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Weighted First-Order Logics over Semirings*

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Dedicated to the memory of Ferenc Gécseg

Abstract

We consider a first-order logic, a linear temporal logic, star-free expressions and counter-free Büchi automata, with weights, over idempotent, zero-divisor free and totally commutative complete semirings. We show the expressive equivalence (of fragments) of these concepts, generalizing in the quantitative setup, the corresponding folklore result of formal language theory.

1 Introduction

The expressive equivalence of monadic second-order logic and finite automata over finite words was established in [5, 16] and over infinite words in [6]. Droste and Gastin, in [8] (cf. also [9]), introduced a weighted monadic second-order logic over semirings and showed that sentences from a fragment of this logic, interpreted over finite words, are equivalent to weighted automata. A corresponding result for infinite words was stated in [13]. Recently in [12], the authors extended the expressive equivalence of monadic second-order logic and automata over more general structures, namely valuation monoids. On the other hand, first-order (*FO* for short) logic (i.e., the logic obtained from monadic second-order one by relaxing second-order quantifiers) is equivalent to linear temporal logic (*LTL* for short), star-free expressions and counter-free Büchi automata (cf. for instance [7]). More interestingly, *LTL* and its alternatives serve as specification languages in model checking for real world applications [3, 22, 31]. The last few years there is also an increasing interest in establishing *FO* logic and its equivalent objects in the quantitative framework. This is motivated by the need to create model checking tools which incorporate quantitative features. In [14], the aforementioned equivalence was established in the weighted setup of arbitrary bounded lattices. Recently, in [26] (cf. also [24]), we introduced a weighted *FO* logic, a weighted *LTL*, ω-star-free series

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and counter-free weighted Büchi automata over the max-plus semiring with discounting and investigated fragments of them satisfying an expressive equivalence. The convergence of infinite sums over nonnegative real numbers was ensured by the existence of discounting parameters.

In this paper, we consider a weighted *FO* logic, a weighted *LTL*, ω-star-free series and counter-free weighted Büchi automata over idempotent, zero-divisor free and totally commutative complete semirings. We show that there are suitable fragments of our objects so that the classes of infinitary series, derived by them, coincide. Our results can be proved for series over finite words as well, though we skip any technical detail.

The structure of our paper is as follows. Except of this introductory section, in Section 2 we recall the notion of totally commutative complete semirings and present notations used in the paper. The underlying structure for all weighted objects considered in the paper will be an arbitrary idempotent, zero-divisor free and totally commutative complete semiring.

In Section 3 we introduce the weighted *LTL* and define the semantics of *LTL* formulas interpreted as infinitary series. We consider a fragment of our *LTL* namely the fragment of U-nesting formulas. We should note that a quantitative *LTL* over De Morgan algebras was introduced for the first time in [21].

In Section 4 we consider the weighted *FO* logic which is in fact the one induced by the weighted *MSO* logic of [8, 9]. Its semantics is interpreted by infinitary series as induced by the semantics of the corresponding weighted *MSO* logic of [13]. We consider the fragment of weakly quantified *FO* logic formulas and in our first main result, in Section 5, we show that every series which is definable by a U-nesting *LTL* formula is definable also by a weakly quantified *FO* logic sentence.

In Section 6 we deal with star-free and ω-star-free series. We recall that the class of star-free languages over an alphabet A is the smallest class of languages over A which contains $\emptyset$, the singleton $\{a\}$ for every $a \in A$, and which is closed under finite union, complementation and concatenation. Furthermore, the class of ω-star-free languages over A is the closure of the empty set under the operations of union, complement and concatenation with star-free languages on the left (cf. for instance [7, 23, 27, 29]). It is worth noting that the application of the star-operation (whenever it is permitted) to star-free languages is implemented by the other operations. However, in the setup of series (over semirings) the complement operation is not "too strong". Therefore, we defined the class ω-star-free series as the least class of infinitary series generated by the monomials (over A and our semiring) by applying finitely many times the operations of sum, Hadamard product, complement, Cauchy product, and iteration and ω-iteration restricted to series of the form $\sum_{a\in A} (k_a)_a$ where, for every $a \in A$, k_a is an element of our semiring. The second main result of the paper, in Section 7, states that the class of definable series by weakly quantified *FO* logic sentences is contained in the class of ω-star-free series.

In Section 8 we introduce counter-free weighted automata and counter-free weighted Büchi automata and investigate closure properties of the classes of their behaviors. We define a fragment of the class of series accepted by counter-free weighted Büchi automata, namely the class of almost simple ω-counter-free series and we show, in Section 9, that this contains the class of ω-star-free series.

Finally, in Section 10 we show that the class of almost simple ω-counter-free series is contained in the class of series which are definable by U-nesting *LTL* formulas. In fact this last inclusion concludes the coincidence of the classes of series definable by U-nesting formulas of the weighted *LTL* and weakly quantified *FO* logic sentences, ω-star-free series and almost simple ω-counter-free series. In the Conclusion we refer to some interesting problems for further research. A preliminary version of this paper appeared in [25].

2 Preliminaries

Let A be an alphabet, i.e., a finite nonempty set. As usually, we denote by A^* the set of all finite words over A and $A^+ = A^* \setminus \{\varepsilon\}$, where ε is the empty word. The set of all infinite sequences with elements in A, i.e., the set of all infinite words over A, is denoted by A^ω. A finite word $w = a_0 \dots a_{n-1}$, where $a_0, \dots, a_{n-1} \in A$ $(n \geq 1)$, is written also as $w = w(0) \dots w(n-1)$ where $w(i) = a_i$ for every $0 \leq i \leq n-1$. For every $0 \leq i \leq n-1$, we denote by $w_{<i}$ (resp. $w_{\leq i}$) the prefix $w(0) \dots w(i-1)$ (resp. $w(0) \dots w(i)$) of w and by $w_{>i}$ (resp. $w_{\geq i}$) the suffix $w(i+1) \dots w(n-1)$ (resp. $w(i) \dots w(n-1)$) of w. For every infinite word $w = a_0 a_1 \dots$ which is written also as $w = w(0)w(1)\dots$, the words $w_{<i}, w_{\leq i}, w_{>i}, w_{\geq i}$ are defined in the same way, with the suffixes $w_{>i}, w_{\geq i}$ being infinite words.

Throughout the paper A will denote an alphabet.

A *semiring* $(K, +, \cdot, 0, 1)$ consists of a set K, two binary operations $+$ and $\cdot$ and two constant elements 0 and 1 such that $\langle K, +, 0\rangle$ is a commutative monoid, $\langle K, \cdot, 1\rangle$ is a monoid, multiplication distributes over addition, and $0 \cdot k = k \cdot 0 = 0$ for every $k \in K$. The semiring is denoted simply by K if the operations and the constant elements are understood.

The semiring K is called *commutative* if $k \cdot k' = k' \cdot k$ for every $k, k' \in K$. It is called *additively idempotent* (or simply *idempotent*), if $k + k = k$ for every $k \in K$. Moreover, the semiring K is *zero-sum free* (resp. *zero-divisor free*) if $k + k' = 0$ implies $k = k' = 0$ (resp. $k \cdot k' = 0$ implies $k = 0$ or $k' = 0$) for every $k, k' \in K$. It is well known that every idempotent semiring is necessarily zero-sum free (cf. [1]).

Next, assume that the semiring K is equipped, for every index set I, with infinitary sum operations $\sum_I : K^I \to K$, such that for every family $(k_i \mid i \in I)$ of elements of K and $k \in K$ we have

$$\sum_{i\in\emptyset} k_i = 0, \quad \sum_{i\in\{j\}} k_i = k_j, \quad \sum_{i\in\{j,l\}} k_i = k_j + k_l \text{ for } j \neq l,$$

$$\sum_{j\in J}\Big(\sum_{i\in I_j} k_i\Big) = \sum_{i\in I} k_i, \text{ if } \bigcup_{j\in J} I_j = I \text{ and } I_j \cap I_{j'} = \emptyset \text{ for } j \neq j',$$

$$\sum_{i\in I}(k \cdot k_i) = k \cdot \Big(\sum_{i\in I} k_i\Big), \quad \sum_{i\in I}(k_i \cdot k) = \Big(\sum_{i\in I} k_i\Big) \cdot k.$$

Then the semiring K together with the operations $\sum_I$ is called *complete* [15, 19].

A complete semiring is said to be *totally complete* [18], if it is endowed with a countably infinite product operation satisfying for every sequence $(k_i \mid i \geq 0)$ of elements of K the subsequent conditions:

$$\prod_{i\geq 0} 1 = 1, \quad \prod_{i\geq 0} k_i = \prod_{i\geq 0} k'_i$$

$$k_0 \cdot \prod_{i\geq 0} k_{i+1} = \prod_{i\geq 0} k_i, \quad \prod_{j\geq 1} \sum_{i\in I_j} k_i = \sum_{(i_1,i_2,\ldots)\in I_1\times I_2\times\ldots} \prod_{j\geq 1} k_{i_j},$$

where in the second equation $k'_0 = k_0 \cdot \ldots \cdot k_{n_1}, k'_1 = k_{n_1+1} \cdot \ldots \cdot k_{n_2}, \ldots$ for an increasing sequence $0 < n_1 < n_2 < \ldots$, and in the last equation $I_1, I_2, \ldots$ are arbitrary index sets.

Furthermore, we will call a totally complete semiring K *totally commutative complete* if it satisfies the statement:

$$\prod_{i\geq 0} (k_i \cdot k'_i) = \left(\prod_{i\geq 0} k_i\right) \cdot \left(\prod_{i\geq 0} k'_i\right).$$

Obviously a totally commutative complete semiring is commutative. For our theory, we shall also need that a totally commutative complete semiring K satisfies the property

$$k \neq 0 \implies \prod_{i\geq 0} k \neq 0$$

for every $k \in K$. Therefore in the sequel, by abusing terminology, when we refer to totally commutative complete semirings we assume that they additionally satisfy the above property.

Example 1. The following semirings are totally commutative complete, and all but the second one are idempotent. Moreover, by excluding the arbitrary completely distributive complete lattices, the remaining ones are zero-divisor free.

- the *boolean semiring* $\mathbb{B} = (\{\mathbf{0}, \mathbf{1}\}, +, \cdot, \mathbf{0}, \mathbf{1})$,
- the semiring $(\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ of *extended natural numbers* [17],
- the *arctical semiring* or *max-plus semiring* $(\mathbb{R}_+ \cup \{\pm\infty\}, \max, +, -\infty, 0)$,
- each completely distributive complete lattice (cf. [2]) with the operations supremum and infimum, in particular each complete chain [20].

Lemma 1. *Let K be an idempotent totally complete semiring and I an index set of size at most continuum. Then, the following statements hold.*

(i) [10, Chap. 5, Lm. 7.3] $\sum_I 1 = 1$.

(ii) $\sum\limits_{I} k = k$ *for every* $k \in K$.

(iii) $\sum\limits_{i \in I} k_i = \sum\limits_{\substack{k \in K \\ \exists i \in I, k_i = k}} k$ *for every family* $(k_i)_{i \in I}$ *in* K.

Proof. (ii) By (i) and distributivity we get $\sum\limits_{I} k = k \cdot \sum\limits_{I} 1 = k \cdot 1 = k$.

(iii) For every $k \in K$ we let $I_k = \{i \in I \mid k_i = k\}$. Then we get

$$\sum_{i \in I} k_i = \sum_{\substack{k \in K \\ \exists i \in I, k_i = k}} \sum_{I_k} k = \sum_{\substack{k \in K \\ \exists i \in I, k_i = k}} k$$

where the second equality follows by (ii). □

In the rest of the paper K *will denote a totally commutative complete, idempotent and zero-divisor free semiring.*

Let Q be a set. A *formal power series* (or simply *series*) *over* Q *and* K is a mapping $s : Q \to K$. For every $v \in Q$ we write (s, v) for the value $s(v)$ and refer to it as the *coefficient of* s *on* v. The *support of* s is the set $\mathrm{supp}(s) = \{v \in Q \mid (s, v) \neq 0\}$. The *constant series* $\widetilde{k}$ ($k \in K$) is defined, for every $v \in Q$, by $\left(\widetilde{k}, v\right) = k$. The *characteristic series* 1_P *of a set* $P \subseteq Q$ is given by $(1_P, v) = 1$ if $v \in P$, and $(1_P, v) = 0$ otherwise. We denote by $K\langle\langle Q \rangle\rangle$ the class of all series over Q and K.

Let $s, r \in K\langle\langle Q \rangle\rangle$ and $k \in K$. The *sum* $s + r$, the *scalar products* ks and sk as well as the *Hadamard product* $s \odot r$ are defined elementwise by $(s + r, v) = (s, v) + (r, v)$, $(ks, v) = k \cdot (s, v)$, $(sk, v) = (s, v) \cdot k$, and $(s \odot r, v) = (s, v) \cdot (r, v)$ for every $v \in Q$. Abusing notations, if $P \subseteq Q$, then we shall identify the restriction $s|_P$ of s on P with the series $s \odot 1_P$. Moreover, if $\mathrm{supp}(s) \subseteq P$, sometimes in the sequel we shall identify $s|_P$ with s. It is a folklore result that the structure $\left(K\langle\langle Q \rangle\rangle, +, \odot, \widetilde{0}, \widetilde{1}\right)$ is a commutative semiring. In our paper, we work with the semirings $K\langle\langle A^* \rangle\rangle$ and $K\langle\langle A^\omega \rangle\rangle$ of finitary and infinitary series over A and K, respectively.

Let B be another alphabet and $h : A^* \to B^*$ be a nondeleting homomorphism, i.e., $h(a) \neq \varepsilon$ for each $a \in A$. Then h can be extended to a mapping $h : A^\omega \to B^\omega$ by letting $h(w) = (h(w(i)))_{i \geq 0}$ for every $w \in A^\omega$. Moreover, h is extended to a mapping $h : K\langle\langle A^* \rangle\rangle \to K\langle\langle B^* \rangle\rangle$ as follows. For every $s \in K\langle\langle A^* \rangle\rangle$ the series $h(s) \in K\langle\langle B^* \rangle\rangle$ is given by $(h(s), u) = \sum_{w \in h^{-1}(u)} (s, w)$ for every $u \in B^*$. Since K is complete, h is also extended to a mapping $h : K\langle\langle A^\omega \rangle\rangle \to K\langle\langle B^\omega \rangle\rangle$ which is defined for every series $s \in K\langle\langle A^\omega \rangle\rangle$ by $(h(s), u) = \sum_{w \in h^{-1}(u)} (s, w)$ for every $u \in B^\omega$. If $r \in K\langle\langle B^* \rangle\rangle$ (resp. $r \in K\langle\langle B^\omega \rangle\rangle$), then the series $h^{-1}(r) \in K\langle\langle A^* \rangle\rangle$ (resp. $h^{-1}(r) \in K\langle\langle A^\omega \rangle\rangle$) is determined by $(h^{-1}(r), w) = (r, h(w))$ for every $w \in A^*$ (resp. $w \in A^\omega$).

3 Weighted linear temporal logic

For every letter $a \in A$ we consider a proposition p_a and we let $AP = \{p_a \mid a \in A\}$. As usually, for every $p \in AP$ we identify $\neg\neg p$ with p.

Definition 1. *The syntax of formulas of the* weighted linear temporal logic *(weighted LTL for short)* over A and K *is given by the grammar*

$$\varphi ::= k \mid p_a \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \bigcirc\varphi \mid \varphi U \varphi \mid \Box\varphi$$

where $k \in K$ *and* $p_a \in AP$.

We denote by $LTL(K, A)$ the set of all such weighted LTL formulas φ. We represent the semantics $\|\varphi\|$ of formulas $\varphi \in LTL(K, A)$ as infinitary series in $K\langle\langle A^\omega\rangle\rangle$.

Definition 2. *Let* $\varphi \in LTL(K, A)$. *The* semantics *of* φ *is a series* $\|\varphi\| \in K\langle\langle A^\omega\rangle\rangle$ *which is defined inductively as follows. For every* $w \in A^\omega$ *we set*

- $(\|k\|, w) = k$,
- $(\|p_a\|, w) = \begin{cases} 1 & \text{if } w(0) = a \\ 0 & \text{otherwise} \end{cases}$,
- $(\|\neg\varphi\|, w) = \begin{cases} 1 & \text{if } (\|\varphi\|, w) = 0 \\ 0 & \text{otherwise} \end{cases}$,
- $(\|\varphi \vee \psi\|, w) = (\|\varphi\|, w) + (\|\psi\|, w)$,
- $(\|\varphi \wedge \psi\|, w) = (\|\varphi\|, w) \cdot (\|\psi\|, w)$,
- $(\|\bigcirc\varphi\|, w) = (\|\varphi\|, w_{\geq 1})$,
- $(\|\varphi U \psi\|, w) = \sum_{i\geq 0} \left(\left(\prod_{0\leq j<i} (\|\varphi\|, w_{\geq j}) \right) \cdot (\|\psi\|, w_{\geq i}) \right)$,
- $(\|\Box\varphi\|, w) = \prod_{i\geq 0} (\|\varphi\|, w_{\geq i})$.

The *eventually* operator is defined as in the classical LTL, i.e., by $\Diamond\varphi := 1U\varphi$, hence we have $(\|\Diamond\varphi\|, w) = \sum_{i\geq 0} (\|\varphi\|, w_{\geq i})$ for every $w \in A^\omega$.

The syntactic boolean fragment $bLTL(K, A)$ of $LTL(K, A)$ is given by the grammar

$$\varphi ::= 0 \mid 1 \mid p_a \mid \neg\varphi \mid \varphi \vee \varphi \mid \bigcirc\varphi \mid \varphi U \varphi$$

where $p_a \in AP$. For every formula $\varphi \in bLTL(K, A)$ it is easily obtained, by structural induction on φ and using idempotency, that $\|\varphi\|$ gets only values in $\{0, 1\}$. By identifying 0 with **0** and 1 with **1** it is trivially concluded that $\|\varphi\|$ coincides with the semantics in the boolean semiring $\mathbb{B}$. The *conjunction* and *always operators* are defined, respectively, by the macros $\varphi \underline{\wedge} \psi := \neg(\neg\varphi \vee \neg\psi)$ and $\underline{\Box}\varphi := \neg\Diamond\neg\varphi$. Clearly, the application of the operators $\underline{\wedge}$ and $\underline{\Box}$ in $bLTL(K, A)$ formulas φ, ψ coincides semantically with the application of the classical operators $\wedge$ and $\Box$ in φ, ψ considered as classical formulas.

We aim to define a further fragment of $LTL(K, A)$. For this we need some preliminary matter. More precisely, an *atomic-step formula* is an $LTL(K, A)$ formula of the form $\bigvee_{a \in A} (k_a \wedge p_a)$ where $k_a \in K$ and $p_a \in AP$ for every $a \in A$. An *LTL-step formula* is an $LTL(K, A)$ formula of the form $\bigvee_{1 \leq i \leq n} (k_i \wedge \varphi_i)$ where $k_i \in K$ and $\varphi_i \in bLTL(K, A)$ for every $1 \leq i \leq n$. We shall denote by $stLTL\,(K, A)$ the class of *LTL*-step formulas over A and K. Furthermore, we shall denote by $abLTL\,(K, A)$ the class of *almost boolean LTL* formulas over A and K, i.e., formulas of the form $\bigwedge_{1 \leq i \leq n} \varphi_i$ with $\varphi_i \in bLTL\,(K, A)$ or $\varphi_i = \bigvee_{a \in A} (k_a \wedge p_a)$, for every $1 \leq i \leq n$.

Definition 3. *The fragment $ULTL\,(K, A)$ of U-*nesting *LTL* formulas over A and K *is the least class of formulas in $LTL\,(K, A)$ which is defined inductively in the following way.*

- $k \in ULTL\,(K, A)$ *for every* $k \in K$.
- $abLTL\,(K, A) \subseteq ULTL\,(K, A)$.
- *If* $\varphi \in ULTL\,(K, A)$, *then* $\neg\varphi \in ULTL\,(K, A)$.
- *If* $\varphi, \psi \in ULTL\,(K, A)$, *then* $\varphi \wedge \psi, \varphi \vee \psi \in ULTL\,(K, A)$.
- *If* $\varphi \in ULTL\,(K, A)$, *then* $\bigcirc\varphi \in ULTL\,(K, A)$.
- *If* $\varphi \in bLTL\,(K, A)$ *or* φ *is an atomic-step formula, then* $\Box\varphi \in ULTL\,(K, A)$.
- *If* $\varphi \in abLTL\,(K, A)$ *and* $\psi \in ULTL\,(K, A)$, *then* $\varphi U \psi \in ULTL\,(K, A)$.

A series $r \in K\langle\langle A^\omega \rangle\rangle$ is called *ω-ULTL-definable* if there is a formula $\varphi \in ULTL\,(K, A)$ such that $r = \|\varphi\|$. We shall denote by ω-$ULTL\,(K, A)$ the class of ω-*ULTL*-definable series over A and K.

4 Weighted first-order logic

In this section, we define the weighted first-order logic (weighted *FO* logic, for short) and consider a syntactic fragment of it. We aim to show that the class of semantics of sentences in this fragment contains the class ω-$ULTL\,(K, A)$.

Definition 4. *The syntax of formulas of the* weighted *FO* logic over A and K *is given by the grammar*

$$\varphi ::= k \mid P_a(x) \mid x \leq y \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x \,.\, \varphi \mid \forall x \,.\, \varphi$$

where $k \in K$ *and* $a \in A$.

We shall denote by $FO(K, A)$ the set of all weighted FO logic formulas over A and K. In order to define the semantics of $FO(K, A)$ formulas, we recall the notions of extended alphabet and valid assignment (cf. for instance [30]). Let $\mathcal{V}$ be a finite set of first-order variables. For an infinite word $w \in A^\omega$ we let $dom(w) = \omega$. A $(\mathcal{V}, w)$-*assignment* σ is a mapping associating variables from $\mathcal{V}$ to elements of ω. For every $x \in \mathcal{V}$ and $i \in \omega$, we denote by $\sigma[x \to i]$ the $(\mathcal{V}, w)$-assignment which associates i to x and acts as σ on $\mathcal{V} \setminus \{x\}$. We encode pairs (w, σ) for every $w \in A^\omega$ and $(\mathcal{V}, w)$-assignment σ, by using the extended alphabet $A_\mathcal{V} = A \times \{0,1\}^\mathcal{V}$. Each word in $A_\mathcal{V}^\omega$ can be considered as a pair (w, σ) where w is the projection over A and σ is the projection over $\{0,1\}^\mathcal{V}$. Then, σ is called a *valid* $(\mathcal{V}, w)$-assignment whenever for every $x \in \mathcal{V}$ the x-row contains exactly one 1. In this case, we identify σ with the $(\mathcal{V}, w)$-assignment so that for every first-order variable $x \in \mathcal{V}$, $\sigma(x)$ is the position of the 1 on the x-row. It is well-known (cf. [7]) that the set $\mathcal{N}_\mathcal{V} = \{(w, \sigma) \mid w \in A^\omega,\ \sigma \text{ is a valid } (\mathcal{V}, w)\text{-assignment}\}$ is an ω-star-free language over $A_\mathcal{V}$. The set $free(\varphi)$ of free variables in a formula $\varphi \in FO(K, A)$ is defined as usual.

