Acta Cybernetica **22** (2015) 393–402.

Local Weighted Tree Languages*

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To the memory of my teacher and colleague Ferenc Gécseg

Abstract

Local weighted tree languages over semirings are introduced. For an arbitrary semiring, a weighted tree language is shown to be recognizable iff it appears as the image of a local weighted tree language under a deterministic relabeling.

1 Introduction

Trees or terms are fundamental concepts among others in computer science. In this paper we consider trees over ranked alphabets. Tree automata were introduced in the 60s of the last century in [6, 18, 20] and since then the theory of tree automata and tree languages has developed rapidly, see [12, 13] and [5] for surveys. Not much later, already in the 80s, quantitative aspects gained attention and weighted tree automata were introduced in [2, 1]. Within the last decades several authors have dealt with different weighted tree automaton models and their behaviour. Among others, for weighted tree automata over semirings, a Kleene-type characterization was obtained in [7], fixed point characterizations in [16, 4], and a characterization by weighted monadic second-order logic in [8, 9]. A summary of these and several other results on weighted tree automata and weighted tree languages can be found in [10] and [11].

Local tree languages were considered first time in [6, 17, 18, 19]. They are defined in the way that the membership of a tree to a local tree language can be decided by checking local properties of that tree. More exactly, for a ranked alphabet Σ, a Σ-fork (shortly: fork) is a tuple $(\sigma_1 \ldots \sigma_k, \sigma)$, where $\sigma \in \Sigma$ is a symbol of arity k and $\sigma_1, \ldots, \sigma_k$ are further symbols in Σ. The fork $(\sigma_1 \ldots \sigma_k, \sigma)$ occurs in a tree if the tree has a σ-node of which the k sons are labeled by $\sigma_1, \ldots, \sigma_k$ from left to right. Let $\mathrm{Fork}(\Sigma)$ be the set of all Σ-forks. Moreover, let us fix a subset $F \subseteq \mathrm{Fork}(\Sigma)$ of admissible forks and a subset $R \subseteq \Sigma$ of admissible roots. Then, a tree $\xi \in T_\Sigma$ belongs to the local tree language determined by the couple (F, R) if and only if all forks in ξ belong to F and the root of ξ belongs to R. A summary

*This work was supported by the NKFI grant K 108 448.

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DOI: 10.14232/actacyb.22.2.2015.10

and the main result of these investigations is presented in [12, Sect. II.9] and [13, Sect. 9]. The main result is a characterization of recognizable tree languages by images of local tree languages under deterministic relabelings, cf. [12, Thm. II.9.5] and [13, Prop. 8.1].

To the best of the author's knowledge, the quantitative aspects of local tree languages has not been investigated yet. In this paper we fill this gap in the theory of weighted tree languages. We introduce the concept of a local weighted tree language over a semiring S in a natural way. Namely, we associate a weight to each fork by a mapping $\varphi : \mathrm{Fork}(\Sigma) \to S$ and to each root by another mapping $\rho : \Sigma \to S$. We note that in both cases the weight can be 0. Then the weight of a tree $\xi \in T_\Sigma$ will be the (semiring) product of the weights associated to the forks in ξ and the weight associated to the root of ξ. The order of the factors is the postorder of the nodes of ξ. Finally, we show that that the mentioned characterization result in the classical (unweighted) case can be generalized to the weighted one. In fact, we prove (cf. Theorem 1) that a weighted tree language over an arbitrary semiring is recognizable if and only if it can be obtained as the image of a local weighted tree language under a deterministic relabeling.

2 Preliminaries

We denote by $\mathbb{N}$ the set of nonnegative integers. Let Q and S be sets, and let $k \in \mathbb{N}$. We will write just $q_1 \dots q_k$ for an element $(q_1, \dots, q_k)$ of Q^k. Hence $Q^0 = \{\varepsilon\}$. We denote the set of all mappings $v : Q \to S$ by S^Q. For each $v \in S^Q$ and $q \in Q$, we abbreviate $v(q)$ by v_q.

