Acta Cybernetica **22** (2016) 807–822.

Bounds on the Stability Number of a Graph via the Inverse Theta Function

Miklós Ujvári*

Abstract

In the paper we consider degree, spectral, and semidefinite bounds on the stability number of a graph. The bounds are obtained via reformulations and variants of the inverse theta function, a notion recently introduced by the author in a previous work.

Keywords: stability number, inverse theta number

1 Introduction

In this paper we provide several new descriptions and variants of the inverse theta function, a notion recently introduced by the author (see [10]). We also present some applications in the stable set problem, bounds on the cardinality of a maximum stable set in a graph.

We start the paper with describing sandwich theorems on the inverse theta number and its predecessor, the theta number (see [4]). First we fix some notation. Let $n \in \mathcal{N}$, and let $G = (V(G), E(G))$ be an undirected graph, with vertex set $V(G) = \{1, \ldots, n\}$, and with edge set $E(G) \subseteq \{\{i, j\} : i \neq j\}$. Let $A(G)$ be the 0-1 adjacency matrix of the graph G, that is let

$$A(G) := (a_{ij}) \in \{0,1\}^{n \times n}, \text{ where } a_{ij} := \begin{cases} 0, & \text{if } \{i,j\} \notin E(G), \\ 1, & \text{if } \{i,j\} \in E(G). \end{cases}$$

The complementary graph $\overline{G}$ is the graph with adjacency matrix

$$A(\overline{G}) := J - I - A(G),$$

where I is the identity matrix, and J denotes the matrix with all elements equal to one. The disjoint union of the graphs G_1 and G_2 is the graph $G_1 + G_2$ with adjacency matrix

$$A(G_1 + G_2) := \begin{pmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{pmatrix}.$$

*H-2600 Vác, Szent János utca 1., Hungary. E-mail: `ujvarim@cs.elte.hu`

DOI: 10.14232/actacyb.22.4.2016.5

Let $(\delta_1, \dots, \delta_n)$ be the sum of the row vectors of the adjacency matrix $A(G)$. The elements of this vector are the degrees of the vertices of the graph G. We define similarly the values $\overline{\delta}_1, \dots, \overline{\delta}_n$ in the complementary graph $\overline{G}$ instead of G. Let Δ_G (resp. μ_G) be the maximum (resp. the arithmetic mean) of the degrees in the graph G. Note that

$$\mu_{\overline{G}} = n - 1 - \mu_G, \; \mu_{G_1+G_2} = \frac{n_1\mu_{G_1} + n_2\mu_{G_2}}{n_1 + n_2}. \tag{1}$$

By Rayleigh's theorem (see [7]) for a symmetric matrix $M = M^T \in \mathcal{R}^{n\times n}$ the minimum and maximum eigenvalue, λ_M resp. Λ_M, can be expressed as

$$\lambda_M = \min_{||u||=1} u^T M u, \; \Lambda_M = \max_{||u||=1} u^T M u.$$

By the Perron-Frobenius theorem (see [6]) for an elementwise nonnegative symmetric matrix $M = M^T \in \mathcal{R}_+^{n\times n}$ the maximum is attained for a nonnegative unit (eigen)vector: we have $\Lambda_M = u^T M u$ for some $u \in \mathcal{R}_+^n$, $u^T u = 1$. Furthermore, if $M = M^T \in \mathcal{R}_+^{n\times n}$, then $-\lambda_M \le \Lambda_M$.

The maximum (resp. minimum) eigenvalue of the adjacency matrix $A(G)$ is denoted by Λ_G (resp. λ_G). By Exercise 11.14 in [5], we have

$$\mu_G, \sqrt{\Delta_G} \le \Lambda_G \le \Delta_G, \sqrt{\mu_G(n-1)}. \tag{2}$$

The set of the n by n real symmetric positive semidefinite matrices will be denoted by $\mathcal{S}_+^n$, that is

$$\mathcal{S}_+^n := \left\{ M \in \mathcal{R}^{n\times n} : M = M^T, \, u^T M u \ge 0 \; (u \in \mathcal{R}^n) \right\}.$$

For example, the Laplacian matrix of the graph G,

$$L(G) := D_{\delta_1,\dots,\delta_n} - A(G) \in \mathcal{S}_+^n.$$

(Here $D_{\delta_1,\dots,\delta_n}$ denotes the diagonal matrix with diagonal elements $\delta_1, \dots, \delta_n$.)

It is well-known (see [7]), that the following statements are equivalent for a symmetric matrix $M = (m_{ij}) \in \mathcal{R}^{n\times n}$: a) $M \in \mathcal{S}_+^n$; b) $\lambda_M \ge 0$; c) M is Gram matrix, that is $m_{ij} = v_i^T v_j$ $(i, j = 1, \dots, n)$ for some vectors $v_1, \dots, v_n$. Furthermore, by Lemma 2.1 in [9], the set $\mathcal{S}_+^n$ can be described as

$$\mathcal{S}_+^n = \left\{ \left(\frac{a_i^T a_j}{(a_i a_j^T)_{11}} - 1 \right)_{i,j=1}^n \;\middle|\; \begin{array}{l} m \in \mathcal{N}, \, a_i \in \mathcal{R}^m \; (1 \le i \le n) \\ a_i^T a_i = 1 \; (1 \le i \le n) \end{array} \right\}. \tag{3}$$

The stability number, $\alpha(G)$, is the maximum cardinality of the (so-called stable) sets $S \subseteq V(G)$ such that $\{i, j\} \subseteq S$ implies $\{i, j\} \notin E(G)$. The chromatic number, $\chi(G)$, is the minimum number of stable sets covering the vertex set $V(G)$.

