Acta Cybernetica **22** (2015) 485–498.

On Shift Radix Systems over Imaginary Quadratic Euclidean Domains*

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Dedicated to the memory of Professor Ferenc Gécseg

Abstract

In this paper we generalize the shift radix systems to finite dimensional Hermitian vector spaces. Here the integer lattice is replaced by the direct sum of imaginary quadratic Euclidean domains. We prove in two cases that the set of one dimensional Euclidean shift radix systems with finiteness property is contained in a circle of radius 0.99 around the origin. Thus their structure is much simpler than the structure of analogous sets.

1 Introduction

For $\mathbf{r} \in \mathbb{R}^n$ the mapping $\tau_{\mathbf{r}} \; : \; \mathbb{Z}^n \mapsto \mathbb{Z}^n$, defined as

$$\tau_{\mathbf{r}}((a_1, \ldots, a_n)) = (a_2, \ldots, a_{n-1}, \lfloor \mathbf{ra} \rfloor),$$

where $\mathbf{ra}$ denotes the inner product, is called *shift radix system*, shortly SRS. This concept was introduced by S. Akiyama et al. [1] and they proved that it is a common generalization of canonical number systems (CNS), first studied by I. Kátai and J. Szabó [8], and the β-expansions, defined by A. Rényi [11]. For computational aspects of CNS we refer to the paper of P. Burcsi and A. Kovács [5].

Among the several generalizations of CNS we cite here only one to polynomials over Gaussian integers by M.A. Jacob and J.P. Reveilles [7]. Generalizing the shift radix systems, H. Brunotte, P. Kirschenhofer and J. Thuswaldner [3] defined GSRS for Hermitian vector spaces. A wider generalization of CNS, namely for polynomials over imaginary quadratic Euclidean domains was studied by the first two authors

*Research supported in part by the OTKA grants NK104208, NK101680. Mario Weitzer is supported by the Austrian Science Fund (FWF): W1230, Doctoral Program "Discrete Mathematics".

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DOI: 10.14232/actacyb.22.2.2015.14

in [10]. It is well known that there are exactly five such domains, which are the ring of integers of the imaginary quadratic fields $\mathbb{Q}(\sqrt{d}), d = 1, 2, 3, 7, 11$. The Euclidean norm function allows not only the division by remainder, but also to define a floor function for complex numbers. This observation leads us to generalize SRS for Hermitian vector spaces endowed by floor functions depending on imaginary quadratic Euclidean domains. Our generalization, which we call ESRS, is uniform for the five imaginary quadratic Euclidean domains. This has the consequence that in case of the Gaussian integers our floor function differs from that used in [3].

The SRS $\tau_{\mathbf{r}}$ is said to have the finiteness property iff for all $\mathbf{a} \in \mathbb{Z}^{\mathbf{n}}$ there exists a $k \geq 1$ such that $\tau_{\mathbf{r}}^k(\mathbf{a}) = \mathbf{0}$. Denote by $\mathcal{D}_n^{(0)}$ the set of $\mathbf{r} \in \mathbb{R}^n$ such that $\tau_{\mathbf{r}}$ has the finiteness property. From numeration point of view these real vectors are most important. It turned out that the structure of $\mathcal{D}_n^{(0)}$ is very complicated already for $n = 2$, see [2], [12] and [13].

The analogue of the two dimensional SRS is the one dimensional GSRS and ESRS. Brunotte et al. [3] studied first the set of one dimensional GSRS with finiteness property, which we denote by $\mathrm{GSRS}^{(0)}$. It turned out that its structure is quite complicated as well. Recently a more precise investigation of M. Weitzer [14] showed that the structure of $GSRS^{(0)}$ is much simpler as that of $\mathcal{D}_2^{(0)}$. Based on extensive computer investigations he conjectures a finite description of $\mathrm{GSRS}^{(0)}$.

Analogously to $\mathcal{D}_n^{(0)}$ we can define $\mathcal{D}_{n,d}^{(0)}, d = 1, 2, 3, 7, 11$ in a straight forward way. We show how one can compute good approximations of $\mathcal{D}_{n,d}^{(0)}$. Performing the computation it turned out that the shape of these objects are quite different. The subjective impression can be misleading, but we were able to prove that $\mathcal{D}_{n,d}^{(0)}$ has no critical points in the cases $d = 2, 11$. More specifically we prove that the circle of radius 0.99 around the origin contains $\mathcal{D}_{n,d}^{(0)}$. In the other cases this is probably not true. It is certainly not true for $\mathcal{D}_2^{(0)}$ and $\mathrm{GSRS}^{(0)}$.

2 Basic concepts

In order to establish a shift radix system over the complex numbers, an imaginary quadratic Euclidean domain will be used as the set of integers, and a floor function is needed which can be determined by making its Euclidean function unique, so choosing the set of fractional numbers from the possible values.

Definition 1. *Let $\mathbb{E}_d = \mathbb{Z}_{\mathbb{Q}[\sqrt{-d}]}$ be an imaginary quadratic Euclidean domain ($d \in \{1, 2, 3, 7, 11\}$, see in [6]). Its* ***canonical integral basis*** *is: $\{1, \omega\}$, where*

$$\omega := \begin{cases} \sqrt{-d} & \text{, if } d \in \{1, 2\}, \\ \frac{1+\sqrt{-d}}{2} & \text{, otherwise.} \end{cases}$$

(In the case of $d = 1$ instead of ω the imaginary unit i is used.)