A *ranked alphabet* is a tuple (Σ, rk) where Σ is a finite set and $rk : \Sigma \to \mathbb{N}$ is a mapping called rank mapping. For every $k \geq 0$, we define $\Sigma_k = \{\sigma \in \Sigma \mid rk(\sigma) = k\}$. Sometimes we write $\sigma^{(k)}$ to mean that $\sigma \in \Sigma_k$. Moreover, let H be a set disjoint with Σ. The set of Σ-*terms over* H, denoted by $T_\Sigma(H)$, is the smallest set T such that (i) $\Sigma_0 \cup H \subseteq T$ and (ii) if $k \geq 1$, $\sigma \in \Sigma_k$, and $\xi_1, \dots, \xi_k \in T$, then $\sigma(\xi_1, \dots, \xi_k) \in T$. We denote $T_\Sigma(\emptyset)$ by T_Σ.

We define the *height* and the *root* of trees as the functions $\mathrm{height} : T_\Sigma \to \mathbb{N}$ and $\mathrm{rt} : T_\Sigma \to \Sigma$, respectively, as follows: (i) for every $\alpha \in \Sigma_0$, we define $\mathrm{height}(\alpha) = 0$, $\mathrm{rt}(\alpha) = \alpha$ and (ii) for every $\xi = \sigma(\xi_1, \dots, \xi_k)$, where $k \geq 1$, we define $\mathrm{height}(\xi) = 1 + \max\{\mathrm{height}(\xi_i) \mid 1 \leq i \leq k\}$ and $\mathrm{rt}(\xi) = \sigma$.

A *semiring* $(S, +, \cdot, 0, 1)$ is an algebra which consists of a commutative monoid $(S, +, 0)$, called the additive monoid of S, and a monoid $(S, \cdot, 1)$, called the multiplicative monoid of S, such that multiplication distributes (from both left and right) over addition, and moreover, $0 \neq 1$ and 0 is absorbing with respect to $\cdot$ (also both from left and right). An introduction to and several details about semirings can be found e.g. in the books [14] and [15].

In the rest of this paper Σ and Δ will denote arbitrary ranked alphabets unless specified otherwise, and S will denote an arbitrary semiring.

A *deterministic relabeling* (for short: drel) is a mapping $\tau : \Sigma \to \Delta$ satisfying

$\tau(\Sigma_k) \subseteq \Delta_k$ for every $k \geq 0$. The mapping τ extends to the tree transformation $\tau' : T_\Sigma \to T_\Delta$ defined by $\tau'(\sigma(\xi_1, \ldots, \xi_k)) = \tau(\sigma)(\tau'(\xi_1), \ldots, \tau'(\xi_k))$ for every $k \geq 0$, $\sigma \in \Sigma_k$, and $\xi_1, \ldots, \xi_k \in T_\Sigma$. In what follows we will call τ' also a drel and write τ for τ'.

A *weighted tree language over* Σ *and* S (for short: weighted tree language) is a mapping $\Phi : T_\Sigma \to S$, and a *weighted tree language over* S is a weighted tree language over Σ and S for some ranked alphabet Σ. For every $\xi \in T_\Sigma$, the element $\Phi(\xi)$ of S is called the *weight* of ξ. Now let $\tau : T_\Sigma \to T_\Delta$ be a drel. We extend τ to weighted tree languages as follows: for every $\Phi : T_\Sigma \to S$, we define $\tau(\Phi) : T_\Delta \to S$ by

$$\tau(\Phi)(\zeta) = \sum_{\xi \in T_\Sigma, \tau(\xi) = \zeta} \Phi(\xi)$$

for every $\zeta \in T_\Delta$. Let $\mathrm{C}(S)$ be a class of weighted tree languages over S. We denote by d-REL($\mathrm{C}(S)$) the class of all weighted tree languages $\tau(\Phi)$, where τ is a drel and $\Phi \in \mathrm{C}(S)$.

A Σ-*algebra* (V, θ) consists of a nonempty set V (*carrier set*) and an arity preserving mapping θ, called the *interpretation*, from Σ to the set operations over V, i.e., $\theta(\sigma) : V^k \to V$ for every $k \geq 0$ and $\sigma \in \Sigma_k$. The Σ-*term algebra* (T_Σ, top), defined by $\mathrm{top}(\sigma)(\xi_1, \ldots, \xi_k) = \sigma(\xi_1, \ldots, \xi_k)$ for $\sigma \in \Sigma_k$ and $\xi_1, \ldots, \xi_k \in T_\Sigma$, is *initial* in the class of all Σ-algebras, i.e., for every Σ-algebra (V, θ), there is a unique Σ-algebra homomorphism from T_Σ to V.