Let us define an *orthonormal representation* of the graph G (shortly, o.r. of G) as a system of vectors $a_1, \dots, a_n \in \mathcal{R}^m$ for some $m \in \mathcal{N}$, satisfying

$$a_i^T a_i = 1 \; (i = 1, \dots, n), \; a_i^T a_j = 0 \; (\{i, j\} \in E(\overline{G})).$$

In the seminal paper [4] L. Lovász proved the following result, now popularly called *sandwich theorem*, see [2]:

$$\alpha(G) \le \vartheta(G) \le \chi(\overline{G}), \tag{4}$$

where $\vartheta(G)$ is the *Lovász number* of the graph G, defined as

$$\vartheta(G) := \inf\left\{\max_{1\le i\le n} \frac{1}{(a_i a_i^T)_{11}} : a_1,\dots,a_n \text{ o.r. of } G\right\}.$$

The Lovász number has several equivalent descriptions, see [4]. For example, by (3) and standard semidefinite duality theory (see e.g. [8]), it is the common optimal value of the Slater-regular primal-dual semidefinite programs

$$(TP) \qquad \min \lambda, \begin{cases} x_{ii} = \lambda - 1 \ (i \in V(G)), \\ x_{ij} = -1 \ (\{i,j\} \in E(\overline{G})), \\ X = (x_{ij}) \in \mathcal{S}_+^n,\ \lambda \in \mathcal{R} \end{cases}$$

and

$$(TD) \qquad \max \operatorname{tr}(JY), \begin{cases} \operatorname{tr}(Y) = 1, \\ y_{ij} = 0 \ (\{i,j\} \in E(G)), \\ Y = (y_{ij}) \in \mathcal{S}_+^n. \end{cases}$$

(Here tr stands for trace.) Reformulating the program (TD), Lovász derived the following dual description of the theta number (Theorem 5 in [4]):

$$\vartheta(G) = \max\left\{\sum_{i=1}^n (b_i b_i^T)_{11} : b_1,\dots,b_n \text{ o.r. of } \overline{G}\right\}. \tag{5}$$

Analogously, the *inverse theta number*, $\iota(G)$, satisfies the *inverse sandwich inequalities*,

$$n^2/\vartheta(\overline{G}), (\alpha(G))^2 + n - \alpha(G) \le \iota(G) \le n\vartheta(G), \tag{6}$$

see [10], and (19) for an extension. Here the inverse theta number, defined as

$$\iota(G) := \inf\left\{\sum_{i=1}^n \frac{1}{(a_i a_i^T)_{11}} : a_1,\dots,a_n \text{ o.r. of } G\right\},$$

equals the common attained optimal value of the primal-dual semidefinite programs

$$(TP^-) \qquad \inf \operatorname{tr}(Z) + n,\ z_{ij} = -1 \ (\{i,j\} \in E(\overline{G})),\ Z = (z_{ij}) \in \mathcal{S}_+^n,$$

$$(TD^-) \qquad \sup \operatorname{tr}(JM), \begin{cases} m_{ii} = 1 \ (i = 1,\dots,n), \\ m_{ij} = 0 \ (\{i,j\} \in E(G)), \\ M = (m_{ij}) \in \mathcal{S}_+^n. \end{cases}$$

Moreover, rewriting the feasible solution M of the program (TD^-) as the Gram matrix $M = (b_i^T b_j)$ for some vectors $b_1,\dots,b_n \in \mathcal{R}^m$, we obtain the following analogue of (5):

$$\iota(G) = \max \left\{ \sum_{i,j=1}^{n} b_i^T b_j : b_1, \ldots, b_n \text{ o.r. of } \overline{G} \right\}. \tag{7}$$

The structure of the paper is as follows: In Section 2 we will describe a refinement of (7) and also several new descriptions of the inverse theta function (with well-known analogues in the theory of the theta function). Some of these results will be applied in Section 3, where we present two new lower bounds for the stability number of a graph, and examine their additivity properties. Finally, in Section 4 we study three variants of the inverse theta function, and derive further bounds in the stable set problem.

2 New descriptions of $\iota(G)$

In this section we will describe three reformulations of the inverse theta number of a graph G. The results have analogues in the theory of the theta function, which we will mention in chronological order.

Let us denote by $\mathcal{A}_G$ the following set of matrices:

$$\mathcal{A}_G := \left\{ A = (a_{ij}) \in \mathcal{R}^{n\times n} \;\middle|\; \begin{array}{l} a_{ii} = 0 \ (i = 1, \ldots, n), \\ a_{ij} = 0 \ (\{i,j\} \in E(G)), \\ a_{ij} = a_{ji} \ (\{i,j\} \in E(\overline{G})) \end{array} \right\}.$$

We will describe bounds for the minimum eigenvalue λ_A with $A \in \mathcal{A}_G$.