For fixed d, the complex numbers $1, \omega$ form a basis of $\mathbb{C}$, as a two dimensional vector space over $\mathbb{R}$. Thus all $z \in \mathbb{C}$ can be uniquely written in the form $z = e_1 + e_2\omega$

with $e_1, e_2 \in \mathbb{R}$. Plainly $z \in \mathbb{E}_d$ iff $e_1, e_2 \in \mathbb{Z}$. Let the functions $Re_d : \mathbb{C} \mapsto \mathbb{R}$ and $Im_d : \mathbb{C} \mapsto \mathbb{R}$ be defined as:

$$Re_d(z) := e_1,\ Im_d(z) := e_2.$$

$Re_d(z)$ and $Im_d(z)$ are called the *real and imaginary parts of* z.
The elements of $\mathbb{E}_d$ will be denoted by $(e_1, e_2)_d$.

Plainly, for all $z \in \mathbb{C}$ we have

$$\begin{aligned} Im_d(z) &= \frac{Im(z)}{Im(\omega)}, \\ Re_d(z) &= Re(z) - Im(z)\frac{Re(\omega)}{Im(\omega)}. \end{aligned}$$

In order to define a floor function, a set of fractional numbers has to be defined. Regarding generalization purposes the absolute value of a fractional number should be less than 1, a fractional number should not be negative in a sense, it is a superset of the fractional numbers for the reals, and the floor function should be unambiguous. From these considerations the following definition will be used to specify the floor function with the set of fractional numbers which will be called fundamental sail tile.

Definition 2. *Let* $d \in \{1, 2, 3, 7, 11\}$. *Let the set*

$$\mathbb{D}_d := \left\{ c \in \mathbb{C} \;\middle|\; |c| < 1 \quad |c+1| \geq 1 \quad -\frac{1}{2} \leq Im_d(c) < \frac{1}{2} \right\}$$

be defined as the ***fundamental sail tile*** *(the set of fractional numbers).*
Let $p \in \mathbb{E}_d$. *The set*

$$\mathbb{D}_d(p) := \left\{ p + c \;\middle|\; c \in \mathbb{C} \quad |c| < 1 \quad |c+1| \geq 1 \quad -\frac{1}{2} \leq Im_d(c) < \frac{1}{2} \right\}$$

is called p***-sail tile*** *and* p *is called its* ***representative integer***.

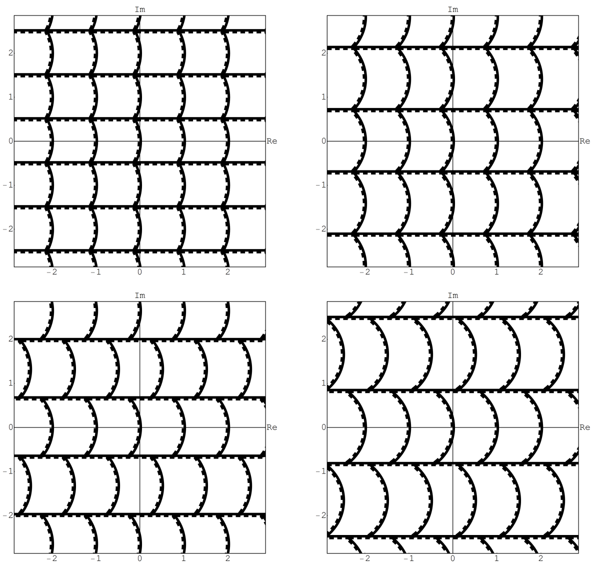

Figure 1: Tilings of $\mathbb{C}$ given by the sets $\mathbb{D}_d(p)$, $d \in \{1, 2, 3, 7, 11\}$.

By using Theorem 1 of [10] one can show that the sets $\mathbb{D}_d(p)$, where p runs through $\mathbb{E}_d$ do not overlap and cover the complex plain $\mathbb{C}$. This justifies the following definition:

Definition 3. *Let the function* $\lfloor\ \rfloor_d : \mathbb{C} \to \mathbb{E}_d$ *be defined as the* ***floor function****. The floor of* e *is the representative integer* p *of the unique* p*-sail tile that contains* e.

The next lemma shows that the above defined floor function can be described with the well-known floor function over the real numbers. We leave its simple proof to the reader.

Lemma 1.

$$\lfloor e \rfloor_d = \begin{cases} \left\lfloor Re(e) - \left\lfloor Im_d(e) + \frac{1}{2} \right\rfloor Re(\omega) \right\rfloor + \omega \left\lfloor Im_d(e) + \frac{1}{2} \right\rfloor, \ \textit{if} \\ \qquad \Big(Re(e) - \left\lfloor Re(e) - \left\lfloor Im_d(e) + \frac{1}{2} \right\rfloor Re(\omega) \right\rfloor - \\ \qquad - \left\lfloor Im_d(e) + \frac{1}{2} \right\rfloor Re(\omega) \Big)^2 + \\ \qquad + \left(Im(e) - \left\lfloor Im_d(e) + \frac{1}{2} \right\rfloor Im(\omega) \right)^2 < 1, \\ \left\lfloor Re(e) - \left\lfloor Im_d(e) + \frac{1}{2} \right\rfloor Re(\omega) \right\rfloor + \omega \left\lfloor Im_d(e) + \frac{1}{2} + 1 \right\rfloor, \ \textit{otherwise.} \end{cases}$$

Equipped with the appropriate floor functions we are in the position to define shift radix systems for Hermitian vectors. The notion depends on the imaginary Euclidean domain.

