}(n+1) - 3\mathrm{hc}(n) &= 2^{\lfloor n/2 \rfloor} \left(2 \binom{n}{\lfloor n/2 \rfloor + 1} - \binom{n}{\lceil n/2 \rceil} \right) \\ &= 2^{\lfloor n/2 \rfloor} \binom{n}{\lceil n/2 \rceil} \geq 2^{\lfloor n/2 \rfloor} 2^n / n. \end{aligned}$$

This number is asymptotically larger than any error in $\mathcal{O}(2^n)$ and, thus, we obtain $\kappa_{\mathcal{G}}(n+1) > 3\kappa_{\mathcal{G}}(n)$ for all odd n greater than some n_0. This concludes the proof of the oscillation property in the case of $\mathcal{C} = \mathcal{G}$. The other cases can be handled with very similar methods. Therefore, as mentioned above, this is left to the reader. □

We calculated the exact values for n_0 in every case, see Table 2. Note that in the cases $(\mathcal{G})$, $(\mathcal{L})$, $(\mathcal{R})$ and $(\mathcal{S})$ $\kappa(n)/\kappa(n-1) > 3$ holds for $4 \leq n < n_0$.

References

[1] J. Brzozowski and G. Davies. Most complex regular right-ideal languages. In H. Jürgensen, J. Karhumäki, and A. Okhotin, editors, *Descriptional Complexity of Formal Systems*, volume 8614 of *Lecture Notes in Computer Science*, pages 90–101. Springer International Publishing, 2014.

[2] J. Brzozowski and H. Tamm. Theory of átomata. *Theoretical Computer Science*, 539:13–27, 2014.

[3] J. A. Brzozowski and S. Davies. Quotient complexities of atoms in regular ideal languages. *Acta Cybernetica*, 22(2):293–311, 2015. Preliminary version: ArXiv e-prints, http://arxiv.org/abs/1503.02208.

[4] J. A. Brzozowski and M. Szykula. Complexity of suffix-free regular languages. In A. Kosowski and I. Walukiewicz, editors, *Fundamentals of Computation Theory - 20th International Symposium, FCT 2015, Gdańsk, Poland, August 17-19, 2015, Proceedings*, volume 9210 of *Lecture Notes in Computer Science*, pages 146–159. Springer, 2015.

[5] J. A. Brzozowski and H. Tamm. Theory of átomata. In G. Mauri and A. Leporati, editors, *Developments in Language Theory - 15th International Conference, DLT 2011, Milan, Italy, July 19-22, 2011. Proceedings*, volume 6795 of *Lecture Notes in Computer Science*, pages 105–116. Springer, 2011.

[6] J. A. Brzozowski and H. Tamm. Quotient complexities of atoms of regular languages. In H. Yen and O. H. Ibarra, editors, *Developments in Language Theory - 16th International Conference, DLT 2012, Taipei, Taiwan, August 14-17, 2012. Proceedings*, volume 7410 of *Lecture Notes in Computer Science*, pages 50–61. Springer, 2012.

[7] J. A. Brzozowski and H. Tamm. Complexity of atoms of regular languages. *International Journal of Foundations of Computer Science*, 24(07):1009–1027, 2013.

[8] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics: A Foundation for Computer Science*. Addison-Wesley, 1994.

[9] L. Schaeffer. Bounds for the atom complexity function. Unpublished, 2012.

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