First, we have for $A \in \mathcal{A}_G$ the lower bounds

$$\lambda_A \geq -\Lambda_{|A|} \geq -\Lambda_{\overline{G}} \cdot \max_{i,j} |a_{ij}|, \tag{8}$$

by Rayleigh's theorem and the Perron-Frobenius theorem. (Here $|A| \in \mathcal{R}^{n\times n}$ denotes the elementwise maximum of the matrices A and $-A$.)

On the other hand, using an equivalent form of the reformulation

$$\vartheta(G) = \max \left\{ \Lambda_M \;\middle|\; \begin{array}{l} m_{ii} = 1 \ (i = 1, \ldots, n), \\ m_{ij} = 0 \ (\{i,j\} \in E(G)), \\ M = (m_{ij}) \in \mathcal{S}_+^n \end{array} \right\},$$

(see for example [2], [10]), L. Lovász proved in Theorem 6 of [4] the upper bound

$$\lambda_A \leq \frac{\Lambda_A}{1 - \vartheta(G)} \ (A \in \mathcal{A}_G). \tag{9}$$

Analogously, as a consequence of the next theorem, we have also the upper bound

$$\lambda_A \leq \frac{\mathrm{tr}\,(JA)}{n - \iota(G)} \ (A \in \mathcal{A}_G). \tag{10}$$

(Note that by Rayleigh's theorem $\mathrm{tr}\,(JA) \le n\Lambda_A$, and by the inverse sandwich theorem $\iota(G) - n \le n(\vartheta(G) - 1)$ so there is no obvious dominance relation between the bounds in (9) and (10).)

Theorem 2.1. *The program*

$$(P_1): \qquad \sup n + \frac{\mathrm{tr}\,(JA)}{-\lambda_A}, \; \{A \in \mathcal{A}_G$$

has attained optimal value $\iota(G)$.

Proof. The variable transformations

$$M_A := I + \frac{1}{-\lambda_A} A, \; A_M := M - I$$

show that programs (TD^-) and (P_1) are equivalent: if A and M are feasible solutions of (P_1) and (TD^-), respectively, then M_A and A_M are feasible solutions of the other program such that between the corresponding values the inequalities

$$\mathrm{tr}\,(JM_A) \ge n + \frac{\mathrm{tr}\,(JA)}{-\lambda_A}, \; n + \frac{\mathrm{tr}\,(JA_M)}{-\lambda_{A_M}} \ge \mathrm{tr}\,(JM)$$

hold. Hence, the two programs have the same (attained) optimal value. □

A different approach leads to another description of the inverse theta number.

Karger, Motwani, and Sudan proved the reformulation

$$\frac{1}{1 - \vartheta(G)} = \min \left\{ \nu \;\middle|\; \begin{array}{l} n_{ii} = 1 \; (i = 1, \dots, n), \\ n_{ij} = \nu \; (\{i, j\} \in E(\overline{G})), \\ N = (n_{ij}) \in \mathcal{S}_+^n, \; \nu \in \mathcal{R} \end{array} \right\},$$

and used a variant of this theorem in their graph colouring algorithm. (See [3] for a summary of related results.) By the inverse sandwich theorem we have the lower bound

$$\frac{1}{1 - \vartheta(G)} \ge \frac{n}{n - \iota(G)}; \tag{11}$$

we will show that this latter value can be obtained as the optimal value of a semidefinite program, too.

Let us consider the primal-dual semidefinite programs

$$(P_2): \qquad \sup -\mathrm{tr}\, B, \; \left\{ \begin{array}{l} b_{11} - b_{ii} = 0 \; (i = 2, \dots, n), \\ b_{ij} = 0 \; (\{i, j\} \in E(G)), \\ \mathrm{tr}\,((J - I)B) = 1, \; B = (b_{ij}) \in \mathcal{S}_+^n, \end{array} \right.$$

$$(D_2): \qquad \inf \gamma, \; \left\{ \begin{array}{l} \mathrm{tr}\, C = n, \\ c_{ij} = \gamma \; (\{i, j\} \in E(\overline{G})), \\ C = (c_{ij}) \in \mathcal{S}_+^n, \; \gamma \in \mathcal{R}. \end{array} \right.$$

The programs have common attained optimal value by standard semidefinite duality theory, see for example [8].

Theorem 2.2. *The programs* (P_2) *and* (D_2) *have (common attained) optimal value* $n/(n-\iota(G))$.

Proof. Similarly as in the proof of Theorem 2.1, the variable transformations

$$M_B := \frac{n}{\operatorname{tr} B} B,\ B_M := \frac{1}{\operatorname{tr}(JM) - n} M$$

show the equivalence of programs (P_2) and $n/(n-(TD^-))$, where the latter program can be obtained from (TD^-) formally exchanging its value function $\operatorname{tr}(JM)$ for $n/(n - \operatorname{tr}(JM))$ and adding the extra constraint $\operatorname{tr}(JM) > n$. □

It is left to the reader to prove that the program

$$\inf \frac{1}{1-\Lambda_R},\ \begin{cases} \operatorname{tr} R = n, \\ r_{ij} = 1\ (\{i,j\} \in E(\overline{G})) \\ R = (r_{ij}) = (r_{ji}) \in \mathcal{R}^{n\times n} \end{cases}$$

is equivalent with both (D_2) and $n/(n-(TP^-))$.