Definition 4. *Let $C := (c_1, \dots, c_n) \in \mathbb{C}^n$ be a complex vector. Let $d \in \{1, 2, 3, 7, 11\}$ and the floor function $\lfloor x \rfloor_d$ defined as above.*
For all vectors $A := (a_1, a_2, \dots, a_n) \in \mathbb{E}_d^n$ let

$$\mathcal{T}_{d,C}(A) := (a_2, \dots, a_n, -q)\,,$$

*where $q = \lfloor c_1 a_1 + c_2 a_2 + \cdots + c_n a_n \rfloor_d$. The mapping $\mathcal{T}_{d,C} : \mathbb{E}_d^n \mapsto \mathbb{E}_d^n$ is called **Euclidean shift radix system with parameter** d or ESRS$_d$ respectively, ESRS for short. If $B := \mathcal{T}_{d,C}(A)$, this mapping will be denoted by*

$$A \underset{d,C}{\Rightarrow} B.$$

If for $A, B \in \mathbb{E}_d^n$ there is a $k \in \mathbb{N}$, such that $\mathcal{T}_{d,C}^k(A) = B$ then this will be indicated by:

$$A \overset{*}{\underset{d,C}{\Longrightarrow}} B.$$

*$\mathcal{T}_{d,C}$ is called **ESRS with finiteness property** iff for all vectors $A \in \mathbb{E}_d^n$*

$$A \overset{*}{\underset{d,C}{\Longrightarrow}} 0,$$

where 0 is the zero vector.

Definition 5. *The following sets form a generalization of the corresponding sets defined in [1]:*

$$\mathcal{D}_{n,d}^0 := \left\{ C \in \mathbb{C}^n \;\middle|\; \forall A \in \mathbb{E}_d^n : A \overset{*}{\underset{d,C}{\Longrightarrow}} 0 \right\},$$

$$\mathcal{D}_{n,d} := \left\{ C \in \mathbb{C}^n \;\middle|\; \forall A \in \mathbb{E}_d^n \text{ the sequence } \left\{ \mathcal{T}_{d,C}^k(A) \right\}_{k \geq 0} \text{ is ultimately periodic} \right\}.$$

$\mathcal{T}_{d,C}$ is ESRS with finiteness property iff $C \in \mathcal{D}_{n,d}^0$.

Remark 1. *The construction defined in this section can be generalized by using a complex number for d.*

3 Basic properties of the one dimensional shift radix systems

This section and the following ones will consider C as a one dimensional vector, i.e. a complex number, which will be denoted by c. In this section we will investigate some properties of the one dimensional case. Theorem 1 can be considered as the generalization of cutout polyhedra defined in [1]. These are areas defined by a closed curve (arcs and lines). Let this area be denoted by P. Let's consider this as **cutout area**.

Theorem 1. *Let $c \in \mathbb{C}$. The number $a_0 \in \mathbb{E}_d$ with (d, c) admits a period*

$$a_0 \underset{d,c}{\Rightarrow} a_1 \underset{d,c}{\Rightarrow} a_2 \underset{d,c}{\Rightarrow} a_3 \ldots \underset{d,c}{\Rightarrow} a_{l-1} \underset{d,c}{\Rightarrow} a_0, \textit{ if and only if}$$

$$c \in \left(\frac{\mathbb{D}_d - a_1}{a_0}\right) \cap \left(\frac{\mathbb{D}_d - a_2}{a_1}\right) \cap \cdots \cap \left(\frac{\mathbb{D}_d - a_{l-1}}{a_{l-2}}\right) \cap \left(\frac{\mathbb{D}_d - a_0}{a_{l-1}}\right).$$

The number l will be called the length of the period.

Proof. The proof is essentially the same as the proof of Theorem 3 in [10]. □

The next theorem shows that if the ESRS associated to c has the finiteness property then it must lie in the closed unit circle.

Theorem 2. *Let $|c| > 1$, $d \in \{1, 2, 3, 7, 11\}$ then $\mathcal{T}_{d,c}$ doesn't have the finiteness property.*

Proof. The basic idea is that we ignore those values of a where the length decreases after applying $\mathcal{T}_{d,c}$, since after finitely many steps it will end in 0 or another value a' the absolute value of which increases by applying the mapping. Investigating the length of a vector after applying the shift radix mapping:

$$a \underset{d,c}{\Rightarrow} ac - r.$$

For the length

$$|a| > |ac - r| \geq |a||c| - |r| > |a||c| - 1,$$
$$|a| < \frac{1}{|c| - 1}.$$

If this inequality holds the length decreases. This is a finite open disk around the origin. For any other a the length will increase, so starting from a applying the shift radix mapping leads to a divergent sequence. □

Plainly $\mathcal{T}_{d,1}$ doesn't have the finiteness property for any d. For finding ESRS with finiteness property, one has to use a well chosen complex number c. Based on Theorem 2, let's start from the closed unit disc around the origin, and let's ignore these cutout areas in order to reach those points which are good to define ESRS with finiteness property:

Remark 2. *The set $\mathcal{D}^0_{n,d}$ can be defined in the following way. Let $S := \{c \in \mathbb{C} |\ |c| \leq 1\}$ and let's consider the areas defined by Theorem 1 as P_i. Then*

$$\mathcal{D}^0_{n,d} = S \setminus \cup P_i.$$

Since cutout areas can be infinitely many, can be disjoint, overlapped by each other or superset and subset of each other, finding the union area of all is a hard problem. The following definition helps to estimate how many cutout areas are around some point in $\mathcal{D}_{n,d}$.

