Acta Cybernetica **23** (2017) 203–228.

An Algebraic Approach to Energy Problems I *-Continuous Kleene ω-Algebras[‡]

Zoltán Ésik[a], Uli Fahrenberg[b], Axel Legay[c], and Karin Quaas[d]

Abstract

Energy problems are important in the formal analysis of embedded or autonomous systems. With the purpose of unifying a number of approaches to energy problems found in the literature, we introduce energy automata. These are finite automata whose edges are labeled with energy functions that define how energy levels evolve during transitions.

Motivated by this application and in order to compute with energy functions, we introduce a new algebraic structure of *-continuous Kleene ω-algebras. These involve a *-continuous Kleene algebra with a *-continuous action on a semimodule and an infinite product operation that is also *-continuous.

We define both a finitary and a non-finitary version of *-continuous Kleene ω-algebras. We then establish some of their properties, including a characterization of the free finitary *-continuous Kleene ω-algebras. We also show that every *-continuous Kleene ω-algebra gives rise to an iteration semiring-semimodule pair.

Keywords: Energy problem, Kleene algebra, *-continuity, *-continuous Kleene ω-algebra

1 Introduction

Energy problems are concerned with the question whether a given system admits infinite schedules during which (1) certain tasks can be repeatedly accomplished and (2) the system never runs out of energy (or other specified resources). These are important in areas such as embedded systems or autonomous systems and,

[‡]This research was supported by grant no. K 108448 from the National Foundation of Hungary for Scientific Research (OTKA), by ANR MALTHY, grant no. ANR-13-INSE-0003 from the French National Research Foundation, and by Deutsche Forschungsgemeinschaft (DFG), projects QU 316/1-1 and QU 316/1-2.

[a]University of Szeged, Hungary (deceased)

[b]École polytechnique, Palaiseau, France. Most of this work was carried out while this author was still employed at Inria Rennes.

[c]Inria Rennes, France

[d]Universität Leipzig, Germany

DOI: 10.14232/actacyb.23.1.2017.13

starting with [4], have attracted some attention in recent years, for example in [20, 27, 3, 5, 28, 7, 6, 23, 9].

With the purpose of generalizing some of the above approaches, we have in [14, 21] introduced *energy automata*. These are finite automata whose transitions are labeled with *energy functions* which specify how energy values change from one system state to another. Using the theory of semiring-weighted automata [10], we have shown in [14] that energy problems in such automata can be solved in a simple static way which only involves manipulations of energy functions.

In order to put the work of [14] on a more solid theoretical footing and with an eye to future generalizations, we have recently introduced a new algebraic structure of **-continuous Kleene ω-algebras* [12, 13].

A continuous (or complete) Kleene algebra is a Kleene algebra in which all suprema exist and are preserved by products. These have nice algebraic properties, but not all Kleene algebras are continuous, for example the semiring of regular languages over some alphabet. Hence a theory of *-continuous Kleene algebras has been developed to cover this and other interesting cases [25].

For infinite behaviors, complete semiring-semimodule pairs involving an infinite product operation have been developed [19]. Motivated by some examples of structures which are not complete in this sense, for example the energy functions of the preceding section, we generalize the notion of *-continuous Kleene algebra to one of *-continuous Kleene ω-algebra. These are idempotent semiring-semimodule pairs which are not necessarily complete, but have enough suprema in order to develop a fixed-point theory and solve weighted Büchi automata (*i.e.,* to compute infinitary power series).

We will define both a finitary and a non-finitary version of *-continuous Kleene ω-algebras. We then establish several properties of *-continuous Kleene ω-algebras, including the existence of the suprema of certain subsets related to regular ω-languages. Then we will use these results in our characterization of the free finitary *-continuous Kleene ω-algebras. We also show that each *-continuous Kleene ω-algebra gives rise to an iteration semiring-semimodule pair.

Structure of the Paper This is the first in a series of two papers which deal with energy problems and their algebraic foundation. In the present paper, we motivate the introduction of our new algebraic structures by two sections on energy automata (Section 2) and on the algebraic structure of energy functions (Section 3). We then pass to introduce continuous Kleene ω-algebras in Section 4 and to expose the free continuous Kleene ω-algebras in Section 5.

In Section 6 we generalize continuous Kleene ω-algebras to our central notion of *-continuous Kleene ω-algebras and finitary *-continuous Kleene ω-algebras. Section 7 exposes the free finitary *-continuous Kleene ω-algebras; the question whether general free *-continuous Kleene ω-algebras exist is left open.

The penultimate Section 8 shows that every *-continuous Kleene ω-algebra is an iteration semiring-semimodule pair, hence techniques from matrix semiring-semimodule pairs apply. This will be important in the second paper of the series. In Section 9 we concern ourselves with least and greatest fixed points and introduce a notion of Kleene ω-algebra, analogous to the concept of Kleene algebra for least fixed points.