Definition 5. *Let $\varphi \in FO(K, A)$ and $\mathcal{V}$ be a finite set of variables with $free(\varphi) \subseteq \mathcal{V}$. The semantics of φ is a series $\|\varphi\|_\mathcal{V} \in K\langle\langle A_\mathcal{V}^\omega\rangle\rangle$. Consider an element $(w, \sigma) \in A_\mathcal{V}^\omega$. If σ is not a valid assignment, then we put $(\|\varphi\|_\mathcal{V}, (w, \sigma)) = 0$. Otherwise, we inductively define $(\|\varphi\|_\mathcal{V}, (w, \sigma)) \in K$ as follows.*

- $(\|k\|_\mathcal{V}, (w, \sigma)) = k,$
- $(\|P_a(x)\|_\mathcal{V}, (w, \sigma)) = \begin{cases} 1 & \text{if } w(\sigma(x)) = a \\ 0 & \text{otherwise} \end{cases},$
- $(\|x \leq y\|_\mathcal{V}, (w, \sigma)) = \begin{cases} 1 & \text{if } \sigma(x) \leq \sigma(y) \\ 0 & \text{otherwise} \end{cases},$
- $(\|\neg\varphi\|_\mathcal{V}, (w, \sigma)) = \begin{cases} 1 & \text{if } (\|\varphi\|_\mathcal{V}, (w, \sigma)) = 0 \\ 0 & \text{otherwise} \end{cases},$
- $(\|\varphi \vee \psi\|_\mathcal{V}, (w, \sigma)) = (\|\varphi\|_\mathcal{V}, (w, \sigma)) + (\|\psi\|_\mathcal{V}, (w, \sigma)),$
- $(\|\varphi \wedge \psi\|_\mathcal{V}, (w, \sigma)) = (\|\varphi\|_\mathcal{V}, (w, \sigma)) \cdot (\|\psi\|_\mathcal{V}, (w, \sigma)),$
- $(\|\exists x \centerdot \varphi\|_\mathcal{V}, (w, \sigma)) = \sum_{i \geq 0} \left(\|\varphi\|_{\mathcal{V}\cup\{x\}}, (w, \sigma[x \to i])\right),$
- $(\|\forall x \centerdot \varphi\|_\mathcal{V}, (w, \sigma)) = \prod_{i \geq 0} \left(\|\varphi\|_{\mathcal{V}\cup\{x\}}, (w, \sigma[x \to i])\right).$

If $\mathcal{V} = free(\varphi)$, then we simply write $\|\varphi\|$ for $\|\varphi\|_{free(\varphi)}$. Moreover, by Prop. 5 in [13], it holds

$$(\|\varphi\|_\mathcal{V}, (w, \sigma)) = \left(\|\varphi\|, \left(w, \sigma|_{free(\varphi)}\right)\right)$$

for every $(w, \sigma) \in \mathcal{N}_\mathcal{V}$.

The syntactic boolean fragment $bFO(K, A)$ of $FO(K, A)$ is defined by the grammar

$$\varphi ::= 0 \mid 1 \mid P_a(x) \mid x \leq y \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists x \centerdot \varphi.$$

For every formula $\varphi \in bFO(K, A)$ it is easily obtained, by structural induction on φ and using idempotency, that $\|\varphi\|$ gets only values in $\{0, 1\}$. By identifying 0 with $\mathbf{0}$ and 1 with $\mathbf{1}$ it is trivially concluded that $\|\varphi\|$ coincides with the semantics in the boolean semiring $\mathbb{B}$. The *conjunction* and *universal quantification* are defined, respectively, by the macros $\varphi \underline{\wedge} \psi := \neg(\neg\varphi \vee \neg\psi)$ and $\underline{\forall} x \centerdot \varphi := \neg\exists x \centerdot \neg\varphi$. Clearly, the application of the operators $\underline{\wedge}$ and $\underline{\forall}$ in $bFO(K, A)$ formulas φ, ψ coincides semantically with the application of the classical operators $\wedge$ and $\forall$ in φ, ψ considered as classical formulas.

Next, we define a fragment of our logic. For this, we recall the notion of an *FO*-step formula from [4]. More precisely, a formula $\varphi \in FO(K, A)$ is an *FO-step formula* if $\varphi = \bigvee_{1 \leq i \leq n} (k_i \wedge \varphi_i)$ with $\varphi_i \in bFO(K, A)$ and $k_i \in K$ for every $1 \leq i \leq n$. Moreover, a formula $\varphi \in FO(K, A)$ is called a *letter-step formula* whenever $\varphi = \bigvee_{a \in A} (k_a \wedge P_a(x))$ with $k_a \in K$ for every $a \in A$. We shall need also the following macros:

- $first(x) := \forall y \centerdot x \leq y$,
- $x = y := x \leq y \wedge y \leq x$,
- $x < y := x \leq y \wedge \neg(x = y)$,
- $z \leq x < y := z \leq x \wedge x < y$,
- $\varphi \rightarrow \psi := \neg\varphi \vee (\varphi \wedge \psi)$.

Definition 6. *A formula $\varphi \in FO(K, A)$ will be called* weakly quantified *if whenever φ contains a subformula of the form $\forall x \centerdot \psi$, then ψ is either a boolean or a letter-step formula with free variable x or a formula of the form $y \leq x \rightarrow \psi'$ or $z \leq x < y \rightarrow \psi'$ where ψ' is a letter-step formula with free variable x.*

We denote by $WQFO(K, A)$ the set of all weakly quantified $FO(K, A)$ formulas over A and K. A series $s \in K\langle\langle A^\omega\rangle\rangle$ is called *ω-wqFO-definable* if there is a sentence $\varphi \in WQFO(K, A)$ such that $s = \|\varphi\|$. We write ω-$wqFO(K, A)$ for the class of *ω-wqFO*-definable series in $K\langle\langle A^\omega\rangle\rangle$.

5 ω-$ULTL$-definable series are ω-$wqFO$-definable

In this section we show that every ω-$ULTL$-definable series over A and K is also ω-*wqFO*-definable. For this, we will prove that for every $\varphi \in ULTL(K, A)$ there exists a sentence $\varphi' \in WQFO(K, A)$ such that $\|\varphi\| = \|\varphi'\|$, using the subsequent technical results.

Lemma 2. *Let $\varphi \in ULTL(K, A)$ such that there exists $\varphi'(y) \in WQFO(K, A)$ with*
$(\|\varphi'(y)\|, (w, [y \rightarrow i])) = (\|\varphi\|, w_{\geq i})$ *for every* $w \in A^\omega, i \geq 0$.
Then $(\|\neg\varphi'(y)\|, (w, [y \rightarrow i])) = (\|\neg\varphi\|, w_{\geq i})$ *for every* $w \in A^\omega, i \geq 0$.

Lemma 3. *Let* $\varphi, \psi \in ULTL(K, A)$ *such that there exist* $\varphi'(y), \psi'(x) \in WQFO(K, A)$ *with* $(\|\varphi'(y)\|, (w, [y \to i])) = (\|\varphi\|, w_{\geq i})$ *and* $(\|\psi'(x)\|, (w, [x \to i])) = (\|\psi\|, w_{\geq i})$ *for every* $w \in A^{\omega}, i \geq 0$. *Then, there exist* $\xi_1(x), \xi_2(x) \in WQFO(K, A)$ *with*

$$(\|\xi_1(x)\|, (w, [x \to i])) = (\|\varphi \wedge \psi\|, w_{\geq i})$$

and

$$(\|\xi_2(x)\|, (w, [x \to i])) = (\|\varphi \vee \psi\|, w_{\geq i})$$

for every $w \in A^{\omega}, i \geq 0$.

Proof. Without any loss, we assume that the variable x does not occur in φ' (otherwise we apply a renaming). We replace every occurrence of y with x in φ', and we let $\xi_1(x) = \varphi'(x) \wedge \psi'(x)$ and $\xi_2(x) = \varphi'(x) \vee \psi'(x)$ which trivially satisfy our claim. □

Lemma 4. *Let* $\varphi \in K \cup abLTL(K, A)$. *Then, there exists* $\varphi'(x) \in WQFO(K, A)$ *such that* $(\|\varphi'(x)\|, (w, [x \to i])) = (\|\varphi\|, w_{\geq i})$ *for every* $w \in A^{\omega}, i \geq 0$.

Proof. Let $\varphi = k \in K$. Then we set $\varphi'(x) = k$. Next, let $\varphi \in abLTL(K, A)$, i.e., $\varphi = \bigwedge_{1 \leq j \leq n} \psi_j$ with $\psi_j \in bLTL(K, A)$ or $\psi_j = \bigvee_{a \in A} (k_a \wedge p_a)$, for every $1 \leq j \leq n$. If $\psi_j \in bLTL(K, A)$, then it is well-known that there exists a formula $\psi_j'(x_j) \in bFO(K, A)$ with one free variable x_j, such that $(\|\psi_j\|, w_{\geq i}) = \left(\left\|\psi_j'(x_j)\right\|, (w, [x_j \to i])\right)$ for every $w \in A^{\omega}, i \geq 0$. Without any loss, we can assume that the variable x_j $(1 \leq j \leq n)$ does not occur in any ψ_k' (whenever $\psi_k' \in bLTL(K, A)$) with $k \neq j$ (if this is not the case, then we apply a renaming of variables). Therefore, we can replace x_j in ψ_j' with a new variable x. In case $\psi_j = \bigvee_{a \in A} (k_a \wedge p_a)$ we consider the $WQFO(K, A)$ letter-step formula $\psi_j'(x) = \bigvee_{a \in A} (k_a \wedge P_a(x))$. Now it is a routine matter to show that the $WQFO(K, A)$ formula $\varphi'(x) = \bigwedge_{1 \leq j \leq n} \psi_j'(x)$ satisfies our claim. □

Lemma 5. *Let* $\varphi \in ULTL(K, A)$ *such that there exists a formula* $\varphi'(y) \in WQFO(K, A)$ *with* $(\|\varphi'(y)\|, (w, [y \to i])) = (\|\varphi\|, w_{\geq i})$ *for every* $w \in A^{\omega}, i \geq 0$. *Then, there exists a* $WQFO(K, A)$ *formula* $\psi(x)$ *such that* $(\|\psi(x)\|, (w, [x \to i])) = (\|\bigcirc \varphi\|, w_{\geq i})$ *for every* $w \in A^{\omega}, i \geq 0$.

Proof. We let $\psi(x) = \exists y. (y = x + 1 \wedge \varphi'(y))$ and we have

$$\begin{aligned}
(\|\psi(x)\|, (w, [x \to i])) &= (\|\exists y. (y = x + 1 \wedge \varphi'(y))\|, (w, [x \to i])) \\
&= \sum_{j \geq 0} (\|y = x + 1 \wedge \varphi'(y)\|, (w, [x \to i, y \to j])) \\
&= (\|y = x + 1 \wedge \varphi'(y)\|, (w, [x \to i, y \to i + 1])) \\
&\quad + \sum_{j \geq 0, j \neq i+1} (\|y = x + 1 \wedge \varphi'(y)\|, (w, [x \to i, y \to j])) \\
&= (\|y = x + 1 \wedge \varphi'(y)\|, (w, [x \to i, y \to i + 1])) \\
&= (\|\varphi'(y)\|, (w, [y \to i + 1])) \\
&= (\|\varphi\|, w_{\geq i+1}) = (\|\bigcirc\varphi\|, w_{\geq i}).
\end{aligned}$$

for every $w \in A^\omega, i \geq 0$, where the fourth equality holds by Lemma 1(ii). □

Lemma 6. *Let $\varphi \in bLTL(K, A)$ or φ be an atomic-step formula. Then, there exists $\psi(y) \in WQFO(K, A)$ such that $(\|\psi(y)\|, (w, [y \to i])) = (\|\Box\varphi\|, w_{\geq i})$ for every $w \in A^\omega, i \geq 0$.*

Proof. If $\varphi \in bLTL(K, A)$, then $\Box\varphi \in bLTL(K, A)$, and thus there exists a formula $\psi(x) \in bFO(K, A)$ with one free variable x, such that $(\|\psi(x)\|, (w, [x \to i])) = (\|\Box\varphi\|, w_{\geq i})$ for every $w \in A^\omega, i \geq 0$. If $\varphi = \bigvee_{a \in A} (k_a \wedge p_a)$, then we consider the $WQFO(K, A)$ letter-step formula $\varphi'(x) = \bigvee_{a \in A} (k_a \wedge P_a(x))$. We also consider the $WQFO(K, A)$ formula $\psi(y) = \forall x. (y \leq x \to \varphi'(x))$. Then, for every $w \in A^\omega, i \geq 0$ we have

$$\begin{aligned}
(\|\psi(y)\|, (w, [y \to i])) &= \prod_{j \geq 0} (\|y \leq x \to \varphi'(x)\|, (w, [y \to i, x \to j])) \\
&= \prod_{j \geq i} (\|y \leq x \wedge \varphi'(x)\|, (w, [y \to i, x \to j])) \\
&= \prod_{j \geq i} (\|\varphi'(x)\|, (w, [x \to j])) \\
&= \prod_{j \geq i} (\|\varphi\|, w_{\geq j}) \\
&= (\|\Box\varphi\|, w_{\geq i})
\end{aligned}$$

where the fourth equality holds by Lemma 4. □

Lemma 7. *Let $\varphi \in abLTL(K, A)$ and $\psi \in ULTL(K, A)$ such that there exists $\psi'(y) \in WQFO(K, A)$ with $(\|\psi'(y)\|, (w, [y \to i])) = (\|\psi\|, w_{\geq i})$ for every $w \in A^\omega, i \geq 0$. Then, there exists $\xi(z) \in WQFO(K, A)$ such that $(\|\xi(z)\|, (w, [z \to i])) = (\|\varphi U \psi\|, w_{\geq i})$ for every $w \in A^\omega, i \geq 0$.*

Proof. Let $\varphi = \bigwedge_{1\leq l\leq m} \varphi_l$. Then, by the proof of Lemma 4, there exists a formula $\varphi'(x) = \bigwedge_{1\leq l\leq m} \varphi_l'(x)$ where for every $1 \leq l \leq m$, $\varphi_l'(x) \in bFO(K, A)$ or it is a letter-step formula with $(\|\varphi_l'(x)\|(w, [x \to i])) = (\|\varphi_l\|, w_{\geq i})$ for every $w \in A^\omega, i \geq 0$. Moreover, we have $(\|\varphi'(x)\|, (w, [x \to i])) = (\|\varphi\|, w_{\geq i})$ for every $w \in A^\omega, i \geq 0$. We consider the $FO(K, A)$ formula $\xi'(z) = \exists y. (\forall x. ((z \leq x < y) \to \varphi'(x)) \wedge (z \leq y) \wedge \psi'(y))$. For every $w \in A^\omega, i \geq 0$ we compute

$$
\begin{aligned}
&(\|\xi'(z)\|, (w, [z \to i])) \\
&= \sum_{j\geq 0} (\|\forall x. ((z \leq x < y) \to \varphi'(x)) \wedge (z \leq y) \wedge \psi'(y)\|, (w, [z \to i, y \to j])) \\
&= \sum_{j\geq 0} (\|\forall x. ((z \leq x < y) \to \varphi'(x)) \wedge \psi'(y)\|, (w, [z \to i, y \to i + j])) \\
&= \sum_{j\geq 0} \left(\left(\prod_{0\leq k<j} (\|\varphi'(x)\|, (w, [x \to i + k])) \right) \cdot (\|\psi'(y)\|, (w, [y \to i + j])) \right) \\
&= \sum_{j\geq 0} \left(\left(\prod_{0\leq k<j} (\|\varphi\|, w_{\geq i+k}) \right) \cdot (\|\psi\|, w_{\geq i+j}) \right) \\
&= (\|\varphi U \psi\|, w_{\geq i}).
\end{aligned}
$$

Now, we consider the formula

$$\xi(z) = \exists y. \left(\bigwedge_{1\leq l\leq m} (\forall x. ((z \leq x < y) \to \varphi_l'(x))) \wedge (z \leq y) \wedge \psi'(y) \right)$$

and for every $w \in A^\omega, i \geq 0$ we get $(\|\xi(z)\|, (w, [z \to i])) = (\|\xi'(z)\|, (w, [z \to i])) = (\|\varphi U \psi\|, w_{\geq i})$. Since $\xi(z) \in WQFO(K, A)$, we conclude our proof. □

Lemma 8. *For every $ULTL(K, A)$ formula φ we can construct a $WQFO(K, A)$ formula $\varphi'(x)$ such that $(\|\varphi'(x)\|, (w, [x \to i])) = (\|\varphi\|, w_{\geq i})$ for every $w \in A^\omega, i \geq 0$.*

Proof. We use Lemmas 2, 3, 4, 5, 6, and 7. □

Proposition 1. *For every $\varphi \in ULTL(K, A)$ we can construct a $WQFO(K, A)$ sentence φ' with $\|\varphi'\| = \|\varphi\|$.*

Proof. Let $\varphi \in ULTL(K, A)$. By the previous lemma, there exists a $WQFO(K, A)$ formula $\psi(x)$ such that $(\|\psi(x)\|, (w, [x \to i])) = (\|\varphi\|, w_{\geq i})$, for every $w \in A^\omega, i \geq 0$. We consider the $WQFO(K, A)$ sentence $\varphi' = \exists x. (first(x) \wedge \psi(x))$ and we get $(\|\varphi'\|, w) = (\|\psi(x)\|, (w, [x \to 0])) = (\|\varphi\|, w)$ for every $w \in A^\omega$, i.e., $\|\varphi'\| = \|\varphi\|$, as required. □

By the above proposition, we get the main result of this section.

Theorem 1. ω-$ULTL\,(K,A) \subseteq \omega$-$wqFO(K,A)$.

The result of the next corollary, which is trivially obtained by the constructive proofs of this section's lemmas and propositions, in fact generalizes the corresponding result that relates boolean *LTL* and *FO* logic.

Corollary 1. *For every $\varphi \in ULTL\,(K,A)$ we can construct a $WQFO(K,A)$ sentence φ', that uses at most three different names of variables, such that $\|\varphi'\| = \|\varphi\|$.*

6 Star-free series

In this section, we introduce the notions of star-free and ω-star-free series over A and K. Let $L \subseteq A^*$ (resp. $L \subseteq A^\omega$). As usually, we denote by 1_L the characteristic series of L. If L is a singleton, i.e., $L = \{w\}$, then we simply write 1_w for $1_{\{w\}}$. Furthermore, we simply denote by k_L the series $k1_L$ for $k \in K$. The *monomials over A and K* are series of the form $(k_a)_a$ for $a \in A$ and $k_a \in K$. For simplicity, we shall consider also the series of the form k_ε with $k \in K$ as monomials. A series $s \in K\langle\langle A^*\rangle\rangle$ is called a *letter-step series* if $s = \sum_{a\in A} (k_a)_a$ where $k_a \in K$ for every $a \in A$. The *complement $\overline{s}$ of a series s* is given by $(\overline{s}, w) = 1$ if $(s, w) = 0$, and $(\overline{s}, w) = 0$ otherwise. Let $r, s \in K\langle\langle A^*\rangle\rangle$. The (*Cauchy*) *product of r and s* is the series $r \cdot s \in K\langle\langle A^*\rangle\rangle$ defined for every $w \in A^*$ by

$(r \cdot s, w) = \sum \{(r, u) \cdot (s, v) \mid u, v \in A^*, w = uv\}$.

The nth-iteration $r^n \in K\langle\langle A^*\rangle\rangle$ $(n \geq 0)$ of a series $r \in K\langle\langle A^*\rangle\rangle$ is defined inductively by

$r^0 = 1_\varepsilon$ and $r^{n+1} = r \cdot r^n$ for $n \geq 0$.

Then, we have $(r^n, w) = \sum \left\{\prod_{1\leq i\leq n}(r, u_i) \mid u_i \in A^*, w = u_1 \ldots u_n\right\}$ for every $w \in A^*$. A series $r \in K\langle\langle A^*\rangle\rangle$ is called *proper* if $(r, \varepsilon) = 0$. If r is proper, then for every $w \in A^*$ and $n > |w|$ we have $(r^n, w) = 0$. The *iteration $r^+ \in K\langle\langle A^*\rangle\rangle$ of a proper series $r \in K\langle\langle A^*\rangle\rangle$* is defined by $r^+ = \sum_{n>0} r^n$. Thus, for every $w \in A^+$ we have $(r^+, w) = \sum\limits_{1\leq n\leq |w|} (r^n, w)$ and $(r^+, \varepsilon) = 0$.

Definition 7. *The class of* star-free series over A and K, *denoted by $SF(K,A)$, is the least class of series containing the monomials (over A and K) and being closed under sum, Hadamard product, complement, Cauchy product, and iteration restricted to letter-step series.*

Next, let $r \in K\langle\langle A^*\rangle\rangle$ be a finitary and $s \in K\langle\langle A^\omega\rangle\rangle$ an infinitary series. Then, the *Cauchy product of r and s* is the infinitary series $r \cdot s \in K\langle\langle A^\omega\rangle\rangle$ defined for every $w \in A^\omega$ by

$(r \cdot s, w) = \sum \{(r, u) \cdot (s, v) \mid u \in A^*, v \in A^\omega, w = uv\}$ [1].

[1] Since the semiring K is idempotent (resp. By Lemma 1(ii)), the notation of the sum in the definition of Cauchy product of two finitary series (resp. of a finitary and an infinitary series), is consistent with the standard definition.

The *ω-iteration of a proper finitary series* $r \in K\langle\langle A^*\rangle\rangle$ is the infinitary series $r^\omega \in K\langle\langle A^\omega\rangle\rangle$ which is defined by

$(r^\omega, w) = \sum\left\{\prod_{i\geq 1}(r, u_i) \mid u_i \in A^*, w = u_1u_2\ldots\right\}$

for every $w \in A^\omega$.

Example 2. Let $r = \sum_{a\in A}(k_a)_a \in K\langle\langle A^*\rangle\rangle$ be a letter-step series. We will show that $(r^+)^+ = r^+$. Moreover, for every $w \in A^\omega$ we have $(r^\omega, w) = \prod_{i\geq 0}(r, w(i))$.

Let $w = w(0)\ldots w(n-1) \in A^+$. Then

$$\begin{aligned}(r^+, w) &= \sum\left\{\prod_{1\leq j\leq k}(r, u_j) \mid w = u_1\ldots u_k, 1\leq k\leq n\right\}\\ &= \prod_{0\leq j\leq n-1}(r, w(j)).\end{aligned}$$

Furthermore, we get

$$\begin{aligned}&\left((r^+)^+, w\right)\\ &= \sum\left\{\prod_{1\leq j\leq k}(r^+, u_j) \mid w = u_1\ldots u_k, 1\leq k\leq n\right\}\\ &= \sum\left\{\prod_{1\leq j\leq k}\left(\prod_{0\leq i_j\leq |u_j|-1}(r, u_j(i_j))\right) \mid w = u_1\ldots u_k, 1\leq k\leq n\right\}\\ &= \prod_{0\leq j\leq n-1}(r, w(j)) = (r^+, w).\end{aligned}$$

Similarly, we can show that $(r^\omega, w) = \prod_{i\geq 0}(r, w(i))$, for every $w \in A^\omega$.