Now, we turn to the third description of the inverse theta number.

We will use the following lemma, a slight modification of (7).

Lemma 2.1. *For any graph* G,

$$\iota(G) = \sup\left\{ \sum_{i,j=1}^{n} \hat{b}_i^T \hat{b}_j \ \middle|\ \begin{array}{l} \hat{b}_1, \ldots, \hat{b}_n \ \textit{o.r. of}\ \overline{G} \\ (\hat{b}_i)_1 = e_1^T \hat{b}_i > 0 \ \textit{for}\ i = 1, \ldots, n \end{array} \right\},$$

with e_1 *denoting the vector* $(1,0,\ldots,0)^T$.

Proof. Let (b_i) be an orthonormal representation of $\overline{G}$ such that

$$\iota(G) = \sum_{i,j=1}^{n} b_i^T b_j$$

(that is an optimal solution in (7)). For $0 < \varepsilon < 1$, let us define an orthonormal representation $(\hat{b}_i(\varepsilon))$ of $\overline{G}$ the following way:

$$(\hat{b}_i(\varepsilon)) := \begin{pmatrix} \sqrt{1-\varepsilon^2} \cdot O \\ \varepsilon b_1, \ldots, \varepsilon b_n \end{pmatrix},$$

where $O \in \mathcal{R}^{n\times n}$ is an orthogonal matrix satisfying $e_1^T O > 0$. Note that then $e_1^T \hat{b}_i(\varepsilon) > 0$ holds for all i. On the other hand, it can easily be verified that

$$\sum_{i,j=1}^{n} \hat{b}_i^T(\varepsilon) \hat{b}_j(\varepsilon) \to \iota(G)\ (\varepsilon \to 1).$$

Hence, we have proved

$$\iota(G) \le \sup \left\{ \sum_{i,j=1}^{n} \hat{b}_i^T \hat{b}_j \;\middle|\; \begin{array}{l} \hat{b}_1, \dots, \hat{b}_n \text{ o.r. of } \overline{G} \\ e_1^T \hat{b}_i > 0 \text{ for } i = 1, \dots, n \end{array} \right\},$$

which is the nontrivial part of the lemma. □

Applying the variable transformation described in (3) to the program in Lemma 2.1, as an immediate consequence we obtain an analogue of Theorem 2.2 in [9].

Theorem 2.3. *The optimal value of the program*

$$(P_3): \qquad \sup \sum_{i,j=1}^{n} \frac{d_{ij}+1}{\sqrt{(d_{ii}+1)\cdot(d_{jj}+1)}}, \left\{ \begin{array}{l} d_{ij} = -1 \ (\{i,j\} \in E(G)), \\ D = (d_{ij}) \in \mathcal{S}_+^n \end{array} \right.$$

equals $\iota(G)$. □

We will apply Theorem 2.3 in the next section for obtaining lower bounds in the stable set problem.

3 Lower bounds on $\alpha(G)$

In this section we will describe two lower bounds on the stability number of a graph G, and examine their additivity properties.

Note that the

$$Z_1 := L(\overline{G}),\ Z_2 := \Lambda_{\overline{G}} I - A(\overline{G})$$

feasible solutions in (TP^-) give the inequalities

$$\sqrt{\iota(G)} \le \sqrt{n(\mu_{\overline{G}}+1)}, \sqrt{n(\Lambda_{\overline{G}}+1)}. \tag{12}$$

By Exercises 11.20 and 11.14 in [5], we have

$$\chi(G) \le \Lambda_G + 1 \le \sqrt{\mu_G(n-1)} + 1,\ \ \mu_G \le \Lambda_G.$$

On the other hand, easy calculation verifies

$$\sqrt{\mu_G(n-1)} + 1 \le \sqrt{n(\mu_G+1)}.$$

Hence, we have besides (12) also

$$\chi(\overline{G}) \le \sqrt{n(\mu_{\overline{G}}+1)} \le \sqrt{n(\Lambda_{\overline{G}}+1)}. \tag{13}$$

On the dual side instead of $\sqrt{\iota(G)}, \chi(\overline{G})$ we can approximate $\iota(G)/n, \alpha(G)$. Note that

$$D_1 := L(G),\ D_2 := \Lambda_G I - A(G)$$

are feasible solutions of the program (P_3) in Theorem 2.3. This fact implies the version of the following theorem, where $\alpha(G)$ is exchanged for $\iota(G)/n$. (For analogous results with $\vartheta(G)$, see [9].)

Theorem 3.1. *For any graph G,*

a)

$$\alpha'(G) := 1 + \sum_{\{i,j\}\in E(\overline{G})} \frac{2/n}{\sqrt{(\delta_i+1)\cdot(\delta_j+1)}} \leq \alpha(G);$$

b)

$$\alpha''(G) := 1 + \frac{\mu_{\overline{G}}}{\Lambda_G+1} \leq \alpha(G).$$

Proof. By Exercise 11.14 in [5] we have $\mu_G \leq \Lambda_G$. Using this relation it is immediate that

$$\frac{n}{\Lambda_G+1} \leq \alpha''(G) \leq \frac{n}{\mu_G+1}. \tag{14}$$

We will show that the inequalities

$$\frac{n}{\mu_G+1} \leq \alpha'(G) \leq \sum_{i=1}^{n} \frac{1}{\delta_i+1} \tag{15}$$

hold also, from which the theorem follows, as

$$\sum_{i=1}^{n} \frac{1}{\delta_i+1} \leq \alpha(G) \tag{16}$$

by the Caro-Wei theorem (see [1], or for another proof [9]).