Definition 6. *Let $c \in \mathcal{D}_{n,d}$.*

- *If there exists an open neighborhood of c which contains only finitely many cutout areas then we call c a **regular point**.*
- *If each open neighborhood of c has nonempty intersection with infinitely many cutout areas then we call c a **weak critical point** for $\mathcal{D}_{n,d}$.*
- *If for each open neighborhood U of c the set $U \setminus \mathcal{D}^0_{n,d}$ cannot be covered by finitely many cutout areas then c is called a **critical point**.*

Let's check what are the conditions to reach a cutout area in the one dimensional case.

Remark 3. *Theorem 1's result for one dimensional case can be used to define cutout areas with periods of any length. $\mathcal{T}_{d,c}$ admits a period $a_0 \underset{d,c}{\Rightarrow} a_1 \underset{d,c}{\Rightarrow} a_2 \underset{d,c}{\Rightarrow} \ldots \underset{d,c}{\Rightarrow} a_n \underset{d,c}{\Rightarrow} a_0$ if and only if*

$$c \in \left(\frac{\mathbb{D}_d - a_1}{a_0}\right) \cap \left(\frac{\mathbb{D}_d - a_2}{a_1}\right) \cap \cdots \cap \left(\frac{\mathbb{D}_d - a_n}{a_{n-1}}\right) \cap \left(\frac{\mathbb{D}_d - a_0}{a_n}\right).$$

The one-step and the two-step cases are really important, since the one-step periods define large sets around -1, and the two-step case appear most likely around 1. The following two lemmata speak about these special cases.

Lemma 2. *Let $|c| < 1$. $\mathcal{T}_{d,c}$ admits a one-step period, if and only if $c \in \frac{\mathbb{D}_d}{a} - 1$ for an $a \in \mathbb{E}_d \setminus \{0\}$.*

Proof. The shift radix mapping leads to the following:

$$a \underset{d,c}{\Rightarrow} -ac + r,$$

$a \in \mathbb{E}_d \setminus \{0\}$. This can be a one-step period, iff $c = \frac{r}{a} - 1$. r is a general element of the fundamental sail tile, so $c \in \frac{\mathbb{D}_d}{a} - 1$. □

Lemma 3. *Let $|c| < 1$. $\mathcal{T}_{d,c}$ admits a two-step period, if and only if $c \in \left(\frac{\mathbb{D}_d - a'}{a}\right) \cap \left(\frac{\mathbb{D}_d - a}{a'}\right)$, where $a, a' \in \mathbb{E}_d \setminus \{0\}$.*

Proof. The shift radix mapping leads to the following:

$$a \underset{d,c}{\Rightarrow} -ac + r,$$

$a \in \mathbb{E}_d \setminus \{0\}$. Let $a' := -ac + r \in \mathbb{E}_d \setminus \{0\}$, $a' \underset{d,c}{\Rightarrow} -a'c + r$. This can be a two-step period, iff $a = -a'c + r$. This means that c has to be in the set

$$c \in \left(\frac{\mathbb{D}_d - a'}{a}\right) \cap \left(\frac{\mathbb{D}_d - a}{a'}\right).$$

□

Theorem 3 shows that only finitely many $a \in \mathbb{E}_d$ have to be investigated to decide the finiteness property of a specific value of c.

Theorem 3. *Let $|c| < 1$. $\mathcal{T}_{d,c}$ is a ESRS with finiteness property, iff for all $a \in \mathbb{E}_d$ where $|a| < \frac{1}{1-|c|}$*

$$a \underset{d,c}{\overset{*}{\Longrightarrow}} 0.$$

Proof.

$$a \underset{d,c}{\Rightarrow} -ac + r, \text{ where}$$

$r \in \mathbb{D}_d$. To decide the finiteness property one has to check only those numbers where the absolute value does not decrease.

$$|a| \leq |-ac + r| \leq |a||c| + |r| < |a||c| + 1, \text{ so}$$

$|a| < \frac{1}{1-|c|}$. □

Now, let's see how the sets $\mathcal{D}_{1,d}^0$ ($d \in \{1, 2, 3, 7, 11\}$) look like. Algorithm 1 defines a searching method, which will approximate the mentioned set using the results of Remark 2 and Theorem 3. The input parameters are $d \in \{1, 2, 3, 7, 11\}$ and rs, which sets how many points in the unit circle will be tested, the result is a superset of $\mathcal{D}_{1,d}^0$.