Definition 8. *The class of* ω-star-free series over A and K, *denoted by* ω-$SF(K, A)$, *is the least class of infinitary series generated by the monomials (over A and K) by applying finitely many times the operations of sum, Hadamard product, complement, Cauchy product, iteration restricted to letter-step series, and ω-iteration restricted to letter-step series.*

The next result is trivially proved by Definitions 7, 8 and standard arguments.

Lemma 9. *Let $r \in SF(K, A)$ (resp. $r \in \omega$-$SF(K, A)$) and $B \subseteq A$. Then $r|_{B^*} \in SF(K, B)$ (resp. $r|_{B^\omega} \in \omega$-$SF(K, B)$).*

In the sequel, we state properties of the classes $SF(K, A)$ and ω-$SF(K, A)$. More precisely, we prove a splitting lemma and the closure of the classes under inverse strict alphabetic epimorphisms and bijections.

Lemma 10. *If $r \in SF(K, A)$ (resp. $r \in \omega$-$SF(K, A)$) and $k \in K$, then $kr \in SF(K, A)$ (resp. $kr \in \omega$-$SF(K, A)$).*

Proof. We have $kr = k_\varepsilon \cdot r$, hence we get the proof of our claim. $\square$

Lemma 11. *Let $L, L' \subseteq A^*$ and $K, K' \subseteq A^\omega$. Then*

- $1_{L\cup L'} = 1_L + 1_{L'}$, $\quad 1_{K\cup K'} = 1_K + 1_{K'}$
- $1_{L\cap L'} = 1_L \odot 1_{L'}$, $\quad 1_{K\cap K'} = 1_K \odot 1_{K'}$
- $1_{LL'} = 1_L \cdot 1_{L'}$, $\quad 1_{LK} = 1_L \cdot 1_K$
- $1_{L^+} = (1_L)^+$ *whenever* $\varepsilon \notin L$
- $1_{L^\omega} = (1_L)^\omega$ *whenever* $\varepsilon \notin L$.

Proof. We use standard arguments and the idempotency property of the semiring K. In particular, for the last statement we use Lemma 1(i). $\square$

The two subsequent results are shown by induction on the structure of star-free (resp. ω-star-free) languages and series using Lemma 11.

Lemma 12. *For every $L \subseteq A^*$ the following statements are equivalent.*

(i) *L is a star-free language.*

(ii) $1_L \in SF(K, A)$.

Lemma 13. *For every $L \subseteq A^\omega$ the following statements are equivalent.*

(i) *L is an ω-star-free language.*

(ii) $1_L \in \omega\text{-}SF(K, A)$.

Since for every $L \subseteq A^*$ (resp. $L \subseteq A^\omega$) and $k \in K$ we have $k_L = k_\varepsilon \cdot 1_L$, by Lemmas 12 and 13, we get Lemma 14 below.

Lemma 14. *Let $L \subseteq A^*$ (resp. $L \subseteq A^\omega$) and $k \in K$. If L is star-free (resp. ω-star-free), then $k_L \in SF(K, A)$ (resp. $k_L \in \omega\text{-}SF(K, A)$).*

Lemma 15. *If $s \in SF(K, A)$ (resp. $s \in \omega\text{-}SF(K, A)$), then* $\mathrm{supp}(s)$ *is a star-free language (resp. an ω-star-free) language over A.*

Proof. Using standard arguments, we state the proof by induction on the structure of s. $\square$

Lemma 16.

(i) *Let $L \subseteq A^*$ be a star-free language and $B, \Gamma \subseteq A$ with $B \cap \Gamma = \emptyset$. Then $1_L|_{B^*\Gamma B^*} = \sum_{1\leq i\leq n} \left(1_{M_i} \cdot \left(1_{\gamma_i} \cdot 1_{M_i'}\right)\right)$ where for every $1 \leq i \leq n$, $M_i, M_i' \subseteq B^*$ are star-free languages, and $\gamma_i \in \Gamma$.*

(ii) Let $L \subseteq A^\omega$ be an ω-star-free language and $B, \Gamma \subseteq A$ with $B \cap \Gamma = \emptyset$. Then $1_L|_{B^*\Gamma B^\omega} = \sum_{1 \le i \le n} \left(1_{M_i} \cdot \left(1_{\gamma_i} \cdot 1_{M_i'}\right)\right)$ where for every $1 \le i \le n$, $M_i \subseteq B^*$ is star-free, $M_i' \subseteq B^\omega$ is ω-star-free, and $\gamma_i \in \Gamma$.

Proof. We prove only (ii); Statement (i) is shown with the same arguments. By the splitting lemma for ω-star-free languages (cf. Lm. 3.2. in [7]), we get $L \cap B^*\Gamma B^\omega = \bigcup_{1 \le i \le n} M_i \gamma_i M_i'$ where for every $1 \le i \le n$, $M_i \subseteq B^*$ is star-free, $\gamma_i \in \Gamma$, and $M_i' \subseteq B^\omega$ is ω-star-free. Since $1_L|_{B^*\Gamma B^\omega} = 1_{L \cap B^*\Gamma B^\omega}$, we complete our proof using Lemma 11. □

Proposition 2 (Splitting lemma for finitary series)**.** *Let $s \in SF(K, A)$ and $B, \Gamma \subseteq A$ with $B \cap \Gamma = \emptyset$. Then $s|_{B^*\Gamma B^*} = \sum_{1 \le i \le n} \left(s_1^{(i)} \cdot \left(s_2^{(i)} \cdot s_3^{(i)}\right)\right)$ where for every $1 \le i \le n$, $s_1^{(i)}, s_3^{(i)} \in SF(K, B)$ and $s_2^{(i)} = (k_i)_{\gamma_i}$ with $\gamma_i \in \Gamma, k_i \in K$.*

Proof. We use induction on the structure of s. Let $s = (k_a)_a$, $a \in A$, be a monomial. Then, if $a \in \Gamma$, we have $s|_{B^*\Gamma B^*} = 1_\varepsilon \cdot ((k_a)_a \cdot 1_\varepsilon)$, otherwise $s|_{B^*\Gamma B^*} = 1_\emptyset \cdot \left((k_\gamma)_\gamma \cdot 1_\emptyset\right)$ for an arbitrary $\gamma \in \Gamma$. If $s = k_\varepsilon$, then again $s|_{B^*\Gamma B^*} = 1_\emptyset \cdot \left((k_\gamma)_\gamma \cdot 1_\emptyset\right)$ for an arbitrary $\gamma \in \Gamma$.

Let $s, r \in SF(K, A)$ satisfying the induction hypothesis. This means that $s|_{B^*\Gamma B^*} = \sum_{1 \le i \le n} \left(s_1^{(i)} \cdot \left(s_2^{(i)} \cdot s_3^{(i)}\right)\right)$ and $r|_{B^*\Gamma B^*} = \sum_{1 \le j \le m} \left(r_1^{(j)} \cdot \left(r_2^{(j)} \cdot r_3^{(j)}\right)\right)$ where for every $1 \le i \le n$ and $1 \le j \le m$, we have $s_1^{(i)}, s_3^{(i)}, r_1^{(j)}, r_3^{(j)} \in SF(K, B)$, $s_2^{(i)} = (k_i)_{\gamma_i}$, $r_2^{(j)} = (l_j)_{\gamma_j'}$, $\gamma_i, \gamma_j' \in \Gamma$, $k_i, l_j \in K$. Obviously, $(s + r)|_{B^*\Gamma B^*}$ has the required form.

Next let $w \in B^*\Gamma B^*$ and $0 \le k \le |w| - 1$ with $w(k) \in \Gamma$. Then $w_{<k}, w_{>k} \in B^*$ and we have

$$\begin{aligned}
(s|_{B^*\Gamma B^*}, w) &= \left(\sum_{1 \le i \le n} \left(s_1^{(i)} \cdot \left(s_2^{(i)} \cdot s_3^{(i)}\right)\right), w\right) \\
&= \sum_{1 \le i \le n} \left(s_1^{(i)} \cdot \left(s_2^{(i)} \cdot s_3^{(i)}\right), w\right) \\
&= \sum_{1 \le i \le n} \left(\left(s_1^{(i)}, w_{<k}\right) \cdot \left(s_2^{(i)}, w(k)\right) \cdot \left(s_3^{(i)}, w_{>k}\right)\right)
\end{aligned}$$

where the third equality holds since for every $1 \le i \le n$ and every decomposition $w = u_1 u_2 u_3$ with $u_2 \ne w(k)$ we have $\left(s_2^{(i)}, u_2\right) = 0$.
Similarly

$$\begin{aligned}
(r|_{B^*\Gamma B^*}, w) &= \left(\sum_{1 \le j \le m} \left(r_1^{(j)} \cdot \left(r_2^{(j)} \cdot r_3^{(j)}\right)\right), w\right) \\
&= \sum_{1 \le j \le m} \left(\left(r_1^{(j)}, w_{<k}\right) \cdot \left(r_2^{(j)}, w(k)\right) \cdot \left(r_3^{(j)}, w_{>k}\right)\right).
\end{aligned}$$

Hence,

$$
\begin{aligned}
&\left((s \odot r)|_{B^*\Gamma B^*}, w\right) = (s|_{B^*\Gamma B^*}, w) \cdot (r|_{B^*\Gamma B^*}, w) \\
&= \sum_{1 \le i \le n} \left(\left(s_1^{(i)}, w_{<k}\right) \cdot \left(s_2^{(i)}, w(k)\right) \cdot \left(s_3^{(i)}, w_{>k}\right) \right) \\
&\cdot \sum_{1 \le j \le m} \left(\left(r_1^{(j)}, w_{<k}\right) \cdot \left(r_2^{(j)}, w(k)\right) \cdot \left(r_3^{(j)}, w_{>k}\right) \right) \\
&= \sum_{\substack{1 \le i \le n \\ 1 \le j \le m}} \left(\left(s_1^{(i)} \odot r_1^{(j)}, w_{<k}\right) \cdot \left(\left(s_2^{(i)} \odot r_2^{(j)}, w(k)\right) \cdot \left(s_3^{(i)} \odot r_3^{(j)}, w_{>k}\right) \right) \right) \\
&= \left(\sum_{\substack{1 \le i \le n \\ 1 \le j \le m}} \left(\left(s_1^{(i)} \odot r_1^{(j)}\right) \cdot \left(\left(s_2^{(i)} \odot r_2^{(j)}\right) \cdot \left(s_3^{(i)} \odot r_3^{(j)}\right) \right) \right), w \right).
\end{aligned}
$$

Since $s_1^{(i)} \odot r_1^{(j)}$, $s_3^{(i)} \odot r_3^{(j)} \in SF(K, B)$, and $s_2^{(i)} \odot r_2^{(j)} = (k_i \cdot l_j)_{\gamma_i}$ if $\gamma_i = \gamma'_j$, and $s_2^{(i)} \odot r_2^{(j)} = 0_\gamma$ for an arbitrary $\gamma \in \Gamma$ otherwise, our claim is true for the Hadamard product.

Furthermore,

$$
\begin{aligned}
\left((s \cdot r)|_{B^*\Gamma B^*}, w\right) &= \sum \left\{ (s|_{B^*\Gamma B^*}, u) \cdot (r, v) \mid u \in B^*\Gamma B^*, v \in B^*, w = uv \right\} \\
&+ \sum \left\{ (s, u) \cdot (r|_{B^*\Gamma B^*}, v) \mid u \in B^*, v \in B^*\Gamma B^*, w = uv \right\}
\end{aligned}
$$

with

$$
\begin{aligned}
&\sum \left\{ (s|_{B^*\Gamma B^*}, u) \cdot (r, v) \mid u \in B^*\Gamma B^*, v \in B^*, w = uv \right\} \\
&= \sum \left\{ \left(\sum_{1 \le i \le n} \left(s_1^{(i)} \cdot \left(s_2^{(i)} \cdot s_3^{(i)} \right) \right), u \right) \cdot (r, v) \mid u \in B^*\Gamma B^*, v \in B^*, w = uv \right\} \\
&= \sum \left\{ \left(\sum_{1 \le i \le n} \left(s_1^{(i)} \cdot \left(s_2^{(i)} \cdot s_3^{(i)} \right) \right), u \right) \cdot (r|_{B^*}, v) \mid u, v \in A^*, w = uv \right\} \\
&= \left(\left(\sum_{1 \le i \le n} \left(s_1^{(i)} \cdot \left(s_2^{(i)} \cdot s_3^{(i)} \right) \right) \right) \cdot r|_{B^*}, w \right) \\
&= \left(\sum_{1 \le i \le n} \left(\left(s_1^{(i)} \cdot \left(s_2^{(i)} \cdot s_3^{(i)} \right) \right) \cdot r|_{B^*} \right), w \right) \\
&= \left(\sum_{1 \le i \le n} \left(s_1^{(i)} \cdot \left(s_2^{(i)} \cdot \left(s_3^{(i)} \cdot r|_{B^*} \right) \right) \right), w \right)
\end{aligned}
$$

where $r|_{B^*} = r \odot 1_{B^*} \in SF(K, B)$, and the fourth equality holds since the Cauchy product distributes over the sum of series. Similarly

$$
\begin{aligned}
&\sum \{(s,u) \cdot (r|_{B^*\Gamma B^*}, v) \mid u \in B^*, v \in B^*\Gamma B^*, w = uv\} \\
&= \sum \left\{ (s,u) \cdot \left(\sum_{1\le j\le m} \left(r_1^{(j)} \cdot \left(r_2^{(j)} \cdot r_3^{(j)} \right) \right), v \right) \mid u \in B^*, v \in B^*\Gamma B^*, w = uv \right\} \\
&= \sum \left\{ (s|_{B^*}, u) \cdot \left(\sum_{1\le j\le m} \left(r_1^{(j)} \cdot \left(r_2^{(j)} \cdot r_3^{(j)} \right) \right), v \right) \mid u, v \in A^*, w = uv \right\} \\
&= \left(s|_{B^*} \cdot \sum_{1\le j\le m} \left(r_1^{(j)} \cdot \left(r_2^{(j)} \cdot r_3^{(j)} \right) \right), w \right) \\
&= \left(\sum_{1\le j\le m} \left(s|_{B^*} \cdot \left(r_1^{(j)} \cdot \left(r_2^{(j)} \cdot r_3^{(j)} \right) \right) \right), w \right) \\
&= \left(\sum_{1\le j\le m} \left(\left(s|_{B^*} \cdot r_1^{(j)} \right) \cdot \left(r_2^{(j)} \cdot r_3^{(j)} \right) \right), w \right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
((s \cdot r)|_{B^*\Gamma B^*}, w) = &\left(\sum_{1\le i\le n} \left(s_1^{(i)} \cdot \left(s_2^{(i)} \cdot \left(s_3^{(i)} \cdot r|_{B^*} \right) \right) \right), w \right) \\
&+ \left(\sum_{1\le j\le m} \left(\left(s|_{B^*} \cdot r_1^{(j)} \right) \cdot \left(r_2^{(j)} \cdot r_3^{(j)} \right) \right), w \right).
\end{aligned}
$$

Therefore, the series $(s \cdot r)|_{B^*\Gamma B^*}$ has the required form.

Now, let s be a letter-step series. Then, $s|_{B^*\Gamma B^*} = s|_\Gamma = \sum_{\gamma\in\Gamma} (k_\gamma)_\gamma$. Let $w \in \mathrm{supp}(s^+) \cap B^*\Gamma B^*$, which implies that there is an index $0 \le k \le |w| - 1$ such that $w_{<k}, w_{>k} \in B^*$ and $w(k) \in \Gamma$. Then

$$
\begin{aligned}
&\left((s^+) |_{B^*\Gamma B^*}, w \right) \\
&= \sum \{ (s^m|_{B^*\Gamma B^*}, w) \mid 1 \leq m \leq |w| \} = \left(s^{|w|}|_{B^*\Gamma B^*}, w \right) \\
&= \prod_{0 \leq j \leq |w|-1} (s, w(j)) \\
&= \left(\prod_{0 \leq j \leq k-1} (s, w(j)) \right) \cdot (s, w(k)) \cdot \left(\prod_{k<j \leq |w|-1} (s, w(j)) \right) \\
&= \left((s|_B)^+ \cdot \left(s|_\Gamma \cdot (s|_B)^+ \right), w \right) \\
&= \left(\sum_{\gamma \in \Gamma} \left((s|_B)^+ \cdot \left((k_\gamma)_\gamma \cdot (s|_B)^+ \right) \right), w \right)
\end{aligned}
$$

and this concludes the induction for letter-step series.

Finally, let $s \in SF(K, A)$. Then $\overline{s} = 1_{\overline{\mathrm{supp}(s)}}$. Since $\mathrm{supp}(s)$ is a star-free language, we get that $\overline{\mathrm{supp}(s)}$ is also star-free. Hence, by Lemma 16(i) we conclude our proof. $\square$

Proposition 3 (Splitting lemma for infinitary series)**.** *Let $s \in \omega\text{-}SF(K, A)$ and $B, \Gamma \subseteq A$ with $B \cap \Gamma = \emptyset$. Then $s|_{B^*\Gamma B^\omega} = \sum_{1 \leq i \leq n} \left(s_1^{(i)} \cdot \left(s_2^{(i)} \cdot s_3^{(i)} \right) \right)$ where for every $1 \leq i \leq n$, $s_1^{(i)} \in SF(K, B)$, $s_3^{(i)} \in \omega\text{-}SF(K, B)$, and $s_2^{(i)} = (k_i)_{\gamma_i}$ with $\gamma_i \in \Gamma, k_i \in K$.*

Proof. Taking into account the definition of ω-star-free series, firstly we embed the proof of Lemma 2. Furthermore, we use arguments of that proof as follows. For the operations of sum and Hadamard product we let $s, r \in \omega\text{-}SF(K, A)$, and for Cauchy product we let $s \in SF(K, A)$ and $r \in \omega\text{-}SF(K, A)$. For the complement operation, we let $s \in \omega\text{-}SF(K, A)$ and we use the corresponding argument for ω-star-free languages and Lemma 16(ii). Finally, let s be a letter-step series. Then, $s|_{B^*\Gamma B^*} = s|_\Gamma = \sum_{\gamma \in \Gamma} (k_\gamma)_\gamma$. Let $w \in \mathrm{supp}(s^\omega) \cap B^*\Gamma B^\omega$, i.e., there exists an index $k \geq 0$ such that $w_{<k} \in B^*, w_{>k} \in B^\omega$, and $w(k) \in \Gamma$. Then we get

$$\begin{aligned}
&((s^\omega)\,|_{B^*\Gamma B^\omega}, w) \\
&= \sum \left\{ \prod_{i\geq 1} (s, u_i) \mid u_i \in A^*, w = u_1 u_2 \ldots \right\} \\
&= \prod_{j\geq 0} (s, w(j)) \\
&= \left(\prod_{0\leq j\leq k-1} (s, w(j)) \right) \cdot (s, w(k)) \cdot \left(\prod_{j>k} (s, w(j)) \right) \\
&= \left((s|_B)^+ \cdot (s|_\Gamma \cdot (s|_B)^\omega)\,, w \right) \\
&= \left(\sum_{\gamma\in\Gamma} \left((s|_B)^+ \cdot \left((k_\gamma)_\gamma \cdot (s|_B)^\omega \right) \right), w \right)
\end{aligned}$$

i.e.,

$$(s^\omega)\,|_{B^*\Gamma B^\omega} = \sum_{\gamma\in\Gamma} \left((s|_B)^+ \cdot \left((k_\gamma)_\gamma \cdot (s|_B)^\omega \right) \right)$$

and this completes our proof. □

Proposition 4. *Let A, B be two alphabets and $h : A \to B$ a bijection. Then $s \in SF(K, A)$ (resp. $s \in \omega\text{-}SF(K, A)$) implies that $h(s) \in SF(K, B)$ (resp. $h(s) \in \omega\text{-}SF(K, B)$).*

Proof. There is an one-to-one correspondence between the words of A^* and B^* (resp. the words of A^ω and B^ω) derived by h. Then, we can easily state our proof by induction on the structure of star-free (resp. ω-star-free) series. □

Proposition 5. *Let A, B be alphabets and $h : A \to B$ a strict alphabetic epimorphism. Then $s \in SF(K, B)$ (resp. $s \in \omega\text{-}SF(K, B)$) implies that $h^{-1}(s) \in SF(K, A)$ (resp. $h^{-1}(s) \in \omega\text{-}SF(K, A)$).*

Proof. We prove our claim by induction on the structure of star-free (resp. ω-star-free) series. Let $s = (k_b)_b$ be a monomial over B and K. Then, $h^{-1}(s)$ is a letter-step series and thus a star-free series over A and K. If $s = k_\varepsilon$, then $h^{-1}(s) = k_\varepsilon$ since h is strict. Next let $s_1, s_2 \in SF(K, B)$ (resp. $s_1, s_2 \in \omega\text{-}SF(K, B)$) such that $h^{-1}(s_1), h^{-1}(s_2) \in SF(K, A)$ (resp. $h^{-1}(s_1), h^{-1}(s_2) \in \omega\text{-}SF(K, A)$). Trivially $h^{-1}(s_1 \odot s_2) = h^{-1}(s_1) \odot h^{-1}(s_2)$ and $h^{-1}(s_1 + s_2) = h^{-1}(s_1) + h^{-1}(s_2)$.

Furthermore, for every $w \in A^*$ we have

$$\begin{aligned}\left(h^{-1}(s_1 \cdot s_2), w\right) &= (s_1 \cdot s_2, h(w)) \\ &= \sum \{(s_1, u_1) \cdot (s_2, u_2) \mid u_1, u_2 \in B^*, u_1 u_2 = h(w)\} \\ &= \sum \{(s_1, h(w_1)) \cdot (s_2, h(w_2)) \mid w_1, w_2 \in A^*, w_1 w_2 = w\} \\ &= \sum \left\{\left(h^{-1}(s_1), w_1\right) \cdot \left(h^{-1}(s_2), w_2\right) \mid w_1, w_2 \in A^*, w_1 w_2 = w\right\} \\ &= \left(h^{-1}(s_1) \cdot h^{-1}(s_2), w\right)\end{aligned}$$

where the fourth equality holds since h is strict alphabetic. Hence $h^{-1}(s_1 \cdot s_2) = h^{-1}(s_1) \cdot h^{-1}(s_2)$. If $s_1 \in SF(K, B)$, $s_2 \in \omega\text{-}SF(K, B)$, and $w \in A^\omega$, then we use the same as above argument, where we write $u_2 \in B^\omega$ and $w_2 \in A^\omega$.

Assume now that s is a letter-step series over B and K. Then, the series $h^{-1}(s)$ is a letter-step series over A and K, hence $h^{-1}(s) \in SF(K, A)$. For every $w \in A^+$ we get

$$\begin{aligned}(h^{-1}(s^+), w) = (s^+, h(w)) &= \prod_{0 \le j \le |w|-1} (s, h(w)(j)) \\ &= \prod_{0 \le j \le |w|-1} (s, h(w(j))) = \prod_{0 \le j \le |w|-1} (h^{-1}(s), w(j)) \\ &= ((h^{-1}(s))^+, w),\end{aligned}$$

i.e., $h^{-1}(s^+) = (h^{-1}(s))^+ \in SF(K, A)$.