First, using the obvious inequality

$$\frac{2}{\sqrt{\delta_i+1}\cdot\sqrt{\delta_j+1}} \leq \frac{1}{\delta_i+1} + \frac{1}{\delta_j+1}, \tag{17}$$

we obtain

$$\begin{aligned} \alpha'(G) &\leq 1 + \frac{1}{n}\cdot\sum_{i=1}^{n} \frac{1}{\delta_i+1}\cdot(n-1-\delta_i) \\ &= \sum_{i=1}^{n} \frac{1}{\delta_i+1}. \end{aligned}$$

On the other hand, we will verify the relation

$$\alpha'(G) \geq \frac{n}{\mu_G+1}. \tag{18}$$

Using the arithmetic mean-harmonic mean inequality, it is easy to show that

$$\begin{aligned} \alpha'(G) &\geq 1 + \frac{4}{n}\cdot\sum_{\{i,j\}\in E(\overline{G})} \frac{1}{\delta_i+1+\delta_j+1} \\ &\geq 1 + \frac{1}{n}(n\mu_{\overline{G}})^2 \Bigg/ \sum_{\{i,j\}\in E(\overline{G})} (\delta_i+1+\delta_j+1)\,. \end{aligned}$$

Hence, to prove (18), it is enough to verify that

$$n\mu_{\overline{G}}(\mu_G + 1) \geq \sum_{\{i,j\} \in E(\overline{G})} (\delta_i + 1 + \delta_j + 1)$$

holds. This inequality can be rewritten as

$$\sum_{i=1}^{n}(n-1-\delta_i) \cdot \sum_{i=1}^{n}(\delta_i + 1) \geq n \cdot \sum_{i=1}^{n}(\delta_i + 1)(n - 1 - \delta_i),$$

and thus is a consequence of the Cauchy-Schwarz inequality. The proof of (18) is complete, as well. □

The following theorem describes additivity properties of the bounds α', α''. (For additivity properties of $\vartheta(G)$, see Sections 18, 19 in [2].)

Theorem 3.2. *With the lower bounds $\ell = \alpha', \alpha''$ we have*
a) $\ell(G_1 + G_2) \leq \ell(G_1) + \ell(G_2)$,
b) $\ell(\overline{G_1 + G_2}) \leq \max\{\ell(\overline{G_1}), \ell(\overline{G_2})\}$,
for any graphs G_1, G_2.

Proof. Case 1: $\ell = \alpha'$. a) Rewriting the statement, we have to verify

$$\sum_{i \in V(G_1),\, j \in V(G_2)} \frac{2}{\sqrt{(\delta_i + 1)(\delta_j + 1)}} \leq \alpha'(G_1)n_2 + \alpha'(G_2)n_1,$$

that is (without loss of generality assuming $G_1 = G_2 = G$)

$$\left(\sum_{i=1}^{n} \frac{1}{\sqrt{\delta_i + 1}} \right)^2 \leq \alpha'(G)n.$$

In other words, we have to prove the inequality

$$\sum_{i=1}^{n} \frac{1}{\delta_i + 1} + \sum_{\{i,j\} \in E(G)} \frac{2}{\sqrt{(\delta_i + 1)(\delta_j + 1)}} \leq n,$$

which follows immediately applying (17).

b) is obvious, as

$$\begin{aligned} \alpha'(\overline{G_1 + G_2}) &\leq \frac{\alpha'(\overline{G_1})n_1 + \alpha'(\overline{G_2})n_2}{n_1 + n_2} \\ &\leq \max\{\alpha'(\overline{G_1}), \alpha'(\overline{G_2})\} \end{aligned}$$

hold.

Case 2: $\ell = \alpha''$. By Rayleigh's theorem the formulas

$$\begin{aligned} \Lambda_{G_1+G_2} &\geq \max\{\Lambda_{G_1}, \Lambda_{G_2}\}, \\ \Lambda_{\overline{G_1+G_2}} &\geq \max\{\Lambda_{\overline{G_1}}, \Lambda_{\overline{G_2}}\} \end{aligned}$$

hold. The statements a) and b), respectively, are straightforward consequences of these inequalities, after applying (1): For example, a) can be reduced this way to the inequality

$$\frac{n_2\mu_{G_1} + n_1\mu_{G_2}}{n_1 + n_2} \leq \max\{\Lambda_{G_1}, \Lambda_{G_2}\},$$

which holds true, as $\mu_G \leq \Lambda_G$ for any graph G, by Exercise 11.14 in [5]. $\square$

Additivity properties of a lower bound on the stability number can be applied for strengthening the bound if the given graph or its complementer is not connected. In fact, if

$$G = G_1 + G_2 \text{ (or } \overline{G} = H_1 + H_2)$$

with some graphs G_1, G_2 (H_1, H_2), then $\alpha(G)$ is equal to

$$\alpha(G_1) + \alpha(G_2) \ (\max\{\alpha(\overline{H_1}), \alpha(\overline{H_2})\}).$$

Hence, both $\ell(G)$ and the, by additivity stronger, bound

$$\ell(G_1) + \ell(G_2) \ (\max\{\ell(\overline{H_1}), \ell(\overline{H_2})\})$$

are lower bounds on $\alpha(G)$.