A *weighted tree automaton (over* Σ *and* S*)* (for short: wta) is a tuple $\mathcal{A} = (Q, \Sigma, S, \delta, \kappa)$ where

- Q is a finite nonempty set, the *set of states*,
- Σ is the *ranked input alphabet*,
- $\delta = (\delta_k \mid k \in \mathbb{N})$ is a *family of transition mappings*[1] $\delta_k : Q^k \times \Sigma_k \times Q \to S$,
- $\kappa : Q \to S$ is the *root weight mapping.*

For every $k \in \mathbb{N}$ and transition $w = (q_1 \ldots q_k, \sigma, q) \in Q^k \times \Sigma_k \times Q$, and $1 \leq i \leq k$, we call q_i the ith input state of w and denote it by $\mathrm{in}_i(w)$. Similarly, we call q the output state of w and denote it by $\mathrm{out}(w)$. Moreover, we call the element $\delta_k(w)$ of S the weight of the transition w.

For $\mathcal{A}$ we consider the Σ-algebra $(S^Q, \delta_\mathcal{A})$ where, for every $k \geq 0$ and $\sigma \in \Sigma_k$, the k-ary operation $\delta_\mathcal{A}(\sigma) : S^Q \times \ldots \times S^Q \to S^Q$ is defined by

$$\delta_\mathcal{A}(\sigma)(v_1, \ldots, v_k)_q = \sum_{q_1, \ldots, q_k \in Q} (v_1)_{q_1} \cdot \ldots \cdot (v_k)_{q_k} \cdot \delta_k(q_1 \ldots q_k, \sigma, q)$$

for every $q \in Q$ and $v_1, \ldots, v_k \in S^Q$. (Here $\sum$ and $\cdot$ denote a finite sum and the multiplication in the semiring S, respectively.) Let us denote the unique Σ-algebra

[1] In the literature δ is also called a *tree representation* and δ_k is given as a mapping of type $\Sigma_k \to S^{Q^k \times Q}$.

homomorphism from T_Σ to S^Q by h_δ. The weighted tree language $\|\mathcal{A}\| : T_\Sigma \to S$ *recognized* by $\mathcal{A}$ is defined by

$$\|\mathcal{A}\|(\xi) = \sum_{q \in Q} h_\delta(\xi)_q \cdot \kappa(q)$$

for every $\xi \in T_\Sigma$. Due to the definitions of $\delta_\mathcal{A}$ and h_δ, we obtain that

$$h_\delta(\sigma(\xi_1,\ldots,\xi_k))_q = \sum_{q_1,\ldots,q_k \in Q} h_\delta(\xi_1)_{q_1} \cdot \ldots \cdot h_\delta(\xi_k)_{q_k} \cdot \delta_k(q_1 \ldots q_k, \sigma, q) \qquad (1)$$

for every $\sigma(\xi_1,\ldots,\xi_k) \in T_\Sigma$ and $q \in Q$. An introduction to the theory of wta over semirings and several results can be found in [10] and [11].

Example 1. (Cf. [3, Example 3.3]) We consider the arctic semiring $\mathsf{Arct} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ and construct the wta $\mathcal{A} = (Q, \Sigma, \mathsf{Arct}, \delta, \kappa)$ which recognizes the weighted tree language height. Let $Q = \{p_1, p_2\}$, $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$, and $\kappa(p_1) = 0$ and $\kappa(p_2) = -\infty$. Moreover, let

$$\begin{array}{lllll} \delta_0(\varepsilon,\alpha,p_1) & = & \delta_0(\varepsilon,\alpha,p_2) & = & 0, \\ \delta_2(p_1p_2,\sigma,p_1) & = & \delta_2(p_2p_1,\sigma,p_1) & = & 1, \\ \delta_2(p_2p_2,\sigma,p_2) & = & 0, & & \end{array}$$

and for every other transition (q_1q_2, σ, q) we have $\delta_2(q_1q_2, \sigma, q) = -\infty$. We consider the tree $\xi = \sigma(\alpha, \alpha)$ and compute $h_\delta(\xi)_{p_1}$ and $h_\delta(\xi)_{p_2}$. Clearly, $h_\delta(\alpha)_{p_1} = \delta_0(\alpha)_{\varepsilon,p_1} = 0$ and $h_\delta(\alpha)_{p_2} = 0$. Then

$$h_\delta(\sigma(\alpha,\alpha))_{p_1} = \max_{q_1,q_2 \in Q}\{h_\delta(\alpha)_{q_1} + h_\delta(\alpha)_{q_2} + \delta_2(q_1q_2,\sigma,p_1)\} = 1$$