Next, let $s \in SF(K, B)$. Then, $\overline{s} = 1_{\overline{\mathrm{supp}(s)}}$ and $\overline{\mathrm{supp}(s)}$ is, by Lemma 12, a star-free language over B. Moreover, the language $h^{-1}\left(\overline{\mathrm{supp}(s)}\right) \subseteq A^*$ is star-free (cf. for instance [28]) hence, the series $h^{-1}(\overline{s}) = h^{-1}\left(1_{\overline{\mathrm{supp}(s)}}\right) = 1_{h^{-1}\left(\overline{\mathrm{supp}(s)}\right)}$ is star-free by Lemma 12. The case $s \in \omega\text{-}SF(K, B)$ is treated similarly.

Finally, assume that s is a letter-step series over B and K. Then, $h^{-1}(s)$ is a letter-step series over A and K. Moreover, for every $w \in A^\omega$ we have

$$\begin{aligned}(h^{-1}(s^\omega), w) = (s^\omega, h(w)) &= \prod_{j \ge 0} (s, h(w)(j)) \\ &= \prod_{j \ge 0} (s, h(w(j))) = \prod_{j \ge 0} (h^{-1}(s), w(j)) \\ &= ((h^{-1}(s))^\omega, w),\end{aligned}$$

i.e., $h^{-1}(s^\omega) = \left(h^{-1}(s)\right)^\omega \in \omega\text{-}SF(A, K)$, and our proof is completed. □

7 ω-$wqFO$-definable series are ω-star-free

In the sequel, we show that every ω-$wqFO$-definable series over A and K is an ω-star-free series, i.e., $\omega\text{-}wqFO(K, A) \subseteq \omega\text{-}SF(K, A)$. For this, we use induction on the structure of $WQFO\,(K,A)$ formulas. We shall need the following auxiliary result.

Lemma 17. *Let $\varphi \in FO\,(K,A)$ and $\mathcal{V}$ be a finite set of first-order variables containing $free\,(\varphi)$. If $\|\varphi\|$ is an ω-star-free series, then $\|\varphi\|_{\mathcal{V}}$ is an ω-star-free series.*

Proof. Let $\|\varphi\|$ be an ω-star-free series and $h : A_{\mathcal{V}} \to A_{free(\varphi)}$ the strict alphabetic epimorphism erasing the x-row for every $x \in \mathcal{V} \setminus free\,(\varphi)$. It holds $\|\varphi\|_{\mathcal{V}} = h^{-1}(\|\varphi\|) \odot 1_{\mathcal{N}_{\mathcal{V}}}$. Then by Proposition 5 we get that $h^{-1}(\|\varphi\|) \in \omega$-$SF\,(K, A_{\mathcal{V}})$, and thus $\|\varphi\|_{\mathcal{V}} \in \omega$-$SF\,(K, A_{\mathcal{V}})$, as wanted. □

Lemma 18. *Let $\varphi \in FO\,(K,A)$ be an atomic formula. Then, $\|\varphi\|$ is an ω-star-free series.*

Proof. If $\varphi = k \in K$, then $\|\varphi\| = k_{A^\omega}$. Next, if $\varphi = P_a(x)$ or $x \leq y$, then φ is a boolean first-order formula, hence $\mathcal{L}(\varphi)$ is an ω-star-free language and $\|\varphi\| = 1_{\mathcal{L}(\varphi)}$ is an ω-star-free series. □

Lemma 19. *Let $\varphi \in FO\,(K,A)$ such that $\|\varphi\|$ is an ω-star-free series. Then, $\|\neg\varphi\|$ is also an ω-star-free series.*

Proof. By definition, we have $\|\neg\varphi\| = \overline{\|\varphi\|}$. □

Lemma 20. *Let $\varphi, \psi \in FO\,(K,A)$. If $\|\varphi\|, \|\psi\|$ are ω-star-free series, then $\|\varphi \wedge \psi\|, \|\varphi \vee \psi\|$ are ω-star-free series.*

Proof. Let $\mathcal{V} = free\,(\varphi) \cup free\,(\psi)$. We have $\|\varphi \wedge \psi\| = \|\varphi\|_{\mathcal{V}} \odot \|\psi\|_{\mathcal{V}}$ and $\|\varphi \vee \psi\| = \|\varphi\|_{\mathcal{V}} + \|\psi\|_{\mathcal{V}}$, hence our claim follows by definition of ω-star-free series and Lemma 17. □

Lemma 21. *Let $\varphi \in FO\,(K,A)$ such that $\|\varphi\|$ is an ω-star-free series. Then, $\|\exists x.\varphi\|$ is also an ω-star-free series.*

Proof. Let $\mathcal{W} = free\,(\varphi) \cup \{x\}$ and $\mathcal{V} = free(\exists x.\varphi) = \mathcal{W} \setminus \{x\}$. We define two subalphabets B, Γ of $A_{\mathcal{W}}$ by letting $B = \{(a,f) \in A_{\mathcal{W}} \mid f(x) = 0\}$ and $\Gamma = \{(a,f) \in A_{\mathcal{W}} \mid f(x) = 1\}$. Since $\|\varphi\|_{\mathcal{W}} \in \omega$-$SF\,(K, A_{\mathcal{W}})$ (by Lemma 17, in case $x \notin free(\varphi)$), by Proposition 3 we get

$$\|\varphi\|_{\mathcal{W}}\,|_{B^*\Gamma B^\omega} = \sum_{1 \leq i \leq n} \left(s_1^{(i)} \cdot \left(s_2^{(i)} \cdot s_3^{(i)} \right) \right)$$

with $s_1^{(i)} \in SF\,(K,B)$, $s_3^{(i)} \in \omega$-$SF\,(K,B)$, and $s_2^{(i)} = (k_i)_{\gamma_i}$, where $k_i \in K$, $\gamma_i \in \Gamma$ for every $1 \leq i \leq n$. We show that

$$\|\exists x.\varphi\| = \left(\sum_{1 \leq i \leq n} \left(h|_B \left(s_1^{(i)} \right) \cdot \left((k_i)_{h(\gamma_i)} \cdot h|_B \left(s_3^{(i)} \right) \right) \right) \right) \odot 1_{\mathcal{N}_{\mathcal{V}}}$$

where $h : A_{\mathcal{W}} \to A_{\mathcal{V}}$ is the strict alphabetic epimorphism assigning $(a, f|_{\mathcal{V}})$ to (a, f) for every $(a, f) \in A_{\mathcal{W}}$.

Let $(w, \sigma) \in \mathcal{N}_{\mathcal{V}}$. Then we have

$$\begin{aligned}
&(\|\exists x.\varphi\|, (w,\sigma)) \\
&= \sum_{j\geq 0} (\|\varphi\|_{\mathcal{W}}, (w, \sigma[x \to j])) \\
&= \sum_{j\geq 0} (\|\varphi\|_{\mathcal{W}}|_{B^*\Gamma B^\omega}, (w, \sigma[x \to j])) \\
&= \sum_{j\geq 0} \left(\sum_{1\leq i\leq n} \left(s_1^{(i)} \cdot \left(s_2^{(i)} \cdot s_3^{(i)}\right)\right), (w, \sigma[x \to j]) \right) \\
&= \sum_{j\geq 0} \left(\sum_{1\leq i\leq n} \left(\begin{array}{r} \left(s_1^{(i)}, (w, \sigma[x \to j])_{<j}\right) \cdot \left(s_2^{(i)}, (w, \sigma[x \to j])(j)\right) \\ \cdot \left(s_3^{(i)}, (w, \sigma[x \to j])_{>j}\right) \end{array} \right)\right) \\
&= \sum_{1\leq i\leq n} \left(\sum_{j\geq 0} \left(\begin{array}{r} \left(s_1^{(i)}, (w, \sigma[x \to j])_{<j}\right) \cdot \left(s_2^{(i)}, (w, \sigma[x \to j])(j)\right) \\ \cdot \left(s_3^{(i)}, (w, \sigma[x \to j])_{>j}\right) \end{array} \right)\right) \\
&= \sum_{1\leq i\leq n} \left(\sum_{j\geq 0} \left(\begin{array}{r} \left(h|_B\left(s_1^{(i)}\right), (w, \sigma)_{<j}\right) \cdot \left((k_i)_{h(\gamma_i)}, (w, \sigma)(j)\right) \\ \cdot \left(h|_B\left(s_3^{(i)}\right), (w, \sigma)_{>j}\right) \end{array} \right)\right) \\
&= \sum_{1\leq i\leq n} \left(h|_B\left(s_1^{(i)}\right) \cdot \left((k_i)_{h(\gamma_i)} \cdot h|_B\left(s_3^{(i)}\right)\right), (w,\sigma)\right)
\end{aligned}$$

where the sixth equality holds since $h\left((k_i)_{\gamma_i}\right) = (k_i)_{h(\gamma_i)}$ and $h|_B : B \to A_{\mathcal{V}}$ is a bijection. On the other hand, for every $(w, \sigma) \in A_{\mathcal{V}}^\omega \setminus \mathcal{N}_{\mathcal{V}}$ we have

$$\left(\sum_{1\leq i\leq n} \left(h|_B\left(s_1^{(i)}\right) \cdot \left((k_i)_{h(\gamma_i)} \cdot h|_B\left(s_3^{(i)}\right)\right)\right) \odot 1_{\mathcal{N}_{\mathcal{V}}}, (w,\sigma) \right) = 0.$$

Hence, $\|\exists x.\varphi\| = \left(\sum_{1\leq i\leq n} \left(h|_B\left(s_1^{(i)}\right) \cdot \left((k_i)_{h(\gamma_i)} \cdot h|_B\left(s_3^{(i)}\right)\right)\right) \right) \odot 1_{\mathcal{N}_{\mathcal{V}}}$. By Proposition 4, for every $1 \leq i \leq n$, we get that $h|_B\left(s_1^i\right) \in SF(K, A_{\mathcal{V}})$, $h|_B\left(s_3^{(i)}\right) \in \omega$-$SF(K, A_{\mathcal{V}})$. Therefore $\|\exists x.\varphi\|$ is an ω-star-free series. $\square$

Lemma 22. *Let $\varphi \in FO(K, A)$ be a boolean, or a letter-step formula with free variable x, or $\varphi = (y \leq x) \to \psi$, or $\varphi = (y \leq x < z) \to \psi$ where ψ is a letter-step formula with free variable x. Then, $\|\forall x.\varphi\|$ is an ω-star-free series.*

Proof. If $\varphi \in bFO(K, A)$, then $\forall x.\varphi \in bFO(K, A)$, hence the language $\mathcal{L}(\forall x.\varphi)$ is ω-star-free and the series $\|\forall x.\varphi\| = 1_{\mathcal{L}(\forall x.\varphi)}$ is ω-star-free.

Next, assume that $\varphi = \bigvee_{a\in A}(k_a \wedge P_a(x))$ is a letter-step formula with $k_a \in K$ for every $a \in A$. We consider the letter-step series $r = \sum_{a\in A}(k_a)_a$. Then for every word $w \in A^\omega$ we have

$$\begin{aligned}(\|\forall x.\varphi\|, w) &= \prod_{i\geq 0}(\|\varphi\|, (w, [x\to i]))\\ &= \prod_{i\geq 0}\left(\left\|\bigvee\nolimits_{a\in A}(k_a \wedge P_a(x))\right\|, (w,[x\to i])\right)\\ &= \prod_{i\geq 0}(r, w(i))\\ &= (r^\omega, w)\end{aligned}$$

where the fourth equality holds by Example 2. Hence, we get $\|\forall x.\varphi\| = r^\omega$ which implies that $\|\forall x.\varphi\|$ is an ω-star-free series.

Next, let $\varphi = (y \leq x) \to \bigvee_{a\in A}(k_a \wedge P_a(x))$. We consider the subset $F = \{(a,0) \mid a \in A\}$ of $A_{\{y\}}$. The language F^* is star-free, hence, the series 1_{F^*} is star-free. Consider the series $s = \sum_{a\in A}\left((k_a)_{(a,0)}\right)$ and $s' = \sum_{a\in A}\left((k_a)_{(a,1)}\right)$ over $A_{\{y\}}$ and K. Now for every $w \in A^\omega$ and $l \geq 0$, we get

$$\begin{aligned}(\|\forall x.\varphi\|, (w,[y\to l])) &= \prod_{j\geq 0}\left(\left\|(y\leq x)\to \bigvee\nolimits_{a\in A}(k_a\wedge P_a(x))\right\|, (w,[x\to j, y\to l])\right)\\ &= \prod_{j\geq l}\left(\left\|\bigvee\nolimits_{a\in A}(k_a\wedge P_a(x))\right\|, (w,[x\to j])\right)\\ &= (s', (w(l),1))\cdot \prod_{j>l}(s, (w(j),0))\\ &= (1_{F^*}\cdot(s'\cdot s^\omega), (w,[y\to l])),\end{aligned}$$

i.e., $\|\forall x.\varphi\| = 1_{F^*}\cdot(s'\cdot s^\omega)$ is an ω-star-free series.

Finally, let $\varphi = (y \leq x < z) \to \bigvee_{a\in A}(k_a \wedge P_a(x))$. We consider the finite languages $F = \{(a,0,0) \mid a\in A\}$, $F_1 = \{(a,1,0) \mid a \in A\}$, $F_2 = \{(a,0,1) \mid a \in A\}$ and $F_3 = \{(a,1,1) \mid a\in A\}$ over $A_{\{y,z\}}$. The languages $F, F_1, F_2, F_3, F^+, F^*$ are star-free, hence the series $1_{F_1}, 1_{F_2}, 1_{F_3}, 1_{F^+}, 1_{F^*}$ are star-free. Since $(F^+)^+ = F^+$ the languages $L = (F^+)^\omega, L' = F_2 L$ are ω-star-free (cf. [28]) and the infinitary series $1_L, 1_{L'}$ are ω-star-free. We consider the series $s = \sum_{a\in A}\left(k_{(a,0,0)}\right)_{(a,0,0)}$ and $s' = \sum_{a\in A}\left(k_{(a,1,0)}\right)_{(a,1,0)}$ over $A_{\{y,z\}}$ and K, where $k_{(a,0,0)} = k_{(a,1,0)} = k_a$ for every $a \in A$. Moreover, we let $r_1 = 1_{F^*}\cdot(s'\cdot((1_\varepsilon + s^+)\cdot 1_{L'}))$, $r_2 = 1_{F^*}\cdot(1_{F_3}\cdot 1_L)$, and $r_3 = 1_{F^*}\cdot(1_{F_2}\cdot(1_{F^*}\cdot(1_{F_1}\cdot 1_L)))$.

Now, for every $w \in A^\omega$ and $j,l \geq 0$ with $j < l$, we have

$(r_2 + r_3, (w, [y \to j, z \to l])) = 0$, and

$$\begin{aligned}
&(\|\forall x.\varphi\|, (w, [y \to j, z \to l])) \\
&= \prod_{i \geq 0} \left(\left\| (y \leq x < z) \to \bigvee\nolimits_{a \in A} (k_a \wedge P_a(x)) \right\|, (w, [x \to i, y \to j, z \to l]) \right) \\
&= \prod_{j \leq i < l} \left(\left\| \bigvee\nolimits_{a \in A} (k_a \wedge P_a(x)) \right\|, (w, [x \to i]) \right) \\
&= (s', (w(j), 1, 0)) \cdot \prod_{j < i < l} (s, (w(i), 0, 0)) \\
&= (r_1, (w, [y \to j, z \to l])) \\
&= (r_1 + (r_2 + r_3), (w, [y \to j, z \to l])) .
\end{aligned}$$

Furthermore, for every $w \in A^\omega$ and $j, l \geq 0$ with $j \geq l$, we get $(r_1, (w, [y \to j, z \to l])) = 0$, and

$$\begin{aligned}
&(\|\forall x.\varphi\|, (w, [y \to j, z \to l])) \\
&= \prod_{i \geq 0} \left(\left\| (y \leq x < z) \to \bigvee\nolimits_{a \in A} (k_a \wedge P_a(x)) \right\|, (w, [x \to i, y \to j, z \to l]) \right) \\
&= \prod_{i \geq 0} (\|\neg (y \leq x < z)\|, (w, [x \to i, y \to j, z \to l])) \\
&= (r_2 + r_3, (w, [y \to j, z \to l])) \\
&= (r_1 + (r_2 + r_3), (w, [y \to j, z \to l])) .
\end{aligned}$$

We conclude that $\|\forall x.\varphi\| = r_1 + (r_2 + r_3)$, hence $\|\forall x.\varphi\|$ is an ω-star-free series, as required. □

Now, we are ready to state the main result of the section.

Theorem 2. *ω-wqFO$(K, A) \subseteq \omega$-SF(K, A).*

Proof. We combine Lemmas 18, 19, 20, 21, and 22. □

8 Counter-free series

In this section, we consider the concept of counter-freeness within weighted (resp. weighted Büchi) automata over A and K. Our models will be nondeterministic. We need first to recall the notions of weighted automata and weighted Büchi automata over A and K. For simplicity reasons, we equip our finitary models with a set of final states instead of a terminal distribution.

A weighted automaton over A and K is a quadruple $\mathcal{A} = (Q, in, wt, F)$ where Q is the *finite state set*, $in : Q \to K$ is the *initial distribution*, $wt : Q \times A \times Q \to K$

is a mapping assigning *weights* to the transitions of the automaton and $F \subseteq Q$ is the *final state set*.

Given a word $w = a_0 \ldots a_{n-1} \in A^*$, a path of $\mathcal{A}$ over w is a finite sequence of transitions $P_w := ((q_i, a_i, q_{i+1}))_{0 \leq i \leq n-1}$. The *running weight of* P_w is the value

$$rwt(P_w) := \prod_{0 \leq i \leq n-1} wt\left((q_i, a_i, q_{i+1})\right)$$

and the *weight of* P_w is given by

$$weight(P_w) := in(q_0) \cdot rwt(P_w).$$

The path P_w is called *successful* if $q_n \in F$. We denote by $\mathrm{succ}(\mathcal{A})$ the set of successful paths of $\mathcal{A}$. The *behavior of* $\mathcal{A}$ is the series $\|\mathcal{A}\| : A^* \to K$ which is defined, for every $w \in A^*$, by $(\|\mathcal{A}\|, w) = \sum\limits_{P_w \in \mathrm{succ}(\mathcal{A})} weight(P_w)$. A series $r \in K\left\langle\left\langle A^* \right\rangle\right\rangle$ is called *recognizable* if it is the behavior of a weighted automaton over A and K.

A weighted Büchi automaton $\mathcal{A} = (Q, in, wt, F)$ over A and K is defined as a weighted automaton. Given an infinite word $w = a_0 a_1 \ldots \in A^\omega$, a path of $\mathcal{A}$ over w is an infinite sequence of transitions $P_w := ((q_i, a_i, q_{i+1}))_{i \geq 0}$. The *running weight of* P_w is the value

$$rwt(P_w) := \prod_{i \geq 0} wt\left((q_i, a_i, q_{i+1})\right)$$

and the *weight of* P_w is given by

$$weight(P_w) := in(q_0) \cdot rwt(P_w).$$

A path P_w is called *successful* if at least one final state occurs infinitely often along P_w. Then, the *behavior of* $\mathcal{A}$ is the infinitary series $\|\mathcal{A}\| : A^\omega \to K$ whose coefficients are given by $(\|\mathcal{A}\|, w) = \sum\limits_{P_w \in \mathrm{succ}(\mathcal{A})} weight(P_w)$, for every $w \in A^\omega$. An infinitary series $r \in K\left\langle\left\langle A^\omega \right\rangle\right\rangle$ is called *ω-recognizable* if it is the behavior of a weighted Büchi automaton over A and K.

We shall need also the following notation. Given a weighted (resp. weighted Büchi) automaton $\mathcal{A} = (Q, in, wt, F)$, a word $w = a_0 \ldots a_{n-1} \in A^*$, and states $q, q' \in Q$, we shall denote by $P_{(q,w,q')}$ a path of $\mathcal{A}$ over w starting at state q and terminating at state q', i.e., $P_{(q,w,q')} = (q, a_0, q_1)\left((q_i, a_i, q_{i+1})\right)_{1 \leq i \leq n-2} (q_{n-1}, a_{n-1}, q')$. Then

$$rwt\left(P_{(q,w,q')}\right) = wt\left((q, a_0, q_1)\right) \cdot \prod\nolimits_{1 \leq i \leq n-2} wt\left((q_i, a_i, q_{i+1})\right) \cdot wt\left((q_{n-1}, a_{n-1}, q')\right).$$

Now, we are ready to introduce our counter-free weighted and counter-free weighted Büchi automata.

Definition 9. *A weighted automaton (resp. weighted Büchi automaton)* $\mathcal{A} = (Q, in, wt, F)$ *over* A *and* K *is called* counter-free *(*cfwa, *resp.* cfwBa*, for short) if for every* $q \in Q$*,* $w \in A^*$*, and* $n \geq 1$*, the relation* $\sum_{P_{(q,w^n,q)}} rwt\left(P_{(q,w^n,q)}\right) \neq 0$ *implies* $\sum_{P_{(q,w^n,q)}} rwt\left(P_{(q,w^n,q)}\right) = \left(\sum_{P_{(q,w,q)}} rwt\left(P_{(q,w,q)}\right)\right)^n$.

A series $r \in K\langle\langle A^*\rangle\rangle$ (resp. $r \in K\langle\langle A^\omega\rangle\rangle$) is called *counter-free* (resp. *ω-counter-free*) if it is accepted by a cfwa (resp. cfwBa) over A and K. We shall denote by $CF(K, A)$ (resp. ω-$CF(K, A)$) the class of all counter-free (resp. ω-counter-free) series over A and K.

A cfwa $\mathcal{A} = (Q, in, wt, F)$ over A and K is called *normalized* if there are two states $q_0, q_t \in Q$ such that $F = \{q_t\}$ and for every $q \in Q$, $a \in A$, we have $in(q) = 1$ if $q = q_0$, and $in(q) = 0$ otherwise, and $wt((q, a, q_0)) = 0 = wt((q_t, a, q))$. We denote a normalized cfwa $\mathcal{A}$ simply by $\mathcal{A} = (Q, q_0, wt, q_t)$.

The following result has been proved for weighted automata in [11].

Lemma 23. *For every cfwa* $\mathcal{A} = (Q, in, wt, F)$ *we can effectively construct a normalized cfwa* $\mathcal{A}' = (Q \cup \{q_0, q_t\}, q_0, wt', q_t)$ *such that* $(\|\mathcal{A}'\|, w) = (\|\mathcal{A}\|, w)$ *for every* $w \in A^+$ *and* $(\|\mathcal{A}'\|, \varepsilon) = 0$.