It is left to the reader to adapt this bound-strengthening method to upper bounds $u(G)$ on the chromatic number $\chi(G)$.

Summarizing, the so-called *weak sandwich theorems* (see [9])

$$\ell(G) \leq \alpha(G) \ \chi(G) \leq u(G)$$

involve the bounds

$$\ell(G) = \alpha'(G), \alpha''(G), \ u(G) := \sqrt{n(\mu_G + 1)}, \sqrt{n(\Lambda_G + 1)}$$

in inverse theta number theory. In the next section we turn to the inverse sandwich theorem and its strengthened version.

4 Upper bounds on $\alpha(G)$

In this section we introduce three variants of the inverse theta number. They constitute bounds for the stability numbers of G and $\overline{G}$.

First, let us derive a bound from the original version $\iota(G)$. Let $S \subseteq V(G)$ be a stable set with cardinality $\#S = \alpha(G)$, and let $\varepsilon > 0$. Let us define the matrix $Z = Z(\varepsilon) \in \mathcal{R}^{n\times n}$ the following way: let $Z := (z_{ij})$, where

$$z_{ij} := \begin{cases} \varepsilon(n - \#S) + 0, & \text{if } i, j \in S, \\ 1/\varepsilon + (n - \#S - 1), & \text{if } i = j \notin S, \\ 0 + (-1), & \text{if } i, j \notin S,\ i \neq j, \\ (-1) + 0 & \text{otherwise.} \end{cases}$$

It can easily be verified using Schur complements (see [6]) that $Z \in \mathcal{S}^n_+$. (This statement holds even without adding the second terms in the definition of the elements z_{ij}.) For $\varepsilon = 1/\sqrt{\#S}$ the value of Z in (TP^-) satisfies

$$\mathrm{tr}\,(Z) + n = \left(n - \alpha(G) + \sqrt{\alpha(G)}\right)^2.$$

As this value is at least $\iota(\overline{G})$, so we obtained

Proposition 4.1. *For any graph G, we have*

$$\iota(\overline{G}) \leq \left(n - \alpha(G) + \sqrt{\alpha(G)}\right)^2, \tag{19}$$

in other words

$$\alpha(G) \leq \frac{1}{4}\left(1 + \sqrt{1 + 4\left(n - \sqrt{\iota(\overline{G})}\right)}\right)^2 \tag{20}$$

holds. □

We remark that the upper bound in (20) is between the values

$$n + 1 - \sqrt{\iota(\overline{G})},\, n + 1 - \frac{\iota(\overline{G})}{n}$$

as it can easily be verified.

Proposition 4.1 allows a strengthening: a ι, ι^+ exchange in (20), where we add the $z_{ij} \geq -1$ constraints in (TP^-). Let us denote by $\iota^+(G)$ the common attained optimal value of the Slater regular primal-dual semidefinite programs

$$(P^+): \qquad \inf n + \mathrm{tr}\, Z^+, \begin{cases} z^+_{ij} = -1 \ (\{i,j\} \in E(\overline{G})), \\ z^+_{ij} \geq -1 \ (\{i,j\} \in E(G)), \\ Z^+ = (z^+_{ij}) \in \mathcal{S}^n_+, \end{cases}$$

$$(D^+): \qquad \sup \mathrm{tr}\,(JM^+), \begin{cases} m^+_{ii} = 1 \ (i = 1, \ldots, n), \\ m^+_{ij} \leq 0 \ (\{i,j\} \in E(G)), \\ M^+ = (m^+_{ij}) \in \mathcal{S}^n_+. \end{cases}$$

(Standard semidefinite duality theory can be found for example in [8].)

Theorem 4.1. *For any graph G,*

$$\alpha(G) \leq \frac{1}{4}\left(1+\sqrt{1+4\left(n-\sqrt{\iota^{+}(\overline{G})}\right)}\right)^{2} \tag{21}$$

holds. □

For an analogue in theta function theory, see the results of Szegedy and Meurdesoif concerning the variant

$$\vartheta^{+}(G) := \sup\left\{ \operatorname{tr}(JY^{+}) \;\middle|\; \begin{array}{l} \operatorname{tr} Y^{+} = 1, \\ y^{+}_{ij} \leq 0 \ (\{i,j\} \in E(G)), \\ Y^{+} = (y^{+}_{ij}) \in \mathcal{S}^{n}_{+} \end{array} \right\},$$

the relations $\vartheta(G) \leq \vartheta^{+}(G) \leq \chi(\overline{G})$, e.g. in [3].