In the second paper of the series [15], we show that one can use matrix operations to solve reachability and Büchi acceptance in weighted automata over *-continuous Kleene ω-algebras, and that energy functions form a *-continuous Kleene ω-algebra. This will allows us to connect the algebraic structures developed in the present paper back to their motivating energy problems.

Acknowledgment The origin of this work is a joint short paper [21] between the last three authors which was presented at the 2012 International Workshop on Weighted Automata: Theory and Applications. After the presentation, the presenter was approached by Zoltán Ésik, who told him that the proper setting for energy problems should be idempotent semiring-semimodule pairs. This initiated a long-lasting collaboration, including several mutual visits, which eventually led to the work presented in this paper and its follow-up [15].

We are deeply indebted to our colleague and friend Zoltán Ésik who taught us all we know about semiring-semimodule pairs and *-continuity. Unfortunately Zoltán could not see this work completed, so any errors are the responsibility of the last three authors.

In honor of Zoltán Ésik, we propose to give the name "*Ésik algebra*" to *-continuous Kleene ω-algebras.

2 Energy Automata

The transition labels on the energy automata which we consider in this paper will be functions which model transformations of energy levels between system states. Such transformations have the (natural) properties that below a certain energy level, the transition might be disabled (not enough energy is available to perform the transition), and an increase in input energy always yields at least the same increase in output energy. Thus the following definition.

Definition 1. *An* energy function *is a partial function $f : \mathbb{R}_{\geq 0} \rightharpoonup \mathbb{R}_{\geq 0}$ which is defined on a closed interval $[l_f, \infty[$ or on an open interval $]l_f, \infty[$, for some lower bound $l_f \geq 0$, and such that for all $x \leq y$ for which f is defined,*

$$yf \geq xf + y - x\,. \tag{$*$}$$

The class of all energy functions is denoted by $\mathcal{F}$.

We will write composition and application of energy functions in diagrammatical order, from left to right. Hence we write $f; g$, or simply fg, for the composition $g \circ f$ and $x; f$ or xf for function application $f(x)$. This is because we will be concerned with *algebras* of energy functions, in which function composition is multiplication, and where it is customary to write multiplication in diagrammatical order.

Thus energy functions are strictly increasing, and in points where they are differentiable, the derivative is at least 1. The inverse functions to energy functions

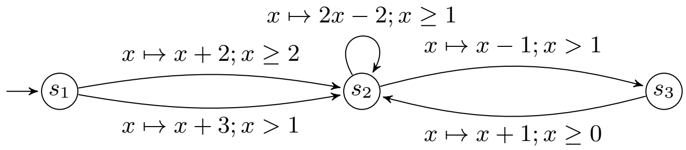

Figure 1: A simple energy automaton.

exist, but are generally not energy functions. Energy functions can be composed, where it is understood that for a composition fg, the interval of definition is $\{x \in \mathbb{R}_{\geq 0} \mid xf \text{ and } xfg \text{ defined}\}$. The following lemma shows an important property of energy functions which we will use repeatedly later, mostly without mention of the lemma.

Lemma 1. *Let $f \in \mathcal{F}$ and $x \in \mathbb{R}_{\geq 0}$.*

- *If $xf < x$, then there is $N \geq 0$ such that xf^n is undefined for all $n \geq N$.*
- *If $xf = x$, then $xf^n = x$ for all $n \geq 0$.*
- *If $xf > x$, then for all $P \in \mathbb{R}$ there is $N \geq 0$ such that $xf^n \geq P$ for all $n \geq N$.*

Proof. In the first case, we have $x - xf = M > 0$. Using $(*)$, we see that $xf^{n+1} \leq xf^n - M$ for all $n \geq 0$ for which xf^{n+1} is defined. Hence the sequence $(xf^n)_{n\geq 0}$ decreases without bound, so that there must be $N \geq 0$ such that xf^N is undefined, and then so is xf^n for any $n > N$.

The second case is trivial. In the third case, we have $xf - x = M > 0$. Again using $(*)$, we see that $xf^{n+1} > xf^n + M$ for all $n \geq 0$. Hence the sequence $(xf^n)_{n\geq 0}$ increases without bound, so that for any $P \in \mathbb{R}$ there must be $N \geq 0$ for which $xf^N \geq P$, and then $xf^n \geq xf^N \geq P$ for all $n \geq N$. $\square$

Example 1. The following example shows that property $(*)$ is not only sufficient for Lemma 1, but in a sense also necessary: Let $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$ and $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be the function $xf = 1 + \alpha x$. Then $yf = xf + \alpha(y - x)$ for all $x \leq y$, so $(*)$ "almost" holds. But $xf^n = \sum_{i=0}^{n-1} \alpha^i + \alpha^n x$ for all $n \in \mathbb{N}$, hence $\lim_{n\to\infty} xf^n = \frac{1}{1-\alpha} < \infty$.