Algorithm 1 Approximation algorithm for the set $\mathcal{D}^0_{1,d}$

1: $d \in \{1, 2, 3, 7, 11\}$ (input parameter)
2: $rs := 1000000$ (input parameter)
3: $res := \frac{1}{\sqrt{rs}}$
4: $S := \{c \in \mathbb{C} |\ \ |c| \le 1\}$
5: $S_{curr} := S$
6: **for** $rad \in \{0, res, 2res \ldots, 1\}$ **do**
7:     **for** $ang \in \{0, res, 2res \ldots, 2\pi\}$ **do**
8:         $c_{curr} := rad \cdot e^{i \cdot ang}$
9:         **if** $c_{curr} \in S_{curr}$ **then**
10:             $A_{curr} := \{a' | a' \in \mathbb{E}_d \ \ |a'| < \frac{1}{1 - |c_{curr}|}\}$
11:             **for** $a_{curr} \in A_{curr}$ **do**
12:                 **if** $\mathcal{T}_{d, c_{curr}}$ admits a period P' starting from a_{curr} **then**
13:                     $S_{curr} = S_{curr} \setminus P'$
14:                     break operation *11*
15:                 **end if**
16:             **end for**
17:         **end if**
18:     **end for**
19: **end for**
20: **return** S_{curr}

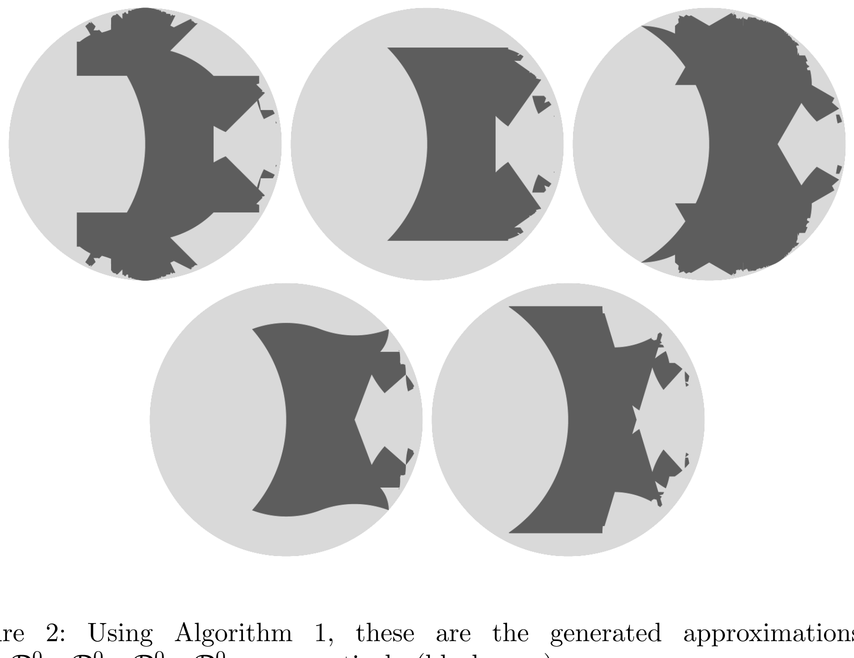

Figure 2: Using Algorithm 1, these are the generated approximations of $\mathcal{D}^0_{1,1}, \mathcal{D}^0_{1,2}, \mathcal{D}^0_{1,3}, \mathcal{D}^0_{1,7}, \mathcal{D}^0_{1,11}$, respectively (black area).

The area close to the origin is the easiest part of the disc to decide the finiteness property, so let's consider the case $|c| < \frac{1}{2}$.

Theorem 4. *Let $|c| < 1 - \frac{1}{\sqrt{4}} = \frac{1}{2}$. The function $\mathcal{T}_{d,c}$ is a ESRS with finiteness property, if $c \in \mathbb{D}_d$. Additionally, if $d = 11$ then*

$$c \notin \left\{ z \in \mathbb{C} \;\middle|\; |(-\omega)z + \omega - 1| \geq 1 \quad -\frac{\sqrt{11}}{4} < Im((-\omega)z + \omega) \right\}, \text{ and}$$

$$c \notin \left\{ z \in \mathbb{C} \;\middle|\; |(-1+\omega)z - \omega| \geq 1 \quad Im((-1+\omega)z + 1 - \omega) \leq \frac{\sqrt{11}}{4} \right\}.$$

Proof. The proof of this theorem only uses basic considerations and the results of this article. □

The following Lemma implies that $\mathcal{D}^0_{1,d}$ and $\mathcal{D}_{1,d}$ reflected at the real axis coincide almost everywhere. Parts where the two sets might not coincide are contained in the union of their respective boundaries.

Lemma 4. *Let $c \in \mathbb{C}$, $a, b \in \mathbb{E}_d$, and $\varphi = (a_1, a_2, \ldots, a_k) \in \mathbb{E}^k_d$. Then $2Im_d(ca)$ is not an odd integer $\Leftrightarrow (\mathcal{T}_c a = b \Leftrightarrow \mathcal{T}_{\overline{c}}\overline{a} = \overline{b})$,*
$2Im_d(ca)$ is an odd integer $\Rightarrow (\mathcal{T}_c a = b \Rightarrow \mathcal{T}_{\overline{c}}\overline{a} - \overline{b} \in \{(0,-1)_d, (1,-1)_d\})$.
In particular, if c is contained in the interior of the cutout area corresponding to φ then
$(a_1, a_2, \ldots, a_k)$ period of $\mathcal{T}_c \Leftrightarrow (\overline{a_1}, \overline{a_2}, \ldots, \overline{a_k})$ period of $\mathcal{T}_{\overline{c}}$.