Now, we turn to the lower variants of $\iota(G)$. Let us consider the primal-dual semidefinite programs

$$(P'): \qquad \inf n + \operatorname{tr} Z', \left\{ \begin{array}{l} z'_{ij} \leq -1 \ (\{i,j\} \in E(\overline{G})), \\ Z' = (z'_{ij}) \in \mathcal{S}^{n}_{+}, \end{array} \right.$$

$$(D'): \qquad \sup \operatorname{tr}(JM'), \left\{ \begin{array}{l} m'_{ii} = 1 \ (i = 1, \ldots, n), \\ m'_{ij} = 0 \ (\{i,j\} \in E(G)), \\ M' = (m'_{ij}) \in \mathcal{S}^{n}_{+} \cap \mathcal{R}^{n\times n}_{+}. \end{array} \right.$$

The programs have common attained optimal value by standard semidefinite duality theory (see for example [8]), we will denote this value by $\iota'(G)$.

Obviously, $\iota'(G) \leq n\vartheta'(G)$, where $\vartheta'(G)$ is a sharpening of the theta number, due to McEliece, Rodemich, Rumsey, and Schrijver ($\alpha(G) \leq \vartheta'(G) \leq \vartheta(G)$, see for example [3]), defined as

$$\vartheta'(G) := \sup\left\{ \operatorname{tr}(JY') \;\middle|\; \begin{array}{l} \operatorname{tr} Y' = 1, \\ y'_{ij} = 0 \ (\{i,j\} \in E(G)), \\ Y' = (y'_{ij}) \in \mathcal{S}^{n}_{+} \cap \mathcal{R}^{n\times n}_{+} \end{array} \right\}.$$

Besides the mentioned relations

$$\vartheta'(G) \geq \iota'(G)/n,\ \alpha(G), \tag{22}$$

we have also

$$\frac{1}{2}\left(1+\sqrt{4(\iota'(G)-n)+1}\right) \geq \iota'(G)/n,\ \alpha(G) \tag{23}$$

as the following theorem shows. (For analogous results with $\iota(G)$, see [10].)

Theorem 4.2. *For any graph G, we have*

$$\iota'(G) \geq \alpha(G)^{2} + n - \alpha(G),$$

in other words

$$\alpha(G) \leq \frac{1}{2}\left(1+\sqrt{4(\iota'(G)-n)+1}\right)$$

holds.

Proof. Let S be a stable set in G with cardinality $\#S = \alpha(G)$. Let us define the matrix $M' := (m'_{ij}) \in \mathcal{R}^{n\times n}$ the following way: let $m'_{ij} := 1$ if $i, j \in S$ or $i = j$, and let $m'_{ij} := 0$ otherwise. Then, the matrix M' is a feasible solution of the program (D') with corresponding value

$$(\#S)^2 + n - (\#S) \le \iota'(G).$$

Hence, the statement follows. □

The bound in Theorem 4.2 implies

$$\alpha(G) \le \sqrt{\iota'(G)}, \tag{24}$$

and also, by $\iota'(G) \le \iota(G)$, the relations

$$\alpha(G) \le \frac{1}{2}\left(1 + \sqrt{4(\iota(G) - n) + 1}\right) \le \sqrt{\iota(G)} \tag{25}$$

from [10]. It is an open problem whether any of these bounds can be less than $\vartheta(G)$ or even $\vartheta'(G)$ for some graphs.

We mention a related result, see also Theorems 3 and 6 in [4] and Proposition 2.1 in [10], where the bounds in (26) appear as lower and upper bounds for $\vartheta(\overline{G})$ and $\sqrt{\iota(\overline{G})}$, respectively.

Proposition 4.2. *For any graph G, the inequalities*

$$1 + \frac{\Lambda_G}{-\lambda_G} \le \sqrt{n\left(1 + \frac{\mu_G}{-\lambda_G}\right)},\ \Lambda_G + 1 \le \sqrt{n(\mu_G + 1)} \tag{26}$$

hold.

Proof. By (2), it suffices to prove (26) after substituting μ_G with $\Lambda_G^2/(n-1)$. Then, the first inequality follows by

$$\frac{1}{2}\left(\frac{n\Lambda_G}{n-1} - 2 + \sqrt{\left(\frac{n\Lambda_G}{n-1} - 2\right)^2 + 4(n-1)}\right) \ge \frac{\Lambda_G}{-\lambda_G}$$

(note that $-\lambda_G \ge 1$ and $\Lambda_G \le n - 1$), the second inequality is immediate. This finishes the proof. □

Finally, we mention another variant of the inverse theta number, which leads to an interesting weak sandwich theorem.

Let us define $\iota''(G)$ as the common attained optimal value of the primal-dual semidefinite programs

$$(P''): \qquad \inf \operatorname{tr}(JM''), \begin{cases} m''_{ii} = 1 \ (i = 1, \dots, n), \\ m''_{ij} = 0 \ (\{i, j\} \in E(G)), \\ M'' = (m''_{ij}) \in \mathcal{S}^n_+, \end{cases}$$

$$(D''): \qquad \sup n - \operatorname{tr} Z'', \begin{cases} z''_{ij} = 1 \ (\{i, j\} \in E(\overline{G})), \\ Z'' = (z''_{ij}) \in \mathcal{S}^n_+. \end{cases}$$

(See, for example, [8] for standard semidefinite duality theory.)

Theorem 4.3. *For any graph G, the inequalities*
a) $\iota''(G) \le \alpha(\overline{G})$,
b) $\iota''(G) \le n - \chi(\overline{G})$
hold.