(note that $\delta_2(p_1p_1, \sigma, p_1) = \delta_2(p_2p_2, \sigma, p_1) = -\infty$ and $-\infty$ is neutral for max) and, similarly, $h_\delta(\sigma(\alpha, \alpha))_{p_2} = 0$. In general, we can prove by structural induction on ξ that $h_\delta(\xi)_{p_1} = \mathrm{height}(\xi)$ and $h_\delta(\xi)_{p_2} = 0$ for every $\xi \in T_\Sigma$. Thus $\|\mathcal{A}\| = \mathrm{height}$ and hence height $\in \mathrm{Rec}(\Sigma, \mathsf{Arct})$.

A wta $\mathcal{A} = (Q, \Sigma, S, \delta, \kappa)$ is *bottom-up deterministic* (for short: bu-deterministic) if for every $k \geq 0$, $\sigma \in \Sigma_k$, and $w \in Q^k$ there is at most one $q \in Q$ such that $\delta_k(w, \sigma, q) \neq 0$. If $\mathcal{A}$ is bu-deterministic, then for every input tree $\xi \in T_\Sigma$, there is at most one $q \in Q$ such that $h_\delta(\xi)_q \neq 0$. In this case the operation $+$ of S is not used for the computation of $\|\mathcal{A}\|$.

A weighted tree language $\Phi : T_\Sigma \to S$ is *recognizable* (resp. bu-*deterministically recognizable*) if there is a wta (resp. bu-deterministic wta) $\mathcal{A}$ such that $\Phi = \|\mathcal{A}\|$. The class of all recognizable weighted tree languages over Σ and S (resp. over S) is denoted by $\mathrm{Rec}(\Sigma, S)$ (resp. $\mathrm{Rec}(S)$). The notation bud-$\mathrm{Rec}(\Sigma, S)$ is introduced analogously.

Finally, we recall that recognizable weighted tree languages are closed under (deterministic) relabelings. A proof can be found, e.g., in [8, Lm. 3.4]. However, in [8] a wta is defined over a commutative semiring and the semantics of a wta is defined in terms of runs. Therefore we give a short proof for our case.

Proposition 1. d-REL(Rec(S)) $\subseteq$ Rec(S).

Proof. Let $\mathcal{A} = (Q, \Sigma, S, \delta, \kappa)$ be a wta and $\tau : T_\Sigma \to T_\Delta$ be a drel. We define the wta $\mathcal{A}' = (Q, \Delta, S, \delta', \kappa)$ by

$$\delta'(q_1 \dots q_k, \omega, q) = \sum_{\sigma \in \Sigma_k, \tau(\sigma)=\omega} \delta(q_1 \dots q_k, \sigma, q)$$

for every $k \geq 0$, $\omega \in \Delta_k$, and $q, q_1, \dots, q_k \in Q$, and show that $\mathcal{A}'$ computes $\tau\big(\|\mathcal{A}\|\big)$. We can show by induction on the height of ζ and using equality (1) that

$$h_{\delta'}(\zeta)_q = \sum_{\xi \in T_\Sigma, \tau(\xi)=\zeta} h_\delta(\xi)_q$$

for every $\zeta \in T_\Delta$ and $q \in Q$. Then we get

$$\|\mathcal{A}\|'(\zeta) = \sum_{q \in Q} h_{\delta'}(\zeta)_q \cdot \kappa(q) = \sum_{q \in Q} \Big(\sum_{\xi \in T_\Sigma, \tau(\xi)=\zeta} h_\delta(\xi)_q \Big) \cdot \kappa(q) =$$
$$\sum_{\xi \in T_\Sigma, \tau(\xi)=\zeta} \Big(\sum_{q \in Q} h_\delta(\xi)_q \cdot \kappa(q) \Big) = \sum_{\xi \in T_\Sigma, \tau(\xi)=\zeta} \|\mathcal{A}\|(\xi) = \tau\big(\|\mathcal{A}\|\big)(\zeta)$$

for each $\zeta \in T_\Delta$, which proves $\|\mathcal{A}\|' = \tau\big(\|\mathcal{A}\|\big)$. □