Proof. We use similar arguments as in the proof of Lm. 7 in [11]. In fact, it remains to show that the normalized weighted automaton $\mathcal{A}'$ is counter-free. Indeed, let $q \in Q \cup \{q_0, q_t\}$, $w \in A^+$, $n \geq 1$, and $P'_{(q,w^n,q)}$ be a path of $\mathcal{A}'$ over w with $rwt(P'_{(q,w^n,q)}) \neq 0$. Since $\mathcal{A}'$ is normalized we get that the states q_0, q_t do not occur in the path $P'_{(q,w^n,q)}$ hence $P'_{(q,w^n,q)}$ is also a path of $\mathcal{A}$. This implies that

$$\begin{aligned}\sum_{P'_{(q,w^n,q)}} rwt\left(P'_{(q,w^n,q)}\right) &= \sum_{P_{(q,w^n,q)}} rwt\left(P_{(q,w^n,q)}\right) = \left(\sum_{P_{(q,w,q)}} rwt\left(P_{(q,w,q)}\right)\right)^n \\ &= \left(\sum_{P'_{(q,w,q)}} rwt\left(P'_{(q,w,q)}\right)\right)^n,\end{aligned}$$

where $P_{(q,w^n,q)}$ denotes a path of $\mathcal{A}$ over w, and this concludes our proof. □

A cfwBa $\mathcal{A} = (Q, in, wt, F)$ over A and K is called *initial weight normalized* if there is a state $q_0 \in Q$ such that for every $q \in Q$ and $a \in A$ we have $in(q) = 1$ if $q = q_0$, and $in(q) = 0$ otherwise, and $wt((q, a, q_0)) = 0$. We denote an initial weight normalized cfwBa $\mathcal{A}$ simply by $\mathcal{A} = (Q, q_0, wt, F)$.

Lemma 24. *For every cfwBa* $\mathcal{A} = (Q, in, wt, F)$ *we can effectively construct an initial weight normalized cfwBa* $\mathcal{A}' = (Q \cup \{q_0\}, q_0, wt', F)$ *such that* $\|\mathcal{A}'\| = \|\mathcal{A}\|$.

Proof. We use the same arguments, as in Lemma 23 for the modification of the initial distribution. □

In the sequel, we prove closure properties of the classes $CF(K,A)$ and ω-$CF(K,A)$. We shall need these properties in order to relate star-free and ω-star-free series with counter-free and ω-counter-free series, nevertheless, these results have also their own interest.

Proposition 6. *The class $CF(K,A)$ contains the monomials and it is closed under sum, Hadamard product, complement, Cauchy product, and iteration restricted to letter-step series.*

Proof. The closure of $CF(K,A)$ under sum, is shown by taking the disjoint union of two cfwa. In this case, any "loop" belongs either to the first or to the second automaton, hence the derived weighted automaton is also counter-free. Since monomials over A and K are obviously counter-free series, we get that letter-step series are also counter-free.

Closure under Hadamard product is proved by using the standard "product construction" of two cfwa. More precisely, let $\mathcal{A}_1=(Q_1, in_1, wt_1, F_1)$ and $\mathcal{A}_2=(Q_2, in_2, wt_2, F_2)$ be two cfwa over A and K. Consider the weighted automaton $\mathcal{A}=(Q, in, wt, F)$ with $Q = Q_1 \times Q_2$, $F = F_1 \times F_2$, and $in((q_1,q_2)) = in_1(q_1) \cdot in_2(q_2)$, $wt(((q_1,q_2),a,(p_1,p_2))) = wt_1((q_1,a,p_1)) \cdot wt_2((q_2,a,p_2))$, for every $(q_1,q_2),(p_1,p_2) \in Q, a \in A$. Then, for every $w \in A^*$ and path P_w of $\mathcal{A}$ over w, there are two unique paths $P_{1,w}$ of $\mathcal{A}_1$ over w, and $P_{2,w}$ of $\mathcal{A}_2$ over w (obtained by projections of P_w on Q_1 and Q_2, respectively, in the obvious way) and vice-versa. Furthermore, we have $weight(P_w) = weight(P_{1,w}) \cdot weight(P_{2,w})$. Now assume that for some $(q_1,q_2) \in Q$, $w \in A^*$, and $n \geq 1$ there is a path $P_{((q_1,q_2),w^n,(q_1,q_2))}$ with $rwt\left(P_{((q_1,q_2),w^n,(q_1,q_2))}\right) \neq 0$. Then

$$
\begin{aligned}
&\left(\sum_{P_{((q_1,q_2),w,(q_1,q_2))}} rwt\left(P_{((q_1,q_2),w,(q_1,q_2))}\right)\right)^n \\
&= \left(\sum_{P_{1,(q_1,w,q_1)},P_{2,(q_2,w,q_2)}} \left(rwt\left(P_{1,(q_1,w,q_1)}\right) \cdot rwt\left(P_{2,(q_2,w,q_2)}\right)\right)\right)^n \\
&= \left(\sum_{P_{1,(q_1,w,q_1)}} rwt\left(P_{1,(q_1,w,q_1)}\right) \cdot \sum_{P_{2,(q_2,w,q_2)}} rwt\left(P_{2,(q_2,w,q_2)}\right)\right)^n \\
&= \left(\sum_{P_{1,(q_1,w,q_1)}} rwt\left(P_{1,(q_1,w,q_1)}\right)\right)^n \cdot \left(\sum_{P_{2,(q_2,w,q_2)}} rwt\left(P_{2,(q_2,w,q_2)}\right)\right)^n \\
&= \sum_{P_{1,(q_1,w^n,q_1)}} rwt\left(P_{1,(q_1,w^n,q_1)}\right) \cdot \sum_{P_{2,(q_2,w^n,q_2)}} rwt\left(P_{2,(q_2,w^n,q_2)}\right) \\
&= \sum_{P_{((q_1,q_2),w^n,(q_1,q_2))}} rwt\left(P_{((q_1,q_2),w^n,(q_1,q_2))}\right)
\end{aligned}
$$

which implies that $\mathcal{A}$ is counter-free, and by construction $\|\mathcal{A}\| = \|\mathcal{A}_1\| \odot \|\mathcal{A}_2\|$.

Next, let $r \in CF(K,A)$ and $\mathcal{A} = (Q, in, wt, F)$ be a cfwa accepting r. We consider the nondeterministic finite automaton $\mathcal{A}' = (Q, A, I, \Delta, F)$ with $I = \{q \in Q \mid in(q) \neq 0\}$ and $\Delta = \{(q,a,q') \in Q \times A \times Q \mid wt((q,a,q')) \neq 0\}$. By construction of $\mathcal{A}'$, and since K is zero-divisor free, we get that for every $q_1, q_2 \in Q$ and $w \in A^*$ the path $P_{(q_1,w,q_2)}$ exists in $\mathcal{A}'$ iff $rwt(P_{(q_1,w,q_2)}) \neq 0$ in $\mathcal{A}$. Therefore, $\mathcal{A}'$ accepts the language $\text{supp}(r)$ and it is trivially counter-free hence, $\text{supp}(r)$ is a counter-free language. Then, $\overline{\text{supp}(r)}$ is a counter-free language and let $\mathcal{B}$ be a counter-free automaton accepting it. We convert $\mathcal{B}$, in the obvious way, to a weighted automaton $\mathcal{B}'$ (with weights only 0 and 1) over A and K. Since K is idempotent, $\mathcal{B}'$ trivially accepts $1_{\overline{\text{supp}(r)}} = \overline{r}$, and it is easily obtained that it is counter-free. We conclude that the series $\overline{r}$ is counter-free, as required.

Let now $\mathcal{A}_1 = (Q_1, in_1, wt_1, F_1)$ and $\mathcal{A}_2 = (Q_2, in_2, wt_2, F_2)$ be two cfwa over A and K. Using Lemma 23 we consider the normalized cfwa $\mathcal{A}'_1 = (Q_1 \cup \{q_{0,1}, q_{t,1}\}, q_{0,1}, wt'_1, q_{t,1})$ and $\mathcal{A}'_2 = (Q_2 \cup \{q_{0,2}, q_{t,2}\}, q_{0,2}, wt'_2, q_{t,2})$ such that $\|\mathcal{A}'_i\|$ coincides with $\|\mathcal{A}_i\|$ on A^+ for $i = 1, 2$. Without any loss, we assume that $(Q_1 \cup \{q_{0,1}, q_{t,1}\}) \cap (Q_2 \cup \{q_{0,2}, q_{t,2}\}) = \emptyset$. We construct the weighted automaton $\mathcal{A} = (Q, q_{0,1}, wt, q_{t,2})$ with $Q = Q_1 \cup \{q_{0,1}\} \cup Q_2 \cup \{q_{0,2}, q_{t,2}\}$ where we identify the states $q_{t,1}$ and $q_{0,2}$, and define the weight assignment mapping wt for every $q, q' \in Q, a \in A$ by

$$wt((q,a,q')) = \begin{cases} wt'_1((q,a,q')) & \text{if } q, q' \in Q_1 \cup \{q_{0,1}\} \\ wt'_2((q,a,q')) & \text{if } q, q' \in Q_2 \cup \{q_{0,2}, q_{t,2}\} \\ wt'_1((q,a,q_{t,1})) & \text{if } q \in Q_1 \cup \{q_{0,1}\} \text{ and } q' = q_{0,2} \\ 0 & \text{otherwise.} \end{cases}$$

It is a routine matter to formally prove that $\|\mathcal{A}\| = \|\mathcal{A}'_1\| \cdot \|\mathcal{A}'_2\|$. Furthermore, the weighted automaton $\mathcal{A}$ is counter-free since, by construction, any "loop" with weight $\neq 0$ belongs either to $\mathcal{A}'_1$ or to $\mathcal{A}'_2$. Now we let $k_i = (\|\mathcal{A}_i\|, \varepsilon)$ for $i = 1, 2$. Then $\|\mathcal{A}_1\| \cdot \|\mathcal{A}_2\| = (\|\mathcal{A}'_1\| \cdot \|\mathcal{A}'_2\|) + ((k_1)_\varepsilon \cdot \|\mathcal{A}'_2\|) + (\|\mathcal{A}'_1\| \cdot (k_2)_\varepsilon) + ((k_1)_\varepsilon \cdot (k_2)_\varepsilon)$. One can trivially construct cfwa accepting $(k_1)_\varepsilon$ and $(k_2)_\varepsilon$ and using simplifications of our previous construction[2] for $\mathcal{A}$ can easily show that the series $(k_1)_\varepsilon \cdot \|\mathcal{A}'_2\|$, $\|\mathcal{A}'_1\| \cdot (k_2)_\varepsilon$, and $(k_1)_\varepsilon \cdot (k_2)_\varepsilon$ are counter-free which implies, by what we have shown, that $\|\mathcal{A}_1\| \cdot \|\mathcal{A}_2\|$ is a counter-free series.

Finally, let $r = \sum_{a \in A} (k_a)_a$ be a letter-step series with $k_a \in K$ for every $a \in A$. We consider the cfwa $\mathcal{A} = (\{q_0, q_t\}, q_0, wt, q_t)$ with $wt((q_0, a, q_t)) = wt((q_t, a, q_t)) = k_a$ for every $a \in A$, and the weight of any other transition is 0. Obviously $r^+ = \|\mathcal{A}\|$, and we are done. □

Proposition 7. *The class ω-$CF(K,A)$ is closed under sum, complement, Cauchy product and ω-iteration restricted to letter-step series.*

Proof. The closure under sum and complement is shown as in Proposition 6. In particular, for the complement we use the property $k \neq 0 \implies \prod_{i \geq 0} k \neq 0$ for every $k \in K$, the fact that the class of counter-free Büchi recognizable (i.e., ω-star-free) languages is closed under complement (cf. [7]), and Lemma 1(i).

[2] In fact the cfwa for $(k_1)_\varepsilon$ and $(k_2)_\varepsilon$ cannot be normalized.

Next, let $s_1 \in CF(K,A)$ and $s_2 \in \omega\text{-}CF(K,A)$, and $\mathcal{A}_1 = (Q_1, in_1, wt_1, F_1)$, $\mathcal{A}_2 = (Q_2, in_2, wt_2, F_2)$ be a cfwa and a cfwBa over A and K accepting s_1 and s_2, respectively. Furthermore, let $\mathcal{A}_1'$=$(Q_1 \cup \{q_{0,1}, q_t\}, q_{0,1}, wt_1', q_t)$ be the normalized automaton derived by $\mathcal{A}_1$ (cf. Lemma 23), and $\mathcal{A}_2'$=$(Q_2 \cup \{q_{0,2}\}, q_{0,2}, wt_2', F_2)$ be the initial weight normalized cfwBa derived by $\mathcal{A}_2$ (cf. Lemma 24). Without any loss, we assume that $(Q_1 \cup \{q_{0,1}, q_t\}) \cap (Q_2 \cup \{q_{0,2}\}) = \emptyset$. Consider the weighted automaton $\mathcal{A} = (Q, q_{0,1}, wt, F_2)$ with $Q = Q_1 \cup \{q_{0,1}\} \cup Q_2 \cup \{q_{0,2}\}$ where we have identified the states q_t and $q_{0,2}$. The weight assignment mapping wt is defined for every $q, q' \in Q$ and $a \in A$ by

$$wt((q,a,q')) = \begin{cases} wt_1'((q,a,q')) & \text{if } q,q' \in Q_1 \cup \{q_{0,1}\} \\ wt_2'((q,a,q')) & \text{if } q,q' \in Q_2 \cup \{q_{0,2}\} \\ wt_1'((q,a,q_t)) & \text{if } q \in Q_1 \cup \{q_{0,1}\} \text{ and } q' = q_{0,2} \\ 0 & \text{otherwise.} \end{cases}$$

Trivially, $\|\mathcal{A}\| = s_1|_{A^+} \cdot s_2$. Furthermore, the weighted Büchi automaton $\mathcal{A}$ is counter-free since every "loop" with weight $\neq 0$ belongs either to $\mathcal{A}_1'$ or to $\mathcal{A}_2'$. Let $(s_1, \varepsilon) = k$. Then $s_1 \cdot s_2 = s_1|_{A^+} \cdot s_2 + k_\varepsilon \cdot s_2$ which concludes our claim since $k_\varepsilon \cdot s_2$ is trivially ω-counter-free.

Finally, let $r = \sum_{a \in A} (k_a)_a$ be a letter-step series with $k_a \in K$ for every $a \in A$. We consider the initial weight normalized cfwBa $\mathcal{A} = (\{q_0, q_t\}, q_0, wt, \{q_t\})$ with $wt((q_0, a, q_t)) = wt((q_t, a, q_t)) = k_a$ for every $a \in A$, and the weight of any other transition is 0. Obviously $r^\omega = \|\mathcal{A}\|$, and our proof is completed. □

Next, we introduce the subclass of almost simple counter-free (resp. almost simple ω-counter-free) series and we show, in Section 9, that it contains the class $SF(K,A)$ (resp. $\omega\text{-}SF(K,A)$).

Definition 10. *A cfwa (resp. cfwBa) $\mathcal{A} = (Q, in, wt, F)$ over A and K is called* simple *if for every $q, q', p, p' \in Q$, and $a \in A$, $in(q) \neq 0 \neq in(q')$ implies $in(q) = in(q')$, and $wt((q,a,q')) \neq 0 \neq wt((p,a,p'))$ implies $wt((q,a,q')) = wt((p,a,p'))$. Furthermore, a series $r \in K\langle\langle A^* \rangle\rangle$ (resp. $r \in K\langle\langle A^\omega \rangle\rangle$) is* simple *if it is the behavior of a simple cfwa (resp. cfwBa) over A and K.*

Proposition 8. *If $r, s \in K\langle\langle A^\omega \rangle\rangle$ are simple infinitary series, then $r \odot s$ is also simple.*

Proof. Let $\mathcal{A}, \mathcal{B}$ be two simple cfwBa accepting r, s, respectively. We let k, l for the weights $\neq 0$ assigned by the initial distributions of $\mathcal{A}, \mathcal{B}$, respectively, and k_a, l_a for the weights $\neq 0$ of the transitions labelled by $a \in A$, in $\mathcal{A}$ and $\mathcal{B}$, respectively. Without any loss, we assume that k_a, l_a exist for every $a \in A$, otherwise we consider a subalphabet of A. The language $L = \mathrm{supp}(\|\mathcal{A}\|) \cap \mathrm{supp}(\|\mathcal{B}\|)$ is ω-counter-free (cf. the proof of Proposition 7), and we get

$$\|\mathcal{A}\| \odot \|\mathcal{B}\| = 1_L \odot \left((k \cdot l) \left(\sum_{a \in A} (k_a \cdot l_a)_a \right)^\omega \right).$$

Let $\mathcal{C} = (Q, A, I, \Delta, F)$ be a counter-free nondeterministic Büchi automaton accepting L and consider the wBa $\mathcal{C}'$=(Q, in, wt, F) where for every $q, q' \in Q, a \in A$ we let $in(q) = k \cdot l$ if $q \in I$, and $in(q) = 0$ otherwise, and $wt((q, a, q')) = k_a \cdot l_a$ if $(q, a, q') \in \Delta$, and $wt((q, a, q')) = 0$ otherwise. Since $\mathcal{C}$ is counter-free, we can easily show, using the idempotency property of K, that $\mathcal{C}'$ is also counter-free. Moreover, by definition $\mathcal{C}'$ is simple, and $\|\mathcal{C}'\| = \|\mathcal{A}\| \odot \|\mathcal{B}\|$ which concludes our proof. $\square$

Definition 11.

- *A series* $r \in K\langle\langle A^*\rangle\rangle$ *is called* almost simple *if* $r = \sum_{1\leq i\leq n}\left(r_1^{(i)} \cdot \ldots \cdot r_{m_i}^{(i)}\right)$ *where, for every* $1 \leq i \leq n$*,* $r_1^{(i)}, \ldots, r_{m_i}^{(i)}$ *are simple counter-free series over* A *and* K*.*
- *A series* $r \in K\langle\langle A^\omega\rangle\rangle$ *is called* almost simple *if* $r = \sum_{1\leq i\leq n}\left(r_1^{(i)} \cdot \ldots \cdot r_{m_i}^{(i)}\right)$ *where, for every* $1 \leq i \leq n$*,* $r_1^{(i)}, \ldots, r_{m_i-1}^{(i)}$ *are simple counter-free series and* $r_{m_i}^{(i)}$ *is a simple* ω*-counter-free series over* A *and* K*.*

From the above definition and Proposition 6 (resp. Proposition 7), we get that a finitary (resp. infinitary) almost simple series is a counter-free (resp. an ω-counter-free) series[3]. We shall denote by $asCF(K, A)$ the class of almost simple counter-free series and by ω-$asCF(K, A)$ the class of almost simple ω-counter-free series over A and K.

9 ω-star-free series are almost simple ω-counter-free

In this section we prove that every star-free (resp. ω-star-free) series is an almost simple counter-free (resp. almost simple ω-counter-free) series.

Theorem 3. $SF(K, A) \subseteq asCF(K, A)$.

Proof. The class $asCF(K, A)$ trivially contains the monomials over A and K. Therefore, it suffices to show that it is closed under sum, Hadamard product, complement, Cauchy product, and iteration restricted to letter-step series.

Closure under sum and Cauchy product is easily obtained by definition of the class of almost simple counter-free series. For the closure under complement, let $r \in asCF(K, A)$, i.e., $r \in CF(K, A)$. Then the weighted automaton $\mathcal{B}'$ in the proof of Proposition 6 is simple and moreover accepts the complement $\overline{r}$ hence, $\overline{r} \in asCF(K, A)$. Trivially, we get that $asCF(K, A)$ contains the letter-step series. Furthermore, the automaton $\mathcal{A}$ accepting r^+ for a letter-step series r, in the proof of Proposition 6, is trivially simple, hence the class $asCF(K, A)$ is closed under iteration restricted to letter-step series. Therefore, it remains to prove

[3]In fact we can define an *almost simple* counter-free weighted (resp. weighted Büchi) automaton, but we do not need it here.

the closure under $\odot$. Since, $\odot$ distributes over sum it suffices to show that if $\mathcal{A}_i = (Q_i, in_i, wt_i, F_i), \mathcal{B}_j = (P_j, in'_j, wt'_j, T_j)$, for $1 \leq i \leq n, 1 \leq j \leq m$, are simple cfwa over A and K, then the counter-free series $(\|\mathcal{A}_1\| \cdot \ldots \cdot \|\mathcal{A}_n\|) \odot (\|\mathcal{B}_1\| \cdot \ldots \cdot \|\mathcal{B}_m\|)$ is almost simple. We proceed by induction on m, hence, assume firstly that $m = 1$. Without any loss, we suppose the state sets Q_i $(1 \leq i \leq n)$ to be pairwise disjoint[4]. For every $p, p' \in P_1$ and $2 \leq i \leq n-1$, we consider the simple cfwa $\mathcal{C}_{1,p} = (Q_1 \times P_1, \overline{in}_1, \overline{wt}_1, F_1 \times \{p\})$, $\mathcal{C}_{i,(p,p')} = (Q_i \times P_1, \overline{in}_{i,(p,p')}, \overline{wt}_i, F_i \times \{p'\})$, and $\mathcal{C}_{n,p} = (Q_n \times P_1, \overline{in}_{n,p}, \overline{wt}_n, F_n \times T_1)$ by

- $\overline{in}_1\left(\left(q^{(1)}, p_1\right)\right) = in_1(q^{(1)}) \cdot in'_1(p_1)$ for every $q^{(1)} \in Q_1, p_1 \in P_1$,
- $\overline{wt}_1\left(\left((q_1^{(1)}, p_1), a, (q_2^{(1)}, p_2)\right)\right) = wt_1\left(\left(q_1^{(1)}, a, q_2^{(1)}\right)\right) \cdot wt'_1\left((p_1, a, p_2)\right)$ for every $q_1^{(1)}, q_2^{(1)} \in Q_1, p_1, p_2 \in P_1, a \in A$, and

for every $2 \leq i \leq n-1$

- $\overline{in}_{i,(p,p')}\left(\left(q^{(i)}, p_1\right)\right) = in_i(q^{(i)})$ if $p_1 = p$, and $\overline{in}_{i,(p,p')}\left(\left(q^{(i)}, p_1\right)\right) = 0$ otherwise, for every $q^{(i)} \in Q_i, p_1 \in P_1$,
- $\overline{wt}_i\left(\left((q_1^{(i)}, p_1), a, (q_2^{(i)}, p_2)\right)\right) = wt_i\left(\left(q_1^{(i)}, a, q_2^{(i)}\right)\right) \cdot wt'_1((p_1, a, p_2))$ for every $q_1^{(i)}, q_2^{(i)} \in Q_i, p_1, p_2 \in P_1, a \in A$, and
- $\overline{in}_{n,p}\left(\left(q^{(n)}, p_1\right)\right) = in_n(q^{(n)})$ if $p_1 = p$, and $\overline{in}_{n,p}\left(\left(q^{(n)}, p_1\right)\right) = 0$ otherwise, for every $q^{(n)} \in Q_n, p_1 \in P_1$,
- $\overline{wt}_n\left(\left((q_1^{(n)}, p_1), a, (q_2^{(n)}, p_2)\right)\right) = wt_n\left(\left(q_1^{(n)}, a, q_2^{(n)}\right)\right) \cdot wt'_1((p_1, a, p_2))$, for every $q_1^{(n)}, q_2^{(n)} \in Q_n, p_1, p_2 \in P_1, a \in A$.