Proof. a) Let us introduce the notation

$$M_S := (m_{ij}) \in \mathcal{R}^{n\times n}, \text{ where } m_{ij} := \begin{cases} 1 & \text{if } i,j \in S,\ i = j, \\ -\frac{1}{\#S-1} & \text{if } i,j \in S,\ i \ne j, \\ 0 & \text{otherwise,} \end{cases}$$

for $S \subseteq V(G)$.

Let $S_1, \dots, S_k$ be a stable set partition of $V(G)$ such that the cardinality of the index set $\{i : \#S_i \ge 2\}$ is maximal. Then,

$$\overline{S} := \cup_{i=1}^k \{S_i : \#S_i = 1\}$$

is a stable set in $\overline{G}$. Furthermore, the matrix

$$\sum_{i=1}^k M_{S_i}$$

is feasible in (P'') with corresponding value $\#\overline{S} \le \alpha(\overline{G})$, which completes the proof of statement a).

b) Let $\overline{S}_1, \dots, \overline{S}_\ell$ be disjoint stable sets in $\overline{G}$ covering the vertex set $V(G)$, where $\ell := \chi(\overline{G})$. Then, there exist non-edges $\overline{e}_{pq} \in E(\overline{G})$ between $\overline{S}_p$ and $\overline{S}_q$ for each $1 \le p < q \le \ell$. Let us define a symmetric matrix $M \in \mathcal{R}^{n\times n}$ by writing in it: 1 on diagonal positions, $-1/(\ell-1)$ on the positions corresponding to $\overline{e}_{pq}$, and 0 otherwise. By Gerschgorin's disc theorem (see [7]) the matrix M is positive semidefinite, a feasible solution of the program (P'') with corresponding value $n - \chi(\overline{G})$. This finishes the proof of statement b), too. $\square$

Summarizing, in this section we obtained the

$$(\alpha(G))^2 + n - \alpha(G) \le \iota'(G)$$
$$\iota'(G) \le \iota(G) \le \iota^+(G)$$
$$\iota^+(G) \le \left(n - \alpha(\overline{G}) + \sqrt{\alpha(\overline{G})}\right)^2$$

inverse sandwich theorem as an analogue of Lovász's sandwich theorem.

In the same context we mention also the well-known

$$\chi(\overline{G}) \le n - \nu(G) \le \frac{n + \alpha(G)}{2} \tag{27}$$

sandwich theorem, where $\nu(G)$ denotes the matching number of G, that is the largest number of pairwise disjoint edges in $E(G)$, see Section 7 in [5]. This fact, together with the formulas

$$\alpha(G) \cdot \chi(G) \geq n,\ \chi(G) + \chi(\overline{G}) \leq n + 1 \tag{28}$$

(see Exercise 9.5 in [5]), makes upper bounds for $\alpha(G)$ particularly useful in deriving other (upper and lower) bounds for $\alpha(G), \chi(G)$, for example

$$\alpha(G) \geq 2\vartheta(G) - n, \frac{n}{n + 1 - \vartheta(G)} \tag{29}$$

and

$$\chi(G) \leq n + 1 - \vartheta(G), \frac{n + \vartheta(\overline{G})}{2} \tag{30}$$

via the sandwich theorem.

5 Conclusion

In the paper we studied the inverse theta function: results analogous to sandwich theorems and their strengthened versions from the theory of Lovász's theta number were derived, based on new descriptions of the inverse theta number. Whether the new bounds on the stability number can be tighter than already known ones remained a partly undecided question.

Acknowledgements

I thank one of the anonymous referees of the paper for calling my attention to several relevant references.

References

[1] Alon, N. and Spencer, J. *The Probabilistic Method.* 4th edition, The probabilistic lens after Chapter 6, Wiley, page 100, Hoboken NJ, 2016.

[2] Knuth, D. The sandwich theorem. *Electronic Journal of Combinatorics*, 1:1–48, 1994.

[3] Laurent, M. and Rendl, F. Semidefinite programming and integer programming. In: Aardal, K. et al., editors., *Handbook on Discrete Optimization*, Elsevier B.V., Amsterdam, pages 393–514, 2005.

[4] Lovász, L. On the Shannon capacity of a graph. *IEEE Transactions on Information Theory*, IT-25(1):1–7, 1979.

[5] Lovász, L. *Combinatorial Problems and Exercises.* Corrected reprint of the 1993 second edition, AMS Chelsea Publishing, Providence, RI, 2007.

[6] Serre, D. *Matrices, Theory and Applications.* Translated from the 2001 French original, Graduate Texts in Mathematics vol. 216, Springer-Verlag, New York, 2002.

[7] Strang, G. *Linear Algebra and its Applications.* Academic Press, New York, 1980.

[8] Ujvári, M. A note on the graph-bisection problem. *Pure Mathematics and Applications* 12(1):119–130, 2002.

[9] Ujvári, M. New descriptions of the Lovász number, and the weak sandwich theorem. *Acta Cybernetica*, 20(4):499–513, 2012.

[10] Ujvári, M. Applications of the inverse theta number in stable set problems. *Acta Cybernetica*, 21(3):481–494, 2014.

Received 26th October 2014