Proof. The proof can be done the same way as the proof of Lemma 3.6 in [3]. □

Definition 7. *Let*

$$\big(((x_{2,1}, y_{2,1}), (a_{2,1}, b_{2,1})), \ldots, ((x_{2,45}, y_{2,45}), (a_{2,45}, b_{2,45})\big) := \Big($$

$$((1,0),\ (-2,0)),\ ((-\tfrac{529}{4023}, \tfrac{22378908}{45415717}),\ (0,1)),\quad ((-\tfrac{8413}{3862276}, \tfrac{6385}{8993}),\ (0,1)),$$
$$((-\tfrac{560}{3763}, \tfrac{166}{229}),\ (0,1)),\ ((\tfrac{11051}{36427}, \tfrac{12022}{16987}),\ (0,1)),\ ((-\tfrac{39833}{139318}, \tfrac{634841}{887952}),\ (0,1)),$$
$$((-\tfrac{587}{32542}, \tfrac{1260970}{1501501}),\ (0,1)),\quad ((\tfrac{20911}{27059}, \tfrac{183}{517}),\ (0,1)),\quad ((-\tfrac{3533}{7022}, \tfrac{1411}{1988}),\ (0,1)),$$
$$((\tfrac{645}{3757}, \tfrac{1432877}{1660169}),\ (0,1)),\quad ((\tfrac{844688}{1266909}, \tfrac{2031}{3445}),\ (0,4)),\quad ((\tfrac{44399}{51256}, \tfrac{4447}{14348}),\ (0,2)),$$
$$((\tfrac{781981}{1137704}, \tfrac{159}{260}),\ (0,4)),$$
$$((\tfrac{3741}{6160}, \tfrac{2237}{3237}),\ (0,2)),\quad ((\tfrac{18563}{132052}, \tfrac{677269}{744909}),\ (0,1)),\quad ((-\tfrac{273}{461}, \tfrac{256}{357}),\ (0,1)),$$
$$((-\tfrac{23531}{44649}, \tfrac{2367}{3041}),\ (0,1)),((-\tfrac{2504}{4903}, \tfrac{53361}{66614}),\ (0,1)),((\tfrac{2295978}{14352937}, \tfrac{128937}{134770}),\ (0,1)),$$
$$((-\tfrac{22537}{155137}, \tfrac{19631}{20469}),\ (0,1)),\ ((-\tfrac{1324}{2503}, \tfrac{85287}{104894}),\ (0,1)),\ ((\tfrac{186647}{247677}, \tfrac{278}{433}),\ (0,2)),$$
$$((\tfrac{81473}{111068}, \tfrac{86419}{129984}),\ (0,2)),((-\tfrac{1087}{2004}, \tfrac{670}{809}),\ (0,1)),((\tfrac{19}{25}, \tfrac{16}{25}),\ (0,2)),((\tfrac{27}{37}, \tfrac{25}{37}),$$
$$(0,2)),\quad ((\tfrac{13}{17}, \tfrac{54}{85}),\ (0,2)),\quad ((\tfrac{7647}{10000}, \tfrac{16}{25}),\ (0,2)),\quad ((\tfrac{7339}{10000}, \tfrac{1347}{2000}),\ (0,2)),$$
$$((\tfrac{1979}{20000}, \tfrac{4961}{5000}),\ (0,1)),\quad ((-\tfrac{1979}{20000}, \tfrac{397}{400}),\ (0,1)),\quad ((-\tfrac{2701}{5000}, \tfrac{8399}{10000}),\ (0,1)),$$
$$((-\tfrac{1097}{2000}, \tfrac{4169}{5000}),\ (0,1)),\quad ((\tfrac{1527}{2000}, \tfrac{6429}{10000}),\ (0,2)),\quad ((\tfrac{3831}{5000}, \tfrac{6413}{10000}),\ (0,2)),$$
$$((\tfrac{3699}{5000}, \tfrac{6711}{10000}),\ (0,2)),\quad ((\tfrac{7321}{10000}, \tfrac{6767}{10000}),\ (0,2)),\quad ((\tfrac{7419}{10000}, \tfrac{1339}{2000}),\ (0,2)),$$
$$((\tfrac{3683}{5000}, \tfrac{3377}{5000}),\ (0,2)),\quad ((-\tfrac{1087}{2000}, \tfrac{4183}{5000}),\ (0,1)),\quad ((-\tfrac{1089}{2000}, \tfrac{8387}{10000}),\ (0,1)),$$
$$((-\tfrac{1089}{2000}, \tfrac{1677}{2000}),\ (0,1)),\qquad\qquad ((\tfrac{1}{10}, \tfrac{7}{5\sqrt{2}}),\ (0,1)),$$
$$((\tfrac{1}{100}(50 + \sqrt{1534}), -\tfrac{-100+\sqrt{1534}}{100\sqrt{2}}),\ (0,1)),\ ((\tfrac{9}{10}, \tfrac{3}{5\sqrt{2}}),\ (0,1))\Big),$$