We claim that

$$(\|\mathcal{A}_1\| \cdot \ldots \cdot \|\mathcal{A}_n\|) \odot \|\mathcal{B}_1\| = \sum_{p_1, \ldots, p_{n-1} \in P_1} \left(\|\mathcal{C}_{1,p_1}\| \cdot \left\|\mathcal{C}_{2,(p_1,p_2)}\right\| \cdot \ldots \cdot \left\|\mathcal{C}_{n-1,(p_{n-2},p_{n-1})}\right\| \cdot \left\|\mathcal{C}_{n,p_{n-1}}\right\|\right).$$

Clearly, it suffices to prove that for every $w \in A^*$, the sum

$$\left(\sum_{w=w_1 \ldots w_n} \left(\prod_{1 \leq i \leq n} \left(\sum_{P_{w_i}^{(i)} \in \mathrm{succ}(\mathcal{A}_i)} weight\left(P_{w_i}^{(i)}\right)\right)\right)\right) \left(\sum_{P_w \in \mathrm{succ}(\mathcal{B}_1)} weight\left(P_w\right)\right)$$

[4] Here, we deal with the case $n > 1$. For $n = m = 1$ we consider the product automaton of two simple cfwa which is trivially simple.

equals to

$$\sum_{p_1,\ldots,p_{n-1}\in P_1}\left(\sum_{w=w_1\ldots w_n}\left(\begin{array}{c}\sum\limits_{\overline{P}_{w_1}\in\mathrm{succ}\left(\mathcal{C}_{1,p_1}\right)} weight(\overline{P}_{w_1})\\ \left(\prod\limits_{1\le i\le n-2}\left(\sum\limits_{\overline{P}_{w_i}\in\mathrm{succ}\left(\mathcal{C}_{i,(p_i,p_{i+1})}\right)} weight(\overline{P}_{w_i})\right)\right)\\ \sum\limits_{\overline{P}_{w_{n-1}}\in\mathrm{succ}\left(\mathcal{C}_{n,p_{n-1}}\right)} weight(\overline{P}_{w_{n-1}})\end{array}\right)\right).$$

To this end, let $w = a_0a_1\ldots a_{m-1} \in \mathrm{supp}((\|\mathcal{A}_1\|\cdot\ldots\cdot\|\mathcal{A}_n\|)\odot\|\mathcal{B}_1\|)$ with $a_0, a_1, \ldots, a_{m-1} \in A$. Let us assume that $w_1, \ldots, w_n \in A^*$, with $w = w_1 \ldots w_n$, and $P^{(1)}_{w_1} : \left(q^{(1)}_0, a_0, q^{(1)}_1\right)\left(q^{(1)}_1, a_1, q^{(1)}_2\right)\ldots\left(q^{(1)}_{i_1}, a_{i_1}, q^{(1)}_{i_1+1}\right)$,

$P^{(2)}_{w_2} : \left(q^{(2)}_{i_1+1}, a_{i_1+1}, q^{(2)}_{i_1+2}\right)\left(q^{(2)}_{i_1+2}, a_{i_1+2}, q^{(2)}_{i_1+3}\right)\ldots\left(q^{(2)}_{i_2}, a_{i_2}, q^{(2)}_{i_2+1}\right)$,

$\vdots$

$P^{(n)}_{w_n} : \left(q^{(n)}_{i_{n-1}+1}, a_{i_{n-1}+1}, q^{(n)}_{i_{n-1}+2}\right)\left(q^{(n)}_{i_{n-1}+2}, a_{i_{n-1}+2}, q^{(n)}_{i_{n-1}+3}\right)\ldots\left(q^{(n)}_{m-1}, a_{m-1}, q^{(n)}_m\right)$,

and $P_w : (p_0, a_0, p_1)(p_1, a_1, p_2)\ldots(p_{m-1}, a_{m-1}, p_m)$,

are successful paths of $\mathcal{A}_1, \mathcal{A}_2 \ldots, \mathcal{A}_n, \mathcal{B}_1$ over $w_1, \ldots, w_n, w$ respectively. By definition of $\mathcal{C}_{1,p_{i_1+1}}, \mathcal{C}_{2,(p_{i_1+1},p_{i_2+1})}, \ldots, \mathcal{C}_{n,p_{i_{n-1}+1}}$, we can construct from $P^{(1)}_{w_1}, \ldots, P^{(n)}_{w_n}$ and P_w the paths $\overline{P}_{w_1}, \ldots, \overline{P}_{w_n}$ of $\mathcal{C}_{1,p_{i_1+1}}, \ldots, \mathcal{C}_{n,p_{i_{n-1}+1}}$ over $w_1, \ldots, w_n$ respectively, as follows.

$$\overline{P}_{w_1} : \left(\left(q^{(1)}_0, p_0\right), a_0, \left(q^{(1)}_1, p_1\right)\right)\left(\left(q^{(1)}_1, p_1\right), a_1, \left(q^{(1)}_2, p_2\right)\right)\ldots\left(\left(q^{(1)}_{i_1}, p_{i_1}\right), a_{i_1}, \left(q^{(1)}_{i_1+1}, p_{i_1+1}\right)\right),$$

$$\overline{P}_{w_2} : \left(\left(q^{(2)}_{i_1+1}, p_{i_1+1}\right), a_{i_1+1}, \left(q^{(2)}_{i_1+2}, p_{i_1+2}\right)\right)\left(\left(q^{(2)}_{i_1+2}, p_{i_1+2}\right), a_{i_1+2}, \left(q^{(2)}_{i_1+3}, p_{i_1+3}\right)\right)\ldots$$
$$\left(\left(q^{(2)}_{i_2}, p_{i_2}\right), a_{i_2}, \left(q^{(2)}_{i_2+1}, p_{i_2+1}\right)\right),$$

$\vdots$

$$\overline{P}_{w_n} : \left(\left(q^{(n)}_{i_{n-1}+1}, p_{i_{n-1}+1}\right), a_{i_{n-1}+1}, \left(q^{(n)}_{i_{n-1}+2}, p_{i_{n-1}+2}\right)\right)$$
$$\left(\left(q^{(n)}_{i_{n-1}+2}, p_{i_{n-1}+2}\right), a_{i_{n-1}+2}, \left(q^{(n)}_{i_{n-1}+3}, p_{i_{n-1}+3}\right)\right)\ldots\left(\left(q^{(n)}_{m-1}, p_{m-1}\right), a_{m-1}, \left(q^{(n)}_m, p_m\right)\right).$$

Then, $weight\left(\overline{P}_{w_1}\right) \cdot weight\left(\overline{P}_{w_2}\right) \cdot \ldots \cdot weight\left(\overline{P}_{w_n}\right) = weight\left(P^{(1)}_{w_1}\right) \cdot weight\left(P^{(2)}_{w_2}\right)\cdot\ldots\cdot weight\left(P^{(n)}_{w_n}\right)\cdot weight(P_w)$. Conversely, let $p_{i_1+1}, \ldots, p_{i_{n-1}+1} \in P_1$ such that $w \in \mathrm{supp}\left(\left\|\mathcal{C}_{1,p_{i_1+1}}\right\|\cdot\ldots\cdot\left\|\mathcal{C}_{n,p_{i_{n-1}+1}}\right\|\right)$. Using similar arguments as above, and keeping the previous notations, we get that for every $w_1, \ldots, w_n \in A^*$ with $w = w_1 \ldots w_n$, and successful paths $\overline{P}_{w_1}, \overline{P}_{w_2}, \ldots, \overline{P}_{w_n}$, there exist successful paths $P^{(1)}_{w_1}, P^{(2)}_{w_2}, \ldots, P^{(n)}_{w_n}, P_w$ such that $weight\left(\overline{P}_{w_1}\right)\cdot weight\left(\overline{P}_{w_2}\right)\cdot\ldots\cdot weight\left(\overline{P}_{w_n}\right) = weight\left(P^{(1)}_{w_1}\right)\cdot weight\left(P^{(2)}_{w_2}\right)\cdot\ldots\cdot weight\left(P^{(n)}_{w_n}\right)\cdot weight(P_w)$. Therefore, by standard computations, we get the equality of the two sums and this concludes our claim for $m = 1$.

For the induction step, for simplicity, we prove our claim for $m = 2$. For every $1 \le i \le n$ and $q^{(i)} \in Q_i$, we define the simple cfwa $\mathcal{A}_{i,q^{(i)}} = (Q_i, in_i, wt_i, \{q^{(i)}\})$

and $\mathcal{A}'_{i,q^{(i)}} = (Q_i, in'_i, wt_i, F_i)$ with $in'_i(q) = 1$ if $q = q^{(i)}$, and $in'_i(q) = 0$ otherwise, for every $q \in Q_i$. Then, with similar as above arguments, we can show that $(\|\mathcal{A}_1\| \cdot \ldots \cdot \|\mathcal{A}_n\|) \odot (\|\mathcal{B}_1\| \cdot \|\mathcal{B}_2\|)$ equals to

$$\sum_{\substack{1\leq i\leq n \\ q^{(i)}\in Q_i}} \left(\left(\left(\|\mathcal{A}_1\| \cdot \ldots \cdot \left\|\mathcal{A}_{i,q^{(i)}}\right\| \right) \odot \|\mathcal{B}_1\| \right) \cdot \left(\left(\left\|\mathcal{A}'_{i,q^{(i)}}\right\| \cdot \ldots \cdot \|\mathcal{A}_n\| \right) \odot \|\mathcal{B}_2\| \right) \right).$$

Hence, by induction hypothesis we conclude our claim. □

Below, in our second main result of the present section, we show that every ω-star-free series is an almost simple ω-counter-free series.

Theorem 4. ω-$SF(K, A) \subseteq \omega$-$asCF(K, A)$.

Proof. By Definition 8 and Theorem 3, it suffices to show that the class ω-$asCF(A, K)$ is closed under sum, Hadamard product, complement, ω-iteration restricted to letter-step series, and if $s_1 \in asCF(K, A)$ and $s_2 \in \omega$-$asCF(K, A)$, then $s_1 \cdot s_2 \in \omega$-$asCF(K, A)$. The last property as well as closure under sum are easily obtained by Definition 11. For the closure under complement, we use a similar argument as in the corresponding part of the proof of Theorem 3. Furthermore, the automaton $\mathcal{A}$ accepting r^ω for a letter-step series r, in the proof of Proposition 7, is trivially simple, hence the class ω-$asCF(K, A)$ is closed under ω-iteration restricted to letter-step series. Again, the most complicated case is to prove the closure under Hadamard product, i.e., to prove that if $\mathcal{A}_i = (Q_i, in_i, wt_i, F_i), \mathcal{B}_j = (P_j, in'_j, wt'_j, T_j)$, for $1 \leq i \leq n-1, 1 \leq j \leq m-1$, are simple cfwa and $\mathcal{A}_n = (Q_n, in_n, wt_n, F_n), \mathcal{B}_m = (P_m, in'_m, wt'_m, T_m)$ are simple cfwBa over A and K, then the ω-counter-free series $(\|\mathcal{A}_1\| \cdot \ldots \cdot \|\mathcal{A}_n\|) \odot (\|\mathcal{B}_1\| \cdot \ldots \cdot \|\mathcal{B}_m\|)$ is almost simple. We state our proof by induction on m, hence, let firstly $m = 1$, i.e., $\mathcal{B}_1 = (P_1, in'_1, wt'_1, T_1)$ be a simple cfwBa (again we assume $n > 1$, otherwise if $n = m = 1$ we get our result by Proposition 8). We keep the notations of Theorem 3 and consider the simple cfwa $\mathcal{C}_{1,p}$, and $\mathcal{C}_{i,(p,p')}$ for every $2 \leq i \leq n-1$. Furthremore, for every $p \in P_1$ we define the wBa $\mathcal{C}_{n,p} = \left(Q_n \times P_1 \times \{0,1,2\}, \overline{in}_{n,p}, \overline{wt}_n, Q_n \times P_1 \times \{2\}\right)$ with the initial distribution $\overline{in}_{n,p}$ given for every $q^{(n)} \in Q_n, p_1 \in P_1, x \in \{0,1,2\}$ by

$$\overline{in}_{n,p}(q^{(n)}, p_1, x) = \begin{cases} in_n(q^{(n)}) & \text{if } p_1 = p, x = 0 \\ 0 & \text{otherwise} \end{cases},$$

and the weight assignment mapping $\overline{wt}_n$ defined for every $q_1^{(n)}, q_2^{(n)} \in Q_n, p_1, p_2 \in P_1, a \in A, x, y \in \{0,1,2\}$ as follows.

$$\overline{wt}_n\left(\left(\left(q_1^{(n)}, p_1, x\right), a, \left(q_2^{(n)}, p_2, y\right)\right)\right) = wt_n\left(\left(q_1^{(n)}, a, q_2^{(n)}\right)\right) \cdot wt'_1\left((p_1, a, p_2)\right)$$

if ($x = y = 0$ or $q_2^{(n)} \in F_n, x = 0, y = 1$ or $p_2 \notin T_1, x = y = 1$ or $p_2 \in T_1, x = 1, y = 2$ or $x = 2, y = 0$), and

$\overline{wt}_n\left(\left(\left(q_1^{(n)},p_1,x\right),a,\left(q_2^{(n)},p_2,y\right)\right)\right)=0$ otherwise[5].

We note that, since $\mathcal{A}_n$ (resp. $\mathcal{B}_1,\mathcal{C}_{n,p}$)[6] is simple, for every $w\in A^\omega$, all the successful paths of $\mathcal{A}_n$ (resp. $\mathcal{B}_1,\mathcal{C}_{n,p}$) over w with weight $\neq 0$ have the same weight. Again we will show that

$$(\|\mathcal{A}_1\|\cdot\ldots\cdot\|\mathcal{A}_n\|)\odot\|\mathcal{B}_1\| = \sum_{p_1,\ldots,p_{n-1}\in P_1}\left(\|\mathcal{C}_{1,p_1}\|\cdot\left\|\mathcal{C}_{2,(p_1,p_2)}\right\|\cdot\ldots\cdot\left\|\mathcal{C}_{n-1,(p_{n-2},p_{n-1})}\right\|\cdot\left\|\mathcal{C}_{n,p_{n-1}}\right\|\right)$$

by proving that for every $w\in A^\omega$ the sum

$$\left(\sum_{\substack{w=w_1\ldots w_n\\ w_1,\ldots,w_{n-1}\in A^*,w_nA^\omega}}\left(\prod_{1\le i\le n}\left(\sum_{P_{w_i}^{(i)}\in\mathrm{succ}(\mathcal{A}_i)}weight\left(P_{w_i}^{(i)}\right)\right)\right)\right)\left(\sum_{P_w\in\mathrm{succ}(\mathcal{B}_1)}weight\left(P_w\right)\right)$$

equals to

$$\sum_{p_1,\ldots,p_{n-1}\in P_1}\left(\sum_{\substack{w=w_1\ldots w_n\\ w_1,\ldots,w_{n-1}\in A^*,w_nA^\omega}}\left(\begin{array}{c}\sum_{\overline{P}_{w_1}\in\mathrm{succ}\left(\mathcal{C}_{1,p_1}\right)}weight(\overline{P}_{w_1})\\ \left(\prod_{1\le i\le n-2}\left(\sum_{\overline{P}_{w_i}\in\mathrm{succ}\left(\mathcal{C}_{i,(p_i,p_{i+1})}\right)}weight(\overline{P}_{w_i})\right)\right)\\ \sum_{\overline{P}_{w_{n-1}}\in\mathrm{succ}\left(\mathcal{C}_{n,p_{n-1}}\right)}weight(\overline{P}_{w_{n-1}})\end{array}\right)\right).$$

To this end, let $w=a_0a_1\ldots\in\mathrm{supp}((\|\mathcal{A}_1\|\cdot\ldots\cdot\|\mathcal{A}_n\|)\odot\|\mathcal{B}_1\|)$ with $a_0,a_1,\ldots\in A$. We fix an analysis $w=w_1\ldots w_{n-1}w_n$ $(w_1,\ldots,w_{n-1}\in A^*,w_n\in A^\omega)$, and we let $P_{w_i}^{(i)}$, for every $1\le i\le n$, to be a successful path of $\mathcal{A}_i$ over w_i, and P_w a successful path of $\mathcal{B}_1$ over w. We keep the notations of the proof of Theorem 3, for the paths $P_{w_i}^{(i)}$ $(1\le i\le n-1)$, and we set

$P_{w_n}^{(n)}:\left(q_{i_{n-1}+1}^{(n)},a_{i_{n-1}+1},q_{i_{n-1}+2}^{(n)}\right)\left(q_{i_{n-1}+2}^{(n)},a_{i_{n-1}+2},q_{i_{n-1}+3}^{(n)}\right)\ldots$, and

$P_w:(p_0,a_0,p_1)\,(p_1,a_1,p_2)\ldots$.

We consider the paths $\overline{P}_{w_i}$ $(1\le i\le n-1)$ as in the proof of Theorem 3, and let

$$\overline{P}_{w_n}:\left(\left(q_{i_{n-1}+1}^{(n)},p_{i_{n-1}+1},x_1\right),a_{i_{n-1}+1},\left(q_{i_{n-1}+2}^{(n)},p_{i_{n-1}+2},x_2\right)\right)\left(\left(q_{i_{n-1}+2}^{(n)},p_{i_{n-1}+2},x_2\right),a_{i_{n-1}+2},\left(q_{i_{n-1}+3}^{(n)},p_{i_{n-1}+3},x_3\right)\right)\ldots$$

where for every $j\ge 1$ the choice of x_j is done as follows. We have ($x_j=0$ and (nondeterministically) $x_{j+1}=1$ if $q_{i_{n-1}+j+1}^{(n)}\in F_n$) or ($x_j=1$ and $x_{j+1}=1$ if $p_{i_{n-1}+j+1}\notin T_1$) or ($x_j=1$ and $x_{j+1}=2$ if $p_{i_{n-1}+j+1}\in T_1$) or ($x_j=2$ and $x_{j+1}=0$). Clearly, by definition of $\mathcal{C}_{1,p_{i_1+1}},\ldots,\mathcal{C}_{n,p_{i_{n-1}+1}}$, the above paths are successful,

[5] For every $p\in P_1$, $\|\mathcal{C}_{n,p}\|=\|\mathcal{A}_n\|\odot\|\mathcal{B}_p\|$, where $\mathcal{B}_p$ is the simple cfwBa derived by $\mathcal{B}_1$ by replacing the initial distribution, with the one assigning the value 1 to p and 0 to any other state. Since, $\mathcal{A}_n,\mathcal{B}_p$ are simple, we conclude by Proposition 8 that $\|\mathcal{C}_{n,p}\|$ is simple.

[6] Abusing the definition, we call the wBa $C_{n,p}$ simple though it is not counter-free.

and we get that $weight\left(P^{(1)}_{w_1}\right)\cdot\ldots\cdot weight\left(P^{(n)}_{w_n}\right)\cdot weight(P_w)=weight\left(\overline{P}_{w_1}\right)\cdot\ldots\cdot weight\left(\overline{P}_{w_n}\right)$. Conversely, for fixed $p_{i_1+1},\ldots,p_{i_{n-1}+1}\in P_1$ such that $w\in\operatorname{supp}\left(\left\|\mathcal{C}_{1,p_{i_1+1}}\right\|\cdot\ldots\cdot\left\|\mathcal{C}_{n,p_{i_{n-1}+1}}\right\|\right)$, and successful paths $\overline{P}_{w_1},\overline{P}_{w_2},\ldots,\overline{P}_{w_n}$, we can determine the successful paths $P^{(1)}_{w_1},P^{(2)}_{w_2},\ldots,P^{(n)}_{w_n},P_w$ such that $weight\left(\overline{P}_{w_1}\right)\cdot\ldots\cdot weight\left(\overline{P}_{w_n}\right)=weight\left(P^{(1)}_{w_1}\right)\cdot\ldots\cdot weight\left(P^{(n)}_{w_n}\right)\cdot weight(P_w)$. By Lemma 1 we conclude the required equality.

Next, for the induction step, again for simplicity, we state our claim for $m=2$. Now, we consider, for every $1\leq i\leq n-1$ and $q^{(i)}\in Q_i$, the simple cfwa $\mathcal{A}_{i,q^{(i)}}=(Q_i,in_i,wt_i,\{q^{(i)}\})$ and $\mathcal{A}'_{i,q^{(i)}}=(Q_i,in'_i,wt_i,F_i)$ with $in'_i(q)=1$ if $q=q^{(i)}$, and $in'_i(q)=0$ otherwise. Moreover, for every $q^{(n)}\in Q_n$ we consider the simple cfwa $\mathcal{A}_{n,q^{(n)}}=(Q_n,in_n,wt_n,\{q^{(n)}\})$ and the simple cfwBa $\mathcal{A}'_{n,q^{(n)}}=(Q_n,in'_n,wt_n,F_n)$ with $in'_n(q)=1$ if $q=q^{(n)}$, and $in'_n(q)=0$ otherwise. Then, we get that the Hadamard product $(\|\mathcal{A}_1\|\cdot\ldots\cdot\|\mathcal{A}_n\|)\odot(\|\mathcal{B}_1\|\cdot\|\mathcal{B}_2\|)$ equals to

$$\sum_{\substack{1\leq i\leq n\\ q^{(i)}\in Q_i}}\left(\left(\left(\|\mathcal{A}_1\|\cdot\ldots\cdot\left\|\mathcal{A}_{i,q^{(i)}}\right\|\right)\odot\|\mathcal{B}_1\|\right)\cdot\left(\left(\left\|\mathcal{A}'_{i,q^{(i)}}\right\|\cdot\ldots\cdot\|\mathcal{A}_n\|\right)\odot\|\mathcal{B}_2\|\right)\right)$$

and, by induction hypothesis and Theorem 3, we are done. □

10 Closing the cycle

In this section, we prove that the class of almost simple ω-counter-free series is included in the class $\omega\text{-}ULTL\,(K,A)$ and we conclude the main result of our paper. For this, we shall need some preliminary matter on our weighted LTL.

For every $\varphi\in LTL\,(K,A)$ and $n\geq 0$ we denote by $\bigcirc^n\varphi$ the n-th repetitive application of the $\bigcirc$ operator on φ, i.e., $\bigcirc^n\varphi:=\underbrace{\bigcirc(\bigcirc\ldots(\bigcirc}_{n\text{ times}}\varphi)\ldots)$, and hence $\bigcirc^0\varphi=\varphi$. Then, for every $w\in A^\omega$ we have $(\|\bigcirc^n\varphi\|,w)=(\|\varphi\|,w_{\geq n})$. The *external next depth* $exnd\,(\varphi)$ of a formula $\varphi\in LTL(K,A)$ is defined as follows. If $\varphi=\bigcirc\psi$, then $exnd\,(\varphi)=exnd\,(\psi)+1$. In any other case, we let $exnd\,(\varphi)=0$. For instance $exnd\,(\bigcirc(\bigcirc(\square(\bigcirc(p_a\wedge 2)))))=2$, and if $\varphi\in LTL\,(K,A)$ with $exnd\,(\varphi)=0$, then $exnd\,(\bigcirc^n\varphi)=n$ for every $n\geq 0$. The following lemma is concluded in a straightforward way by the definition of $stLTL\,(K,A)$ formulas.