3 The result

A Σ-*fork* (shortly: fork) is a tuple $(\sigma_1 \dots \sigma_k, \sigma)$ for some $k \geq 0$, where $\sigma \in \Sigma_k$ and $\sigma_1, \dots, \sigma_k$ are further symbols in Σ. The fork $(\sigma_1 \dots \sigma_k, \sigma)$ occurs in a tree if the tree has a σ-node of which the k sons are labeled by $\sigma_1, \dots, \sigma_k$ from left to right. We consider the family $\mathrm{Fork}(\Sigma) = (\mathrm{Fork}_k(\Sigma) \mid k \geq 0)$, where

$$\mathrm{Fork}_k(\Sigma) = \{(\sigma_1 \dots \sigma_k, \sigma) \mid \sigma_1, \dots, \sigma_k \in \Sigma, \sigma \in \Sigma_k\}.$$

Note that $\mathrm{Fork}_k(\Sigma) = \Sigma^k \times \Sigma_k$, hence $\mathrm{Fork}_0(\Sigma) = \Sigma_0$.

A *weighted local system (over Σ and S)* (for short: wls) is a pair $\mathcal{L} = (\Sigma, S, \varphi, \rho)$, where φ is a family of mappings $(\varphi_k \mid k \geq 0)$ with $\varphi_k : \mathrm{Fork}_k(\Sigma) \to S$ and $\rho : \Sigma \to S$ is a further mapping. Intuitively, we associate a weight, i.e., an element of S to each fork and also to each symbol in Σ. Note that this weight can be 0.

Next we define the weighted tree language determined by $\mathcal{L}$. For this, we extend φ to the mapping $\varphi' : T_\Sigma \to S$ defined by induction as follows:

(i) $\varphi'(\sigma) = \varphi_0(\sigma)$ for every $\sigma \in \Sigma_0$,

(ii) $\varphi'(\sigma(\xi_1, \dots, \xi_k)) = \varphi'(\xi_1) \cdot \ldots \cdot \varphi'(\xi_k) \cdot \varphi_k(\mathrm{rt}(\xi_1) \dots \mathrm{rt}(\xi_k), \sigma)$ for every $k \geq 1$, $\sigma \in \Sigma_k$, and $\xi_1, \dots, \xi_k \in T_\Sigma$.

In the following we write φ for φ'. The *weighted tree language* $\|\mathcal{L}\| : T_\Sigma \to S$ *determined by* $\mathcal{L}$ is defined by $\|\mathcal{L}\|(\xi) = \varphi(\xi) \cdot \rho(\mathrm{rt}(\xi))$ for every $\xi \in T_\Sigma$. Note that, like for deterministic wta, the operation $+$ of S is not used for the definition of $\|\mathcal{L}\|$.

Thus, $\varphi(\xi)$ is the (semiring) product of the weights associated to the forks in ξ. The order of the factors is the postorder of the nodes of ξ. Moreover, the weight $\|\mathcal{L}\|(\xi)$ of ξ is the product of $\varphi(\xi)$ and the weight associated to the root of ξ.

A weighted tree language $\Phi : T_\Sigma \to S$ is called *local* if there is a wls $\mathcal{L}$ such that $\Phi = \|\mathcal{L}\|$. The class of all local weighted tree languages over Σ and S (resp. over S) is denoted by $\mathrm{Loc}(\Sigma, S)$ (resp. $\mathrm{Loc}(S)$).

Example 2. We consider again the ranked alphabet $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. We define the wls $\mathcal{L} = (\Sigma, \mathsf{Arct}, \varphi, \rho)$ by

- $\varphi_2(\sigma\alpha, \sigma) = \varphi_2(\alpha\alpha, \sigma) = 1$ and in every other case $\varphi_2(\_\_, \sigma) = 0$,
- $\varphi_0(\varepsilon, \alpha) = 0$, and by $\rho(\sigma) = \rho(\alpha) = 0$.