$$
\begin{gathered}
\big(((x_{11,1},y_{11,1}),(a_{11,1},b_{11,1})),\dots,((x_{11,47},y_{11,47}),(a_{11,47},b_{11,47})\big) := \Big(\\
((1,0),\ (-2,0))\ \big(\big(-\tfrac{529}{4023},\tfrac{22378908}{45415717}\big),\ (0,1)\big), \quad \big(\big(\tfrac{25699}{75158},\tfrac{11951}{22586}\big),\ (2,0)\big), \\
\big(\big(\tfrac{122233}{192089},\tfrac{5593}{12399}\big),\ (0,1)\big), \quad \big(\big(\tfrac{6229}{23994},\tfrac{22353}{28738}\big),\ (0,9)\big), \quad \big(\big(\tfrac{2039}{57213},\tfrac{17365}{20941}\big),\ (0,1)\big), \\
\big(\big(\tfrac{3099}{4183},\tfrac{442047}{1060847}\big),\ (0,1)\big), \quad \big(\big(-\tfrac{39923}{156499},\tfrac{22371}{26896}\big),\ (0,1)\big), \quad \big(\big(\tfrac{4038}{5203},\tfrac{4722}{11383}\big),\ (0,1)\big), \\
\big(\big(\tfrac{285}{406},\tfrac{752}{1417}\big),\ (0,1)\big), \quad \big(\big(\tfrac{15765}{22453},\tfrac{431}{725}\big),\ (0,1)\big), \quad \big(\big(\tfrac{2023}{7895},\tfrac{2634}{2981}\big),\ (0,1)\big), \\
\big(\big(-\tfrac{810241}{3496246},\tfrac{662044}{743591}\big),\ (0,1)\big), \quad \big(\big(\tfrac{127129}{185005},\tfrac{42539}{67882}\big),\ (0,4)\big), \quad \big(\big(-\tfrac{109151}{435226},\tfrac{1106}{1235}\big), \\
(0,1)\big),\big(\big(\tfrac{1499}{5037},\tfrac{10953}{12284}\big),\ (0,1)\big),\big(\big(-\tfrac{8495}{29356},\tfrac{259913}{290617}\big),\ (0,1)\big),\big(\big(\tfrac{755}{851},\tfrac{3083}{7406}\big),\ (0,1)\big), \\
\big(\big(-\tfrac{15483}{32584},\tfrac{4513239}{5265740}\big),\ (0,1)\big),\big(\big(-\tfrac{39752315}{80135632},\tfrac{1130}{1337}\big),\ (0,1)\big),\big(\big(-\tfrac{45318560}{90412991},\tfrac{235960}{280199}\big), \\
(0,1)\big), \quad \big(\big(-\tfrac{422566}{838723},\tfrac{6443}{7665}\big),\ (0,1)\big), \quad \big(\big(-\tfrac{7361}{14390},\tfrac{105082}{125711}\big),\ (0,1)\big), \\
\big(\big(-\tfrac{724614}{1438463},\tfrac{2019}{2369}\big),\ (0,1)\big),\big(\big(-\tfrac{4861}{9600},\tfrac{1020}{1199}\big),\ (0,1)\big),\big(\big(-\tfrac{1064}{2059},\tfrac{166081}{196678}\big),\ (0,1)\big), \\
\big(\big(-\tfrac{545}{1034},\tfrac{168253}{200773}\big),\ (0,1)\big),\ \big(\big(\tfrac{13}{50},\tfrac{24}{25}\big),\ (0,1)\big),\ \big(\big(\tfrac{13}{51},\tfrac{49}{51}\big),\ (0,1)\big),\ \big(\big(-\tfrac{45}{82},\tfrac{34}{41}\big), \\
(0,1)\big),\ \big(\big(-\tfrac{1135}{2048},\tfrac{1699}{2048}\big),\ (0,1)\big),\ \big(\big(-\tfrac{1125}{2048},\tfrac{851}{1024}\big),\ (0,1)\big),\ \big(\big(-\tfrac{1123}{2048},\tfrac{1701}{2048}\big), \\
(0,1)\big),\ \big(\big(-\tfrac{1083}{2048},\tfrac{869}{1024}\big),\ (0,1)\big),\ \big(\big(-\tfrac{1075}{2048},\tfrac{433}{512}\big),\ (0,1)\big),\ \big(\big(-\tfrac{1069}{2048},\tfrac{873}{1024}\big), \\
(0,1)\big),\big(\big(-\tfrac{531}{1024},\tfrac{1745}{2048}\big),\ (0,1)\big),\big(\big(-\tfrac{529}{1024},\tfrac{875}{1024}\big),\ (0,1)\big),\big(\big(\tfrac{505}{2048},\tfrac{991}{1024}\big),\ (0,1)\big), \\
\big(\big(\tfrac{511}{2048},\tfrac{1983}{2048}\big),\ (0,1)\big), \quad \big(\big(\tfrac{513}{2048},\tfrac{991}{1024}\big),\ (0,1)\big), \quad \big(\big(\tfrac{135}{512},\tfrac{987}{1024}\big),\ (0,1)\big), \\
\big(\big(\tfrac{129106}{516339},\tfrac{2147435}{2219844}\big),\ (0,1)\big), \quad \big(\big(\tfrac{1}{212}\big(-140+\sqrt{573}\big),\tfrac{\sqrt{11}}{4}\big), \\
(0,3)\big), \quad \Big(\Big(\tfrac{-550-\sqrt{42130}}{1500},\tfrac{\sqrt{11}\left(-25+2\sqrt{42130}\right)}{1500}\Big),\ (0,1)\Big), \\
\big(\big(\tfrac{1}{48}\big(-33+\sqrt{93}\big),\tfrac{1}{48}\sqrt{11}\big(3+\sqrt{93}\big)\big),\ (0,1)\big), \quad \Big(\Big(\tfrac{1639+\sqrt{10021}}{6600},\tfrac{539+\sqrt{10021}}{200\sqrt{11}}\Big), \\
(0,1)\big)\Big),
\end{gathered}
$$

and let $C_0^{(2)}(k)$ *denote the ultimate period of the orbit of* $(a_{2,k},b_{2,k})_2$ *under* $\mathcal{T}_{2,(x_{2,k},y_{2,k})}$ *for all* $k \in \{1,\dots 45\}$ *and* $C_0^{(11)}(k)$ *the ultimate period of the orbit of* $(a_{11,k},b_{11,k})_{11}$ *under* $\mathcal{T}_{11,(x_{11,k},y_{11,k})}$ *for all* $k \in \{1,\dots 47\}$*. Furthermore let for all* $k \in \mathbb{Z}$*:*

$$
\begin{aligned}
C_1^{(d)}(k) &:= ((-k,1)_d,(k,-1)_d) \\
C_2^{(d)}(k) &:= ((-k,1)_d,(k+1,-1)_d).
\end{aligned}
$$