Lemma 25. *Let $\psi\in stLTL\,(K,A)$. Then $exnd\,(\psi)=0$.*

For every $n\geq 0$, we denote by $stLTL\,(\bigcirc,n,\wedge)$ the class of all $LTL\,(K,A)$ formulas of the form $\bigwedge_{0\leq j\leq m}\bigcirc^{k_j}\psi_j$ with $m\geq 0$, $\max_{0\leq j\leq m}(k_j)=n$, and $\psi_j\in stLTL\,(K,A)$ for every $0\leq j\leq m$. We let $stLTL\,(\bigcirc,\wedge)=\bigcup_{n\geq 0}stLTL\,(\bigcirc,n,\wedge)$. Furthermore, for every $m\geq 0$, we let U_m to be the set of all $(m+1)$-tuples of the form $((\varphi_0,k_0),(\xi_1,\varphi_1,k_1),\ldots,(\xi_m,\varphi_m,k_m))$ where $\varphi_i\in stLTL\,(\bigcirc,k_i,\wedge)$ and $\xi_j\in abLTL\,(K,A)$ for every $0\leq i\leq m$ and $1\leq j\leq m$.

Definition 12. *Let* $T = ((\varphi_0, k_0), (\xi_1, \varphi_1, k_1), \ldots, (\xi_m, \varphi_m, k_m)) \in U_m$. *For every* $w \in A^\omega$ *and* $j \geq 0$ *we define the value* $\langle T,w,j\rangle \in K$ *as follows. If* $j \leq k_0 + \ldots + k_m$, *we set* $\langle T,w,j\rangle = 0$. *Otherwise, for every* $i_1, i_2, \ldots, i_m \in \mathbb{N}$ *and* $0 \leq l \leq m$ *we define the sum* $S_l = k_0 + i_1 + k_1 + \ldots + i_l + k_l$ *with the restriction that* $S_m = j - 1$. *Then, we let*

$$\langle T,w,j\rangle = \sum_{\substack{i_1,i_2,\ldots,i_m \in \mathbb{N}\\ S_m = j-1}} \left((\|\varphi_0\|, w) \cdot \prod_{1\leq l\leq m} \left(\prod_{0\leq j_l < i_l} \left(\|\xi_l\|, w_{\geq S_{l-1}+j_l}\right) \cdot \left(\|\varphi_l\|, w_{\geq S_{l-1}+i_l}\right)\right)\right).$$

Note that in case $m = 0$, the restriction $S_0 = j - 1$, i.e., $k_0 = j - 1$ implies that $\langle T,w,j\rangle = 0$ for every $j > k_0 + 1$. Therefore, if $m = 0$, then $\langle T,w,j\rangle = 0$ for every $j \neq k_0 + 1$, and $\langle T,w,k_0+1\rangle = (\|\varphi_0\|, w)$.

Composition algorithm. Let $T_1 = ((\varphi_0, k_0), (\xi_1, \varphi_1, k_1), \ldots, (\xi_m, \varphi_m, k_m)) \in U_m$ and $T_2 = ((\psi_0, l_0), (\theta_1, \psi_1, l_1), \ldots, (\theta_n, \psi_n, l_n)) \in U_n$ with $\psi_0 = \bigwedge_{0\leq j\leq h} \bigcirc^{p_j} \varrho_j$. We consider the formula $\varrho = \varphi_m \wedge \left(\bigwedge_{0\leq j\leq h} \bigcirc^{k_m+p_j+1} \varrho_j\right)$ in $stLTL(\bigcirc, k_m + l_0 + 1, \wedge)$. Then, if $m = 0$ we let

$$T = ((\varrho, k_0 + l_0 + 1), (\theta_1, \psi_1, l_1), \ldots, (\theta_n, \psi_n, l_n)),$$

otherwise we let

$$T = ((\varphi_0, k_0), (\xi_1, \varphi_1, k_1), \ldots, (\xi_m, \varrho, k_m + l_0 + 1), (\theta_1, \psi_1, l_1), \ldots, (\theta_n, \psi_n, l_n)).$$

Clearly $T \in U_{m+n}$, and we claim that

$$\langle T, w, j\rangle = \sum_{0\leq i\leq j} (\langle T_1, w, i\rangle \cdot \langle T_2, w_{\geq i}, j - i\rangle) \tag{1}$$

for every $w \in A^\omega$, $j \geq 0$. Assume firstly that $m = n = 0$. If $j \neq k_0 + l_0 + 2$, then both sides of the above relation equal to 0. If $j = k_0 + l_0 + 2$, then $\langle T, w, j\rangle = (\|\varrho\|, w) = (\|\varphi_0\|, w) \cdot (\|\psi_0\|, w_{\geq k_0+1}) = \langle T_1, w, k_0 + 1\rangle \cdot \langle T_2, w_{\geq k_0+1}, j - (k_0 + 1)\rangle = \sum_{0\leq i\leq j} (\langle T_1, w, i\rangle \cdot \langle T_2, w_{\geq i}, j - i\rangle)$.

Next, assume that $n \neq 0$ or $m \neq 0$. Then, if $j > k_0 + k_1 + \ldots + k_m + 1 + l_0 + \ldots + l_n$, we assign to $\langle T, w, j\rangle$ the sum of the products of the form

$$\left((\|\varphi_0\|, w) \cdot \prod_{1\leq l\leq m} \left(\begin{array}{l} \prod_{0\leq j_l<i_l} \left(\|\xi_l\|, w_{\geq S_{l-1}+j_l}\right) \\ \cdot \left(\|\varphi_l\|, w_{\geq S_{l-1}+i_l}\right) \end{array} \right)\right)$$
$$\cdot \left((\|\psi_0\|, w_{\geq S_m+1}) \cdot \prod_{1\leq h\leq n} \left(\begin{array}{l} \prod_{0\leq j_h<i'_h} \left(\|\theta_h\|, w_{\geq S_m+1+S'_{h-1}+j_h}\right) \\ \cdot \left(\|\psi_h\|, w_{\geq S_m+1+S'_{h-1}+i'_h}\right) \end{array} \right)\right)$$

where the sum is taken over all $i_1, \ldots, i_m, i'_1, \ldots, i'_n \in \mathbb{N}$ with $k_0 + i_1 + k_1 + \ldots + i_m + k_m + 1 + l_0 + i'_1 + l_1 + \ldots + i'_n + l_n = j - 1$.

On the other side, for every $0 \leq i \leq j$, we get the value $\langle T_1, w, i\rangle$ by summing up the products

$$(\|\varphi_0\|, w) \cdot \prod_{1 \leq l \leq m} \left(\prod_{0 \leq j_l < i_l} \left(\|\xi_l\|, w_{\geq S_{l-1}+j_l}\right) \cdot \left(\|\varphi_l\|, w_{\geq S_{l-1}+i_l}\right) \right) \tag{2}$$

for every $i_1, \ldots, i_m \in \mathbb{N}$ with $S_m = k_0 + i_1 + k_1 + \ldots + i_m + k_m = i - 1$. Similarly, we obtain the value $\langle T_2, w_{\geq i}, j - i\rangle$ as the sum of the products

$$(\|\psi_0\|, w_{\geq i}) \cdot \prod_{1 \leq h \leq n} \left(\prod_{0 \leq j_h < i'_h} \left(\|\theta_h\|, w_{\geq i + S'_{h-1}+j_h}\right) \cdot \left(\|\psi_h\|, w_{\geq i+S'_{h-1}+i'_h}\right) \right) \tag{3}$$

for every $i'_1, \ldots, i'_n \in \mathbb{N}$ with $S'_n = l_0 + i'_1 + l_1 + \ldots + i'_n + l_n = (j - i) - 1$. By a straightforward calculation in the right-hand side of (1) we conclude our claim. Finally, assume that $j \leq k_0 + k_1 + \ldots + k_m + 1 + l_0 + \ldots + l_n$. Then, $\langle T, w, j\rangle = 0$, and for every $0 \leq i \leq j$ at least one of the following is true: $i \leq k_0 + \ldots + k_m$ which implies that $\langle T_1, w, i\rangle = 0$, or $j - i \leq l_0 + \ldots + l_n$, which implies that $\langle T_2, w_{\geq i}, j - i\rangle = 0$.

In the sequel, we recall an alternative definition for star-free languages which does not involve the closure under complementation. For this, we shall need the notion of bounded synchronization delay. More precisely, let $k \geq 0$ be an integer. A prefix-free set $L \subseteq A^+$ has *bounded synchronization delay* if $uvw \in L^*$ implies $uv, w \in L^*$ for every $u, w \in A^*$ and $v \in L^k$. The least integer $k \geq 0$ satisfying the aforementioned property is called the *synchronization delay of* L.

Lemma 26. *[27] A prefix-free set of delay* 0 *is also of delay* 1.

It is well-known (cf. for instance [27, Thm. 6.3]) that the class of star-free languages over A is the smallest class of languages over A containing $\emptyset$ and $\{a\}$ for every $a \in A$, and which is closed under union, concatenation and star operation restricted to prefix-free sets with bounded synchronization delay.

For every $L, F \subseteq A^\omega$ we define the infinitary language (cf. [27]) $LUF = \{w \in A^\omega \mid w = uv \text{ where } u \in A^*,\ v \in F \text{ and } u'v \in L \text{ for each nonempty suffix } u' \text{ of } u\}$. It should be clear that $\operatorname{supp}(1_L U 1_F) = LUF$, where the operation U among two series $r, s \in K\langle\langle A^\omega\rangle\rangle$, is defined for every $w \in A^\omega$, by

$$(rUs, w) = \sum_{i \geq 0} \left(\prod_{0 \leq j < i} (r, w_{\geq j}) \cdot (s, w_{\geq i}) \right).$$

The two subsequent lemmas are proved in [27]. Here we present a slight modification of them and for completeness shake we state their proofs.

Lemma 27. *Let $L \subseteq A^+$ be a prefix-free set with bounded synchronization delay $k \geq 1$. Let $u \in A^*$, $v \in L^{2k}$, and $w \in Y \subseteq A^\omega$ such that*

(i) $uvw \in L^k A^\omega$, *and*

(ii) $u'vw \in L^{k+1}A^\omega \cup (A^\omega \backslash L^k A^\omega)$ *for every suffix u' of u.*

Then $uv \in L^+$.

Proof. We follow the inductive proof of Lm. 6.11 (pg. 371) in [27]. The induction is on the length of u. We let first $|u| = 0$, then $uv = v \in L^{2k}$ and since $\varepsilon \notin L$, we have $L^{2k} \subseteq L^+$. Next, assume that our claim holds for $|u| \leq n-1$ and let $|u| = n$. Condition (ii) holds for $u' = u$, and hence we get $uvw = u_1u_2 \ldots u_{k+1}r$ with $u_1, u_2, \ldots, u_{k+1} \in L$ and $r \in A^\omega$. We point out the following cases.

- The word u_1 is a prefix of u. Then, $u = u_1 q$ with $q \in A^*$, and we get $uvw = u_1u_2 \ldots u_{k+1}r \Rightarrow u_1qvw = u_1u_2 \ldots u_{k+1}r \Rightarrow qvw = u_2 \ldots u_{k+1}r$. Thus, we can apply the induction hypothesis to (q, v, w). We conclude that $qv \in L^+$, and thus $uv = u_1qv \in L^+$.
- The word uv is a prefix of $u_1u_2 \ldots u_{k+1}$. Then, $u_1u_2 \ldots u_{k+1} = uvr$ with $r \in A^*$. Since L has delay k, and $v \neq \varepsilon$ we obtain that $uv \in L^+$.
- We have $|u| < |u_1|$ and $|u_1u_2 \ldots u_{k+1}| < |uv|$. Then, $u_1 = up$ and $uv = u_1u_2 \ldots u_{k+1}q = upu_2 \ldots u_{k+1}q$ for some $p, q \in A^*$, which implies that $v = pu_2 \ldots u_{k+1}q$. Since L has delay k, we have $q \in L^*$. Thus, $uv = u_1u_2 \ldots u_{k+1}q$ is in L^+, as wanted.

□

Lemma 28. *Let $L \subseteq A^+$ be a prefix-free set with bounded synchronization delay $k \geq 1$ and $Y \subseteq A^\omega$. Then*

$$\left(L^+\right) Y = LY \cup \ldots \cup L^{2k-1}Y \cup R$$

with $R = L^k A^\omega \cap \left(\left(L^{k+1}A^\omega \cup \left(A^\omega \setminus L^k A^\omega\right)\right) U L^{2k} Y\right)$.

Proof. Again we follow the proof of Lm. 6.12 (pg. 372) in [27]. Let $Z = LY \cup \ldots \cup L^{2k-1}Y \cup R$. First we prove that $Z \subseteq (L^+) Y$. Clearly, it suffices to show that $R \subseteq (L^+) Y$. Let $z \in R$. Then $z \in L^k A^\omega$ and $z = uvw$ with $u \in A^*$, $v \in L^{2k}$ and $w \in Y$ with $u'vw \in L^{k+1}A^\omega \cup (A^\omega \backslash L^k A^\omega)$ for each nonempty suffix u' of u. Clearly, for $u' = \varepsilon$ it holds $vw \in L^{2k}A^\omega \subseteq L^{k+1}A^\omega \cup (A^\omega \backslash L^k A^\omega)$. By the previous lemma we get $uv \in L^+$, which implies that $uvw \in (L^+) Y$.

We show now the opposite inclusion. Let $z \in L^n Y$ for some $n > 0$. If $n < 2k$, then $z \in Z$. Let now $z = uvw$ with $u \in L^*$, $v \in L^{2k}$ and $w \in Y$. Clearly $z \in L^k A^\omega$. Hence, it remains to prove that $u'vw \in L^{k+1}A^\omega \cup (A^\omega \backslash L^k A^\omega)$ for each nonempty suffix u' of u. Equivalently, it suffices to prove that $u'vw \in L^k A^\omega$ implies $u'vw \in L^{k+1}A^\omega$. Suppose that $u'vw \in L^k A^\omega$. Then $u'vw = xq$ with $x \in L^k$ $(1 \leq i \leq k)$ and $q \in A^\omega$. We point out the following two cases.

- x is a proper prefix of $u'v$. Let $u = pu'$, $p \in A^*$. Since $z = pu'vw = pxq$, there is a word $s \in A^+$ such that $pxs = pu'v = uv \in L^*$. Since $x \in L^k$, we have $s \in L^+$. More precisely, since $s \neq \varepsilon$, it holds $s \in L^+$, i.e., $u'v = xs$ is in $L^{k+1}A^*$, and $u'vw$ is in $L^{k+1}A^\omega$.

- $u'v$ is a prefix of x. Then $x = u'vs$ for some $s \in A^*$. Since $v \in L^{2k}$ there exist $v_1, v_2 \in L^k$ with $v = v_1v_2$. We have $x \in L^k$ and $v_1 \in L^k$, which implies that $u'v_1 \in L^+$. Hence $u'v_1v_2w \in L^{k+1}A^\omega$, and we are done.

□

Due to the idempotency of K, the subsequent result is a straightforward conclusion from the last lemma above.

Lemma 29. *Let $L \subseteq A^+$ be a prefix-free set with bounded synchronization delay $k \geq 1$ and $Y \subseteq A^\omega$. Then*

$$1_{L^+} \cdot 1_Y = (1_L \cdot 1_Y) + \ldots + (1_{L^{2k-1}} \cdot 1_Y) + r$$

with $r = 1_{L^kA^\omega} \odot \left(1_{L^{k+1}A^\omega \cup (A^\omega \setminus L^kA^\omega)} U \left(1_{L^{2k}} \cdot 1_Y\right)\right)$.

Lemma 30. *Let $L \subseteq A^+$ be a star-free language. Then, there exists an integer $n > 0$ and $T_i \in U_{m_i}$ $(m_i \geq 0)$ for every $1 \leq i \leq n$, such that for every $w \in A^\omega$ and $j \geq 0$ we have $(1_L, w_{<j}) = \sum\limits_{1 \leq i \leq n} \langle T_i, w, j\rangle$.*

Proof. We state the proof by induction on the structure of L. For the empty set the tuple $T = (0, 0) \in U_0$ satisfies our claim. Let $L = \{a\}$ for $a \in A$. We consider the tuple $T = (p_a, 0) \in U_0$ [7]. Then $S_0 = 0$ and since $S_0 = j - 1$ we get that $\langle T, w, j\rangle = 0$ for $j \neq 1$. Moreover, $\langle T, w, 1\rangle = 1$ if $w(0) = a$, and $\langle T, w, 1\rangle = 0$ otherwise. Therefore $\langle T, w, j\rangle = (1_a, w_{<j})$ for every $w \in A^\omega, j \geq 0$.

Next, assume that the induction hypothesis holds for the star-free languages $L_1, L_2 \subseteq A^+$. Then, there exist $n, m, l_i, h_k \in \mathbb{N}$, and $T_i \in U_{l_i}, T'_k \in U_{h_k}$, $(1 \leq i \leq n, 1 \leq k \leq m)$ such that for every $w \in A^\omega, j \geq 0$ we have $(1_{L_1}, w_{<j}) = \sum\limits_{1 \leq i \leq n} \langle T_i, w, j\rangle$ and $(1_{L_2}, w_{<j}) = \sum\limits_{1 \leq k \leq m} \langle T'_k, w, j\rangle$. Firstly, let $L = L_1 \cup L_2$. Then $(1_L, w_{<j}) = (1_{L_1} + 1_{L_2}, w_{<j}) = \sum\limits_{1 \leq i \leq n} \langle T_i, w, j\rangle + \sum\limits_{1 \leq k \leq m} \langle T'_k, w, j\rangle$, as wanted.

Next, let $L = L_1L_2$. Then $1_{L_1L_2} = 1_{L_1} \cdot 1_{L_2}$. For every $1 \leq i \leq n$ and $1 \leq k \leq m$ we derive from T_i, T'_k the tuple $T_{i,k} \in U_{l_i+h_k}$ by applying the *Composition*

[7] In fact we transform p_a to the equivalent $stLTL\,(\bigcirc, 0, \wedge)$ formula $1 \wedge p_a$.

algorithm. Then, we get

$$
\begin{aligned}
(1_{L_1} \cdot 1_{L_2}, w_{<j}) &= \sum_{0 \le p \le j} \Big((1_{L_1}, w_{<p}) \cdot \Big(1_{L_2}, (w_{\ge p})_{<j-p} \Big) \Big) \\
&= \sum_{0 \le p \le j} \left(\sum_{1 \le i \le n} \langle T_i, w, p \rangle \cdot \sum_{1 \le k \le m} \langle T'_k, w_{\ge p}, j-p \rangle \right) \\
&= \sum_{0 \le p \le j} \left(\sum_{1 \le i \le n, 1 \le k \le m} \left(\langle T_i, w, p \rangle \cdot \langle T'_k, w_{\ge p}, j-p \rangle \right) \right) \\
&= \sum_{1 \le i \le n, 1 \le k \le m} \left(\sum_{0 \le p \le j} \left(\langle T_i, w, p \rangle \cdot \langle T'_k, w_{\ge p}, j-p \rangle \right) \right) \\
&= \sum_{1 \le i \le n, 1 \le k \le m} \langle T_{i,k}, w, j \rangle
\end{aligned}
$$

for every $w \in A^\omega, j \ge 0$.

Finally, let L be a star-free prefix-free set with bounded synchronization delay $k \ge 0$ satisfying the induction hypothesis. By Lemma 26, it suffices to consider the case $k \ge 1$. We will prove our claim for L^+. By Lemma 29, for $Y = A^\omega$, we get

$$
\begin{aligned}
1_{L^+} \cdot 1_{A^\omega} = &(1_L \cdot 1_{A^\omega}) + \ldots + (1_{L^{2k-1}} \cdot 1_{A^\omega}) + \\
&\left(1_{L^k A^\omega} \odot \left(1_{L^{k+1}A^\omega \cup (A^\omega \setminus L^k A^\omega)} U \left(1_{L^{2k}} \cdot 1_{A^\omega}\right)\right)\right).
\end{aligned}
$$

We denote $2k$ simply by p. By what we have shown above, the induction hypothesis, and same arguments with the ones used in the previous inductive step, we can prove that for every $1 \le h \le p$ there exist an $n_h \in \mathbb{N}$, so that the following hold. For every $1 \le i \le n_h$ there exist an $m_i \ge 0$ and a $T_{h,i} \in U_{m_i}$ with $(1_{L^h}, w_{<j}) = \sum_{1 \le i \le n_h} \langle T_{h,i}, w, j \rangle$, for every $w \in A^\omega, j \ge 0$.

Let $\varphi', \widetilde{\varphi} \in bLTL\,(K, A)$ with semantics $1_{L^k A^\omega}$, $1_{L^{k+1}A^\omega \cup (A^\omega \setminus L^k A^\omega)}$, respectively. We set $\overline{\varphi} = \varphi'$ if $\varphi' \in stLTL\,(\bigcirc, 0, \wedge)$ and $\overline{\varphi} = 1 \wedge \varphi'$, otherwise. Clearly, φ' and $1 \wedge \varphi'$ are equivalent and $1 \wedge \varphi' \in stLTL\,(\bigcirc, 0, \wedge)$. We fix an $1 \le i \le n_p$, and we denote for simplicity $T_{p,i}, U_{m_i}$ (where $T_{p,i} \in U_{m_i}$) with T, U_m, respectively. Let

$$T = ((\psi_0, l_0), (\varphi_1, \psi_1, l_1), \ldots, (\varphi_m, \psi_m, l_m))$$

and define the tuple $T' \in U_{m+1}$ by

$$T' = ((\overline{\varphi}, 0), (\widetilde{\varphi}, \psi_0, l_0), (\varphi_1, \psi_1, l_1), \ldots, (\varphi_m, \psi_m, l_m)).$$

Then, for every $w \in A^\omega, j > l_0 + \ldots + l_m$ we have

$$\langle T', w, j \rangle = \sum_{0 \le q < j - (l_0 + \ldots + l_m)} \left((\|\overline{\varphi}\|, w) \cdot \left(\prod_{0 \le h < q} (\|\widetilde{\varphi}\|, w_{\ge h}) \right) \cdot \langle T, w_{\ge q}, j - q \rangle \right) \tag{4}$$

and $\langle T', w, j\rangle = 0$ for every $j \leq l_0 + \ldots + l_m$. We repeat the same procedure for every $1 \leq i \leq n_p$ and we get the corresponding $(m_i + 1)$-tuple $T'_{p,i}$.

Now, we show that for every $w \in A^\omega, j \geq 0$ we have

$$(1_{L^+}, w_{<j}) = \sum_{1\leq h\leq p-1} \left(\sum_{1\leq i\leq n_h} \langle T_{h,i}, w, j\rangle \right) + \sum_{1\leq i\leq n_p} \left\langle T'_{p,i}, w, j\right\rangle .$$

To this end, let $w_{<j} \in L^+$, hence either $w_{<j} \in \bigcup_{1\leq h\leq p-1} L^h$ or $w_{<j} \in \bigcup_{h\geq p} L^h$. In the first case $\sum_{1\leq h\leq p-1} (1_{L^h}, w_{<j}) = 1$ and so $\sum_{1\leq h\leq p-1} \left(\sum_{1\leq i\leq n_h} \langle T_{h,i}, w, j\rangle \right) = 1$. In the latter case, $\exists u \in L^*, v \in L^p$ such that $w_{<j} = uv$. Since $v = \left(w_{\geq|u|}\right)_{<|v|}$ and $(1_{L^p}, v) = 1$, by induction hypothesis, we get that $\sum_{1\leq i\leq n_p} \left\langle T_{p,i}, w_{\geq|u|}, |v|\right\rangle = \sum_{1\leq i\leq n_p} \left\langle T_{p,i}, w_{\geq|u|}, j - |u|\right\rangle = 1$. Then, by the proof of Lemma 28, we get that for every suffix u' of u we have $u'vw_{\geq j} \in L^{k+1}A^\omega \cup \left(A^\omega \setminus L^k A^\omega\right)$. Hence, $(\|\overline{\varphi}\|, w) \cdot \left(\prod_{0\leq h<|u|} (\|\widetilde{\varphi}\|, w_{\geq h}) \right) \cdot \left\langle T_{p,i}, w_{\geq|u|}, j - |u|\right\rangle = 1$ for some $1 \leq i \leq n_p$. By this and relation (4), we conclude that $\sum_{1\leq i\leq n_p} \left\langle T'_{p,i}, w, j\right\rangle = 1$. Therefore, $(1_{L^+}, w_{<j}) = 1$ implies $\sum_{1\leq h\leq p-1} \left(\sum_{1\leq i\leq n_h} \langle T_{h,i}, w, j\rangle \right) = 1$ or $\sum_{1\leq i\leq n_p} \left\langle T'_{p,i}, w, j\right\rangle = 1$, as required.