It should be clear that $\|\mathcal{L}\|(\xi)$ is the number of the occurrences of the pattern $\sigma(\_, \alpha)$ in ξ, where '_' is a placeholder which may be filled by either σ or α. We note that in [11, Example 3.4] a wta is given over the semiring of natural numbers which recognizes $\|\mathcal{L}\|$.

Next we show that local weighted tree languages are bu-deterministically recognizable.

Lemma 1. $\mathrm{Loc}(\Sigma, S) \subseteq \text{bud-Rec}(\Sigma, S)$.

Proof. Let $\mathcal{L} = (\Sigma, S, \varphi, \rho)$ by a wls over Σ and S. We construct a wta $\mathcal{A} = (Q, \Sigma, S, \delta, \kappa)$ such that $\|\mathcal{A}\| = \|\mathcal{L}\|$. For this, we define

- $Q = \{\overline{\sigma} \mid \sigma \in \Sigma\}$,
- for every $k \geq 0$, $\sigma_1 \dots \sigma_k \in \Sigma^k$, $\sigma \in \Sigma_k$, and $\omega \in \Sigma$,
$$\delta_k(\overline{\sigma_1} \dots \overline{\sigma_k}, \sigma, \overline{\omega}) = \begin{cases} \varphi_k(\sigma_1 \dots \sigma_k, \sigma) & \text{if } \omega = \sigma \\ 0 & \text{otherwise,} \end{cases}$$
- $\kappa(\overline{\sigma}) = \rho(\sigma)$ for every $\sigma \in \Sigma$.

It is clear that $\mathcal{A}$ is bu-deterministic. Next we show the following statement by induction on ξ: for every $\xi \in T_\Sigma$ and $\omega \in \Sigma$, we have

$$h_\delta(\xi)_{\overline{\omega}} = \begin{cases} \varphi(\xi) & \text{if } \omega = \mathrm{rt}(\xi) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\xi = \sigma(\xi_1, \dots, \xi_k)$ for some $k \geq 0$, $\sigma \in \Sigma_k$, and $\xi_1, \dots, \xi_k \in T_\Sigma$. We have

$$h_\delta(\sigma(\xi_1, \dots, \xi_k))_{\overline{\omega}} =$$

$$\sum_{\sigma_1, \dots, \sigma_k \in \Sigma} h_\delta(\xi_1)_{\overline{\sigma_1}} \cdot \ldots \cdot h_\delta(\xi_k)_{\overline{\sigma_k}} \cdot \delta_k(\overline{\sigma_1} \dots \overline{\sigma_k}, \sigma, \overline{\omega}) =$$

$$\varphi(\xi_1) \cdot \ldots \cdot \varphi(\xi_k) \cdot \delta_k(\overline{\mathrm{rt}(\xi_1)} \dots \overline{\mathrm{rt}(\xi_k)}, \sigma, \overline{\omega}) =$$

$$\varphi(\xi_1) \cdot \ldots \cdot \varphi(\xi_k) \cdot \varphi_k(\mathrm{rt}(\xi_1) \dots \mathrm{rt}(\xi_k), \sigma) \text{ if } \omega = \sigma \text{ and } 0 \text{ otherwise} =$$

$$\varphi(\sigma(\xi_1, \dots, \xi_k)) \text{ if } \omega = \sigma \text{ and } 0 \text{ otherwise,}$$

where the second equality is justified by the I. H. and the third one by the definition of δ_k. Note that the case $k = 0$ covers also the base of the induction. Finally, let $\xi \in T_\Sigma$. Then we get

$$\|\mathcal{A}\|(\xi) = \sum_{\omega \in \Sigma} h_\delta(\xi)_{\overline{\omega}} \cdot \kappa(\overline{\omega}) = \varphi(\xi) \cdot \kappa(\overline{\mathrm{rt}(\xi)}) = \varphi(\xi) \cdot \rho(\mathrm{rt}(\xi)) = \|\mathcal{L}\|(\xi),$$

where the second equality follows from the statement and the other ones from the corresponding definitions. □