Theorem 5. *The sets* $\mathcal{D}_{1,2}^{(0)}$ *and* $\mathcal{D}_{1,11}^{(0)}$ *do not contain any weakly critical points (and thus no critical points)* r *satisfying* $r \in \overline{\mathcal{D}_{1,2}^{(0)}}$ *and* $r \in \overline{\mathcal{D}_{1,11}^{(0)}}$ *respectively. More precisely the circle of radius* 0.99 *around the origin contains the sets* $\mathcal{D}_{1,2}^{(0)}$ *and* $\mathcal{D}_{1,11}^{(0)}$*.*

Proof. For any cycle π of complex numbers let $\overline{\pi}$ denote the cycle one gets if all elements of π are replaced by their complex conjugates. The cutout sets of the cycles $C_1^{(2)}(k), C_2^{(2)}(k),\ k \in \mathbb{Z},\ C_0^{(2)}(1),\dots,C_0^{(2)}(45),\ \overline{C_0^{(2)}}(1),\dots,\overline{C_0^{(2)}}(45)$, and $C_1^{(11)}(k), C_2^{(11)}(k),\ k \in \mathbb{Z},\ C_0^{(11)}(1),\dots,C_0^{(11)}(47),\ \overline{C_0^{(11)}}(1),\dots,\overline{C_0^{(11)}}(47)$ respectively, completely cover the ring centered at the origin in the complex plane with inner radius $\frac{99}{100}$ and outer radius 1. Figures 3 and 3 show the cutout sets for the cases $d = 2$ and $d = 11$ respectively. The list has been found by a combination of a variant of Algorithm 1 with manual search. $\square$

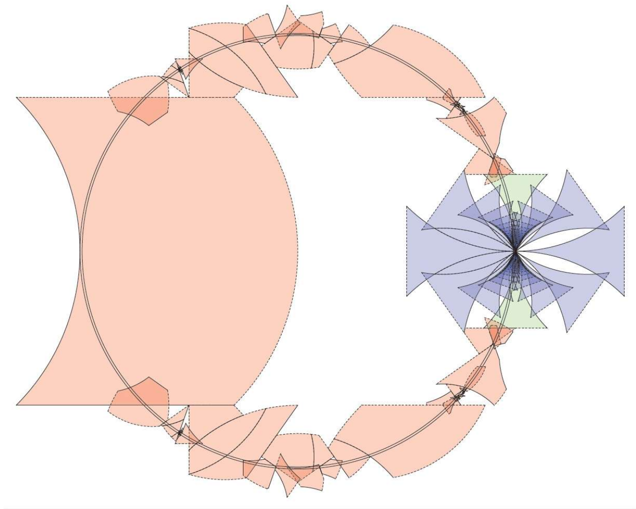

Figure 3: Cutout areas of $\mathcal{D}_{1,2}$ which covers the annulus with radii 99/100 and 1. The green area represents the first cutout area, the blue ones are the two infinite sequences.

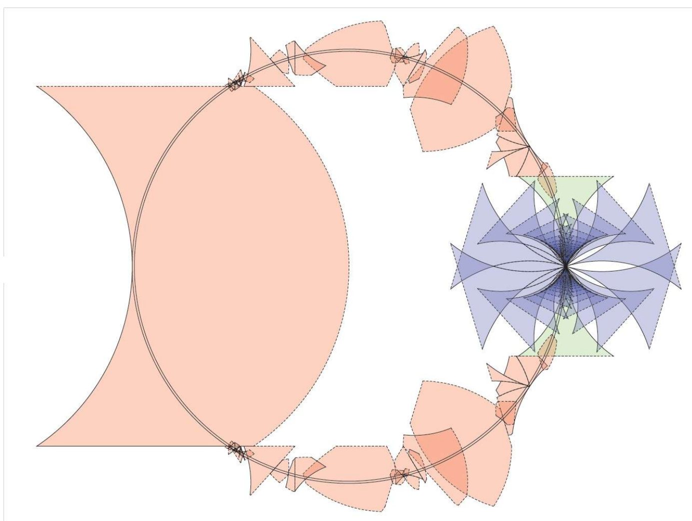

Figure 4: Cutout areas of $\mathcal{D}_{1,11}$ which covers the annulus with radii 99/100 and 1. The green area represents the first cutout area, the blue ones are the two infinite sequences.

4 Conclusion and further work

In this paper shift radix systems have been defined over the complex field (Definition 4), and the one dimensional case has been investigated more precisely.
This can be continued to investigate polynomials and vectors with greater degree, Hausdorff dimensions can be calculated more precisely, or SRS over other Euclidean domains can be investigated as well.

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Received 23rd June 2015