Conversely, assume that $\sum_{1\leq h\leq p-1} \left(\sum_{1\leq i\leq n_h} \langle T_{h,i}, w, j\rangle \right) = 1$ or $\sum_{1\leq i\leq n_p} \left\langle T'_{p,i}, w, j\right\rangle = 1$. If the first one is true, then $\sum_{1\leq h\leq p-1} (1_{L^h}, w_{<j}) = 1$. Otherwise, if the latter case holds, then there is an $1 \leq i \leq n_p$ such that $\left\langle T'_{p,i}, w, j\right\rangle = 1$. This implies that $j > l_0 + \ldots + l_{m_i}$, and by relation (4) we get

$$\left\langle T'_{p,i}, w, j\right\rangle = \sum_{0\leq q<j-(l_0+\ldots+l_{m_i})} \left((\|\overline{\varphi}\|, w) \cdot \prod_{0\leq h<q} (\|\widetilde{\varphi}\|, w_{\geq h}) \cdot \langle T_{p,i}, w_{\geq q}, j - q\rangle \right) = 1.$$

Therefore, $(\|\overline{\varphi}\|, w) = 1$, and for some $0 \leq q < j - (l_0 + \ldots + l_{m_i})$ we have $(\|\widetilde{\varphi}\|, w_{\geq h}) = \left(1_{L^{k+1}A^\omega \cup (A^\omega \setminus L^k A^\omega)}, w_{\geq h}\right) = 1$ for every $0 \leq h < q$, and $\langle T_{p,i}, w_{\geq q}, j - q\rangle = \left(1_{L^p}, (w_{\geq q})_{<j-q}\right) = 1$. We set $u = w_{<q}$, and $v = (w_{\geq q})_{<j-q}$. Then $w = uvw_{\geq j}$ and the requirements of Lemma 27 are fulfilled. We conclude that $w_{<j} = uv \in L^+$, i.e., $(1_{L^+}, w_{<j}) = 1$, and our proof is completed. □

Remark 1. By the above inductive proof, we get that for every star-free language $L \subseteq A^+$ we can find a unique integer $n > 0$ and a unique (up to formulas' equivalence) set of tuples $(T_i)_{1\leq i\leq n}$, with $T_i \in U_{m_i}$ $(m_i \geq 0)$ for every $1 \leq i \leq n$, satisfying

Lemma 30. More interestingly, we get that $\sum_{1\leq i\leq n}\langle T_i,w,j\rangle=\sum_{1\leq i\leq n}\langle T_i,w',j\rangle$ for every $w,w'\in A^\omega$ with $w_{<j}=w'_{<j}$.

Example 3. Let $A=\{a,b\}$ and $L=\{ab\}$. Clearly, L is a prefix-free set with bounded synchronization delay $k=1$. Following the inductive construction of the previous proof we get: $\varphi'=p_a\wedge\bigcirc p_b$, $\varphi_{L^2A^\omega}=p_a\wedge\bigcirc p_b\wedge\bigcirc^2 p_a\wedge\bigcirc^3 p_b$, $\varphi_{A^\omega\setminus LA^\omega}=\neg\left(p_a\wedge\bigcirc p_b\right)$ and $\widetilde{\varphi}=\varphi_{L^2A^\omega}\vee\varphi_{A^\omega\setminus LA^\omega}$. We set $T_1=(\varphi',1)$ and $T_2=\left((1\wedge\varphi',0),(\widetilde{\varphi},\varphi_{L^2A^\omega},3)\right)$. Then, $(1_{L^+},w_{<j})=\langle T_1,w,j\rangle+\langle T_2,w,j\rangle$ for every $w\in A^\omega$, $j\geq 0$. For instance, for every $w\in A^\omega$, $\langle T_1,w,j\rangle=1$ iff $(j=2$ and $w_{<2}=ab)$. Let now $w=abababu$ where $u\in A^\omega$. Then,

$$\begin{aligned}\langle T_2,w,6\rangle&=\sum_{\substack{i_1\in\mathbb{N}\\0+i_1+3=5}}\left((\|\varphi'\|,w)\cdot\prod_{0\leq j_1<i_1}(\|\widetilde{\varphi}\|,w_{\geq j_1})\cdot(\|\varphi_{L^2A^\omega}\|,w_{\geq i_1})\right)\\&=(\|\varphi'\|,w)\cdot(\|\widetilde{\varphi}\|,w)\cdot(\|\widetilde{\varphi}\|,w_{\geq 1})\cdot(\|\varphi_{L^2A^\omega}\|,w_{\geq 2})\\&=1=(1_{L^+},w_{<6}).\end{aligned}$$

Similarly,

$$\begin{aligned}\langle T_2,w,5\rangle&=\sum_{\substack{i_1\in\mathbb{N}\\0+i_1+3=4}}\left((\|\varphi'\|,w)\cdot\prod_{0\leq j_1<i_1}(\|\widetilde{\varphi}\|,w_{\geq j_1})\cdot(\|\varphi_{L^2A^\omega}\|,w_{\geq i_1})\right)\\&=(\|\varphi'\|,w)\cdot(\|\widetilde{\varphi}\|,w)\cdot(\|\varphi_{L^2A^\omega}\|,w_{\geq 1})\\&=0=(1_{L^+},w_{<5}).\end{aligned}$$

It should be clear that the values obtained by the semantics of the formulas $\varphi',\widetilde{\varphi},\varphi_{L^2A^\omega}$ that appear in the computation of $\langle T_2,w,6\rangle$ do not depend on the suffix $u=w_{\geq 6}$ of w, but only on the prefix $w_{<6}$. This implies that for $w'=abababu'$ where $u'\neq u$ $(u'\in A^\omega)$ we get that $\langle T_2,w',6\rangle=\langle T_2,w,6\rangle$. A similar observation can be made for $\langle T_2,w,5\rangle$.

Example 4. Let $A=\{a,b\}$ and $L=a^+b$. For every $w\in A^\omega, j\geq 0$ it holds $(1_b,w_{<j})=\langle T_1,w,j\rangle$ and $(1_{a^+},w_{<j})=\langle T_2,w,j\rangle+\langle T_3,w,j\rangle$ where $T_1=(p_b,0)$, $T_2=(p_a,0)$, and $T_3=\left((p_a,0),((p_a\wedge\bigcirc p_a)\vee\neg p_a,p_a\wedge\bigcirc p_a,1)\right)$. We apply the composition algorithm to T_3 and T_1 (resp. T_2 and T_1) and derive the tuple $T_4=\left((p_a,0),\left((p_a\wedge\bigcirc p_a)\vee\neg p_a,p_a\wedge\bigcirc p_a\wedge\bigcirc^2 p_b,2\right)\right)$ (resp. $T_5=(p_a\wedge\bigcirc p_b,1)$). Then $(1_L,w_{<j})=\langle T_4,w,j\rangle+\langle T_5,w,j\rangle$. Indeed, consider $w=aabu$ with $u\in A^\omega$. It holds $\langle T_5,w,3\rangle=0$ and

$$\begin{aligned}&\langle T_4,w,3\rangle=\\&\sum_{\substack{i_1\in\mathbb{N}\\0+i_1+2=2}}\left((\|p_a\|,w)\cdot\prod_{0\leq j_1<i_1}(\|(p_a\wedge\bigcirc p_a)\|,w_{\geq j_1})\cdot\left(\left\|p_a\wedge\bigcirc p_a\wedge\bigcirc^2 p_b\right\|,w_{\geq i_1}\right)\right)\\&=(\|p_a\|,w)\cdot\left(\left\|p_a\wedge\bigcirc p_a\wedge\bigcirc^2 p_b\right\|,w\right)\\&=1=(1_L,w_{<3}),\end{aligned}$$

i.e., $(1_L, w_{<3}) = 1 = \langle T_4, w, 3\rangle + \langle T_5, w, 3\rangle$, as wanted.

Proposition 9. *Let $L \subseteq A^+$ be a star-free language and $r \in K\langle\langle A^*\rangle\rangle$ a letter-step series. Then, for every $\varphi \in ULTL(K, A)$ the infinitary series $(1_L \odot r^+) \cdot \|\varphi\|$ is ω-ULTL-definable.*

Proof. Let $r = \sum_{a\in A} (k_a)_a$ where $k_a \in K$ for every $a \in A$. We set $\zeta = \bigvee_{a\in A} (k_a \wedge p_a)$. By the previous lemma there exist an $n > 0$ and $T_q \in U_{m_q}$ $(m_q \geq 0)$ for every $1 \leq q \leq n$, such that for every $w \in A^\omega, j \geq 0$ we have $(1_L, w_{<j}) = \sum\limits_{1\leq q\leq n} \langle T_q, w, j\rangle$. We fix a $1 \leq q \leq n$ and let us assume that

$$T_q = \left((\varphi_0, k_0), (\xi_1, \varphi_1, k_1), \ldots, \left(\xi_{m_q}, \varphi_{m_q}, k_{m_q}\right)\right).$$

We define the tuple $T'_q \in U_{m_q}$ by

$$T'_q = \left((\varphi'_0, k_0), (\xi'_1, \varphi'_1, k_1), \ldots, \left(\xi'_{m_q}, \varphi'_{m_q}, k_{m_q}\right)\right)$$

as follows.

- If $m_q = 0$, then $\varphi'_0 = \varphi_0 \wedge \left(\bigwedge\limits_{0\leq h\leq k_0} \bigcirc^h \zeta\right)$.
- If $m_q > 0$, then $\xi'_l = \xi_l \wedge \zeta$ for every $1 \leq l \leq m_q$. Moreover, for every $0 \leq l \leq m_q - 1$, if $k_l \neq 0$, then we let $\varphi'_l = \varphi_l \wedge \left(\bigwedge\limits_{0\leq h\leq k_l-1} \bigcirc^h \zeta\right)$, otherwise $\varphi'_l = \varphi_l$. Finally, we set $\varphi'_{m_q} = \varphi_{m_q} \wedge \left(\bigwedge\limits_{0\leq h\leq k_{m_q}} \bigcirc^h \zeta\right)$.

We show that $\langle T'_q, w, j\rangle = \langle T_q, w, j\rangle \cdot (r^+, w_{<j})$ for every $w \in A^\omega, j \geq 0$. Indeed, assume firstly that $m_q = 0$. Then, for every $j \neq k_0 + 1$ we get $\langle T'_q, w, j\rangle = \langle T_q, w, j\rangle = 0$ which implies that $\langle T'_q, w, j\rangle = \langle T_q, w, j\rangle \cdot (r^+, w_{<j})$. For $j = k_0 + 1$ we have

$$\begin{aligned}
\langle T'_q, w, k_0 + 1\rangle &= \left(\left\|\varphi_0 \wedge \left(\bigwedge_{0\leq h\leq k_0} \bigcirc^h \zeta\right)\right\|, w\right) \\
&= (\|\varphi_0\|, w) \cdot \left(\left\|\bigwedge_{0\leq h\leq k_0} \bigcirc^h \zeta\right\|, w\right) \\
&= \langle T_q, w, k_0 + 1\rangle \cdot \prod_{0\leq h\leq k_0} \left(\sum_{a\in A} (k_a)_a, w(h)\right) \\
&= \langle T_q, w, k_0 + 1\rangle \cdot \left(r^+, w_{<k_0+1}\right).
\end{aligned}$$

Next let $m_q > 0$. For every $j \leq k_0 + \ldots + k_{m_q}$ we have $\left\langle T'_q, w, j \right\rangle = \langle T_q, w, j \rangle = 0$, i.e., $\left\langle T'_q, w, j \right\rangle = \langle T_q, w, j \rangle \cdot (r^+, w_{<j})$. For every $j > k_0 + \ldots + k_{m_q}$ it holds

$$\left\langle T'_q, w, j \right\rangle = \sum_{\substack{i_1, i_2, \ldots, i_{m_q} \in \mathbb{N} \\ S_{m_q} = j-1}} \left((\|\varphi'_0\|, w) \cdot \prod_{1 \leq l \leq m_q} \left(\begin{array}{c} \prod_{0 \leq j_l < i_l} \left(\|\xi'_l\|, w_{\geq S_{l-1}+j_l}\right) \\ \cdot \left(\|\varphi'_l\|, w_{\geq S_{l-1}+i_l}\right) \end{array} \right) \right).$$

By definition we have

- $(\|\varphi'_0\|, w) = (\|\varphi_0\|, w) \cdot \prod_{0 \leq h \leq k_0 - 1} (r, w(h))$,
- $\left(\|\xi'_l\|, w_{\geq S_{l-1}+j_l}\right) = \left(\|\xi_l\|, w_{\geq S_{l-1}+j_l}\right) \cdot (r, w(S_{l-1} + j_l))$ for every $1 \leq l \leq m_q$ and $0 \leq j_l < i_l$,
- $\left(\|\varphi'_l\|, w_{\geq S_{l-1}+i_l}\right) = \left(\|\varphi_l\|, w_{\geq S_{l-1}+i_l}\right) \cdot \prod_{0 \leq h \leq k_l - 1} (r, w(S_{l-1} + i_l + h))$ for every $1 \leq l \leq m_q - 1$, and
- $\left(\left\|\varphi'_{m_q}\right\|, w_{\geq S_{m_q-1}+i_{m_q}}\right) = \left(\left\|\varphi_{m_q}\right\|, w_{\geq S_{m_q-1}+i_{m_q}}\right) \cdot \prod_{0 \leq h \leq k_{m_q}} \left(r, w\left(S_{m_q-1} + i_{m_q} + h\right)\right)$.

Hence

$$\begin{aligned} &\left\langle T'_q, w, j \right\rangle \\ &= \sum_{\substack{i_1, i_2, \ldots, i_{m_q} \in \mathbb{N} \\ S_{m_q} = j-1}} \left(\begin{array}{c} (\|\varphi_0\|, w) \cdot \prod_{1 \leq l \leq m_q} \left(\begin{array}{c} \prod_{0 \leq j_l < i_l} \left(\|\xi_l\|, w_{\geq S_{l-1}+j_l}\right) \\ \cdot \left(\|\varphi_l\|, w_{\geq S_{l-1}+i_l}\right) \end{array} \right) \\ \cdot \prod_{0 \leq h \leq S_{m_q}} (r, w(h)) \end{array} \right) \\ &= \sum_{\substack{i_1, i_2, \ldots, i_{m_q} \in \mathbb{N} \\ S_{m_q} = j-1}} \left((\|\varphi_0\|, w) \cdot \prod_{1 \leq l \leq m_q} \left(\begin{array}{c} \prod_{0 \leq j_l < i_l} \left(\|\xi_l\|, w_{\geq S_{l-1}+j_l}\right) \\ \cdot \left(\|\varphi_l\|, w_{\geq S_{l-1}+i_l}\right) \end{array} \right) \right) \cdot \left(r^+, w_{<j}\right) \\ &= \langle T_q, w, j \rangle \cdot \left(r^+, w_{<j}\right). \end{aligned}$$

Therefore, we get

$$\begin{aligned} (1_L, w_{<j}) \cdot \left(r^+, w_{<j}\right) &= \left(\sum_{1 \leq q \leq n} \langle T_q, w, j \rangle \right) \cdot \left(r^+, w_{<j}\right) \\ &= \sum_{1 \leq q \leq n} \left(\langle T_q, w, j \rangle \cdot \left(r^+, w_{<j}\right) \right) \\ &= \sum_{1 \leq q \leq n} \left\langle T'_q, w, j \right\rangle. \end{aligned}$$

For every $1 \leq q \leq n$, we define now the formula $\zeta_q \in ULTL\,(K, A)$ by

$$\zeta_q = \varphi_0' \wedge \bigcirc^{k_0} \left(\xi_1' U \left(\varphi_1' \wedge \bigcirc^{k_1} \left(\xi_2' U \left(\varphi_2' \wedge \bigcirc^{k_2} \left(\ldots U \left(\varphi_{m_q}' \wedge \bigcirc^{k_{m_q}+1} \varphi\right)\right)\right)\right)\right)\right).$$

By induction on m_q, with straightforward calculations, we can show that

$$(\|\zeta_q\|, w) = \sum_{j \geq 0} \left(\left\langle T_q', w, j\right\rangle \cdot (\|\varphi\|, w_{\geq j})\right)$$

for every $w \in A^\omega$. Therefore, we conclude

$$\begin{aligned}
\left(\sum_{1 \leq q \leq n} (\|\zeta_q\|, w)\right) &= \sum_{1 \leq q \leq n} \left(\sum_{j \geq 0} \left(\left\langle T_q', w, j\right\rangle \cdot (\|\varphi\|, w_{\geq j})\right)\right) \\
&= \sum_{j \geq 0} \left(\sum_{1 \leq q \leq n} \left(\left\langle T_q', w, j\right\rangle \cdot (\|\varphi\|, w_{\geq j})\right)\right) \\
&= \sum_{j \geq 0} \left(\left(\sum_{1 \leq q \leq n} \left\langle T_q', w, j\right\rangle\right) \cdot (\|\varphi\|, w_{\geq j})\right) \\
&= \sum_{j \geq 0} \left(\left(1_L \odot r^+, w_{<j}\right) \cdot (\|\varphi\|, w_{\geq j})\right) \\
&= \left(\left(1_L \odot r^+\right) \cdot \|\varphi\|, w\right),
\end{aligned}$$

and our proof is completed. $\square$

Our next result states that the almost simple ω-counter-free series are ω-*ULTL*-definable, and in fact concludes our theory.

Theorem 5. *ω-asCF$(K, A) \subseteq \omega$-ULTL(K, A).*

Proof. Clearly it suffices to show that whenever $\mathcal{A}_1, \ldots, \mathcal{A}_{n-1}$ are simple cfwa and $\mathcal{A}_n$ is a simple cfwBa over A and K, then $\|\mathcal{A}_1\| \cdot \ldots \cdot \|\mathcal{A}_n\| \in \omega\text{-}ULTL(K, A)$. We let $r_i = \|\mathcal{A}_i\|$, and denote by k_i the initial weight $\neq 0$ and $k_a^{(i)}$ the weight $\neq 0$ of the transitions of $\mathcal{A}_i$ $(1 \leq i \leq n)$ labelled by $a \in A$. Since $\mathrm{supp}\,(r_n)$ is an ω-counter-free language it is also ω-*LTL*-definable hence, there is formula $\varphi \in bLTL(K, A)$ with $\|\varphi\| = 1_{\mathrm{supp}(r_n)}$. We let $\varphi_n = k_n \wedge \varphi \wedge \left(\square \left(\bigvee_{a \in A} \left(k_a^{(n)} \wedge p_a\right)\right)\right)$ and we trivially get $r_n = \|\varphi_n\|$. By construction $\varphi_n \in ULTL\,(K, A)$. Furthermore, for every $1 \leq i \leq n-1$, the language $\mathrm{supp}\,(r_i) \setminus \{\varepsilon\} \subseteq A^*$ is counter-free hence, star-free. Since

$$r_i|_{A^+} = 1_{\mathrm{supp}(r_i) \setminus \{\varepsilon\}} \odot \left(k_i \left(\textstyle\sum_{a \in A} \left(k_a^{(i)}\right)_a\right)^+\right)$$

for every $1 \leq i \leq n-1$, and

$$r_{n-1}|_{A^+} \cdot r_n = k_{n-1}\left(\left(1_{\mathrm{supp}(r_{n-1})\setminus\{\varepsilon\}} \odot \left(\sum_{a\in A}\left(k_a^{(n-1)}\right)_a\right)^+\right)\cdot r_n\right),$$

by applying Proposition 9, we get that

$$\left(1_{\mathrm{supp}(r_{n-1})\setminus\{\varepsilon\}} \odot \left(\sum_{a\in A}\left(k_a^{(n-1)}\right)_a\right)^+\right)\cdot r_n \in \omega\text{-}ULTL(K,A)$$

which implies that there exists a $ULTL\,(K,A)$ formula φ^+_{n-1} such that

$$\left(1_{\mathrm{supp}(r_{n-1})\setminus\{\varepsilon\}} \odot \left(\sum_{a\in A}\left(k_a^{(n-1)}\right)_a\right)^+\right)\cdot r_n = \left\|\varphi^+_{n-1}\right\|.$$

Hence, $r_{n-1}|_{A^+} \cdot r_n = \left\|k_{n-1} \wedge \varphi^+_{n-1}\right\|$. We let $\varphi_{n-1} = \left(k_{n-1} \wedge \varphi^+_{n-1}\right) \vee ((r_{n-1},\varepsilon) \wedge \varphi_n) \in ULTL\,(K,A)$ and we have $\|\varphi_{n-1}\| = r_{n-1}\cdot r_n$. Thus $r_{n-1}\cdot r_n \in \omega\text{-}ULTL(K,A)$. We proceed in the same way, and we show that $r_i \cdot \ldots \cdot r_n \in \omega\text{-}ULTL(K,A)$, for every $1 \leq i \leq n-2$, which concludes our proof. □

Now we are ready to state the coincidence of the classes of ω-*ULTL*-definable, ω-*wqFO*-definable, ω-star-free, and almost simple ω-counter-free series. More precisely, by Theorems 1, 2, 4, and 5 we get our main result.

Theorem 6 (Main theorem)**.**

$$\omega\text{-}ULTL\,(K,A) = \omega\text{-}wqFO(K,A) = \omega\text{-}SF(K,A) = \omega\text{-}asCF(K,A).$$

Conclusion

We showed the coincidence of the classes of series definable in a fragment of the weighted *LTL*, series definable in a fragment of the weighted *FO* logic, ω-star-free series, and almost simple ω-counter-free series. Our underlying semiring required to be idempotent, zero-divisor free and totally commutative complete satisfying an additional property. It is an open problem whether we can relax the idempotency and/or the zero-divisor freeness property of the semiring. Our results can be proved for series over finite words. In this case we do not need completeness axioms anymore. As a future research we state two main directions. The first one is the development of our theory in the probabilistic setup, i.e., to investigate the expressive equivalence (of fragments) of probabilistic *LTL*, probabilistic *FO* logic, probabilistic ω-star-free expressions, and counter-free probabilistic Büchi automata, where the last two concepts have not been defined yet. The latter concerns the development of our theory in the setup of more general structures than semirings. For instance, in [12] the authors studied weighted automata and weighted *MSO* logics over valuation monoids which capture operations that play an important role in practical applications.

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