One can easily find a semiring S and a ranked alphabet Σ such that the inclusion in Lemma 1 is strict. For instance, consider the Boolean semiring $\mathbb{B} = (\{0,1\}, \vee, \wedge, , 0, 1)$, the ranked alphabet $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\}$ and the weighted tree language Φ defined by $\Phi\big(\gamma(\gamma(\alpha))\big) = 1$ and $\Phi(\xi) = 0$ for every other $\xi \in T_\Sigma$. It is easy to show that $\Phi \in (\text{bud-Rec}(\Sigma, \mathbb{B}) \setminus \text{Loc}(\Sigma, \mathbb{B}))$. Another example can be found for the Boolean case on [12, p. 107].

Finally, we give a characterization of recognizable weighted tree languages by images of local weighted tree languages under deterministic relabelings.

Theorem 1. $\text{Rec}(S) = \text{d-REL}(\text{Loc}(S))$.

Proof. The inclusion from right to left follows from Proposition 1 and Lemma 1. Therefore, we prove the other inclusion.

For this, let $\mathcal{A} = (Q, \Sigma, S, \delta, \kappa)$ be a wta. We will construct a ranked alphabet Δ, a wls $\mathcal{L} = (\Delta, S, \varphi, \rho)$, and a deterministic relabeling $\tau : T_\Delta \to T_\Sigma$ such that $\|\mathcal{A}\| = \tau(\|\mathcal{L}\|)$.

Let $\Delta_k = Q^k \times \Sigma_k \times Q$ for every $k \geq 0$. Moreover, let us define φ and ρ as follows. For every $k \geq 0$, $\omega_1 \ldots \omega_k \in \Delta$ and $(q_1 \ldots q_k, \sigma, q) \in \Delta_k$, let

$$\varphi_k\big(\omega_1 \ldots \omega_k, (q_1 \ldots q_k, \sigma, q)\big) = \begin{cases} \delta_k(q_1 \ldots q_k, \sigma, q) & \text{if } \mathrm{out}(\omega_i) = q_i \\ & \text{for all } 1 \leq i \leq k \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\rho((q_1 \ldots q_k, \sigma, q)) = \kappa(q).$$

Finally, let $\tau_k : \Delta_k \to \Sigma_k$ be defined by $\tau((q_1 \ldots q_k, \sigma, q)) = \sigma$ for every $k \geq 0$ and $(q_1 \ldots q_k, \sigma, q) \in \Delta_k$.

First we prove the following statement by induction: for every $\xi \in T_\Sigma$ and $q \in Q$, we have

$$h_\delta(\xi)_q = \sum_{\substack{\zeta \in T_\Delta, \tau(\zeta) = \xi \\ \mathrm{out}(\mathrm{rt}(\zeta)) = q}} \varphi(\zeta).$$

Let $\xi = \sigma(\xi_1, \ldots, \xi_k)$ for some $k \geq 0$, $\sigma \in \Sigma_k$, and $\xi_1, \ldots, \xi_k \in T_\Sigma$. In the following computation we abbreviate a product of the form $a_1 \cdot \ldots \cdot a_k$ by $\prod_{i=1}^k a_i$, where

$a_1, \ldots, a_k \in S$. Then

$$h_\delta(\sigma(\xi_1, \ldots, \xi_k))_q =$$

$$\sum_{q_1,\ldots,q_k \in Q} \Big(\prod_{i=1}^{k} h_\delta(\xi_i)_{q_i} \Big) \cdot \delta_k(q_1 \ldots q_k, \sigma, q) =$$

$$\sum_{q_1,\ldots,q_k \in Q} \Big(\prod_{i=1}^{k} \Big(\sum_{\substack{\zeta_i \in T_\Delta, \tau(\zeta_i)=\xi_i \\ \mathrm{out}(\mathrm{rt}(\zeta_i))=q_i}} \varphi(\zeta_i) \Big) \Big) \cdot \delta_k(q_1 \ldots q_k, \sigma, q) =$$

$$\sum_{q_1,\ldots,q_k \in Q} \Big(\sum_{\substack{\forall 1 \le i \le k : \zeta_i \in T_\Delta, \\ \tau(\zeta_i)=\xi_i, \mathrm{out}(\mathrm{rt}(\zeta_i))=q_i}} \prod_{i=1}^{k} \varphi(\zeta_i) \Big) \cdot \delta_k(q_1 \ldots q_k, \sigma, q) =$$

$$\sum_{q_1,\ldots,q_k \in Q} \Big(\sum_{\substack{\forall 1 \le i \le k : \zeta_i \in T_\Delta, \\ \tau(\zeta_i)=\xi_i, \mathrm{out}(\mathrm{rt}(\zeta_i))=q_i}} \big(\prod_{i=1}^{k} \varphi(\zeta_i) \big) \cdot \delta_k(q_1 \ldots q_k, \sigma, q) \Big) =$$

$$\sum_{\substack{\forall 1 \le i \le k : \zeta_i \in T_\Delta, \\ \tau(\zeta_i)=\xi_i}} \big(\prod_{i=1}^{k} \varphi(\zeta_i) \big) \cdot \delta_k\big(\mathrm{out}(\mathrm{rt}(\zeta_1)) \ldots \mathrm{out}(\mathrm{rt}(\zeta_k)), \sigma, q\big) =$$

$$\sum_{\substack{\forall 1 \le i \le k : \zeta_i \in T_\Delta, \\ \tau(\zeta_i)=\xi_i}} \big(\prod_{i=1}^{k} \varphi(\zeta_i) \big) \cdot \varphi_k\big(\mathrm{rt}(\zeta_1) \ldots \mathrm{rt}(\zeta_k), \big(\mathrm{out}(\mathrm{rt}(\zeta_1)) \ldots \mathrm{out}(\mathrm{rt}(\zeta_k)), \sigma, q\big)\big) =$$

$$\sum_{\substack{\forall 1 \le i \le k : \zeta_i \in T_\Delta, \\ \tau(\zeta_i)=\xi_i, q_i \in Q}} \big(\prod_{i=1}^{k} \varphi(\zeta_i) \big) \cdot \varphi_k\big(\mathrm{rt}(\zeta_1) \ldots \mathrm{rt}(\zeta_k), (q_1 \ldots q_k, \sigma, q)\big) =$$

$$\sum_{\substack{\forall 1 \le i \le k : \zeta_i \in T_\Delta, \\ \tau(\zeta_i)=\xi_i, q_i \in Q}} \varphi\big((q_1 \ldots q_k, \sigma, q)(\zeta_1, \ldots \zeta_k)\big) =$$

$$\sum_{\substack{\zeta \in T_\Delta, \tau(\zeta)=\sigma(\xi_1,\ldots,\xi_k) \\ \mathrm{out}(\mathrm{rt}(\zeta))=q}} \varphi(\zeta).$$

The first, second, and the sixth equality follows from (1), the I. H., and the definition of φ, respectively. Finally, the seventh one follows from the fact that if $q_i \neq \mathrm{out}(\mathrm{rt}(\zeta_i))$ for some $1 \le i \le k$, then $\varphi_k\big(\mathrm{rt}(\zeta_1) \ldots \mathrm{rt}(\zeta_k), (q_1 \ldots q_k, \sigma, q)\big) = 0$.

Finally, for every $\xi \in T_\Sigma$, we have

$$\|\mathcal{A}\|(\xi) = \sum_{q \in Q} h_\delta(\xi)_q \cdot \kappa(q) = \sum_{q \in Q} \Bigg(\sum_{\substack{\zeta \in T_\Delta, \tau(\zeta)=\xi \\ \mathrm{out}(\mathrm{rt}(\zeta))=q}} \varphi(\zeta) \cdot \kappa(q) \Bigg) =$$

$$\sum_{\zeta \in T_\Delta, \tau(\zeta)=\xi} \varphi(\zeta) \cdot \kappa(\mathrm{out}(\mathrm{rt}(\zeta))) = \sum_{\zeta \in T_\Delta, \tau(\zeta)=\xi} \varphi(\zeta) \cdot \rho(\mathrm{rt}(\zeta)) =$$

$$\sum_{\zeta \in T_\Delta, \tau(\zeta)=\xi} \|\mathcal{L}\|(\zeta) = \tau(\|\mathcal{L}\|)(\xi),$$

where the second equality is justified by the statement proved above.

□

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Received 15th June 2015