Definition 2. *An* energy automaton (S, s_0, T, F) *consists of a finite set S of states, with initial state $s_0 \in S$, a finite set $T \subseteq S \times \mathcal{F} \times S$ of transitions labeled with energy functions, and a subset $F \subseteq S$ of acceptance states.*

Example 2. Figure 1 shows a simple energy automaton. Here we have used inequalities to give the definition intervals of energy functions, so that for example, the function labeling the loop at s_2 is given by $f(x) = 2x - 2$ for $x \geq 1$ and undefined for $x < 1$.

A finite *path* in an energy automaton is a finite sequence of transitions $\pi = (s_0, f_1, s_1), (s_1, f_2, s_2), \dots, (s_{n-1}, f_n, s_n)$. We use f_π to denote the combined energy function $f_\pi = f_1 f_2 \cdots f_n$ of such a finite path. We will also use infinite paths, but note that these generally do not allow for combined energy functions.

A *global state* of an energy automaton is a pair $q = (s, x)$ with $s \in S$ and $x \in \mathbb{R}_{\geq 0}$. A transition between global states is of the form $((s, x), f, (s', x'))$ such that $(s, f, s') \in T$ and $x' = f(x)$. A (finite or infinite) *run* of (S, T) is a path in the graph of global states and transitions.

We are ready to state the decision problems with which our main concern will lie. As the input to a decision problem must be in some way finitely representable, we will state them for subclasses $\mathcal{F}' \subseteq \mathcal{F}$ of *computable* energy functions; an $\mathcal{F}'$-automaton is an energy automaton (S, s_0, T, F) with $T \subseteq S \times \mathcal{F}' \times S$. Note that we give no technical meaning to the term "computable" here; we simply need to take care that the input be finitely representable.

Problem 1 (State reachability)**.** Given an $\mathcal{F}'$-automaton $A = (S, s_0, T, F)$ and a computable initial energy $x_0 \in \mathbb{R}_{\geq 0}$: does there exist a finite run of A from (s_0, x_0) which ends in a state in F?

Problem 2 (Coverability)**.** Given an $\mathcal{F}'$-automaton $A = (S, s_0, T, F)$, a computable initial energy $x_0 \in \mathbb{R}_{\geq 0}$ and a computable function $z : F \to \mathbb{R}_{\geq 0}$: does there exist a finite run of A from (s_0, x_0) which ends in a global state (s, x) such that $s \in F$ and $x \geq sz$?

Problem 3 (Büchi acceptance)**.** Given an $\mathcal{F}'$-automaton $A = (S, s_0, T, F)$ and a computable initial energy $x_0 \in \mathbb{R}_{\geq 0}$: does there exist an infinite run of A from (s_0, x_0) which visits F infinitely often?

As customary, a run such as in the statements above is said to be *accepting*. The special case of Problem 3 with $F = S$ is the question whether there *exists an infinite run* in the given energy automaton. This is what is usually referred to as *energy problems* in the literature; our extension to general Büchi conditions has not been treated before.

3 The Algebra of Energy Functions

Let $[0, \infty]_\perp = \{\perp\} \cup [0, \infty]$ denote the complete lattice of non-negative real numbers together with extra elements $\perp$ and ∞, with the standard order on $\mathbb{R}_{\geq 0}$ extended by $\perp < x < \infty$ for all $x \in \mathbb{R}_{\geq 0}$. Also, $\perp + x = \perp - x = \perp$ for all $x \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $\infty + x = \infty - x = \infty$ for all $x \in \mathbb{R}_{\geq 0}$.

Definition 3. *An* extended energy function *is a mapping $f : [0, \infty]_\perp \to [0, \infty]_\perp$, for which $\perp f = \perp$ and $yf \geq xf + y - x$ for all $x \leq y$. Moreover, $\infty f = \infty$, unless $xf = \perp$ for all $x \in [0, \infty]_\perp$. The class of all extended energy functions is denoted $\mathcal{E}$.*

This means, in particular, that $xf = \perp$ implies $yf = \perp$ for all $y \leq x$, and $xf = \infty$ implies $yf = \infty$ for all $y \geq x$. Hence, except for the extension to ∞, these functions are indeed the same as the energy functions from Definition 1. More precisely, every energy function $f : \mathbb{R}_{\geq 0} \rightharpoonup \mathbb{R}_{\geq 0}$ as of Definition 1 gives rise to an extended energy function $\tilde{f} : [0, \infty]_\perp \to [0, \infty]_\perp$ given by $\perp\tilde{f} = \perp$, $x\tilde{f} = \perp$ if xf is undefined, $x\tilde{f} = xf$ otherwise for $x \in \mathbb{R}_{\geq 0}$, and $\infty\tilde{f} = \infty$.

Composition of extended energy functions is defined as before, but needs no more special consideration about its definition interval.

We define a partial order on $\mathcal{E}$, by $f \leq g$ iff $xf \leq xg$ for all $x \in [0, \infty]_\perp$. We will need three special energy functions, $\underline{\perp}$, id and $\overline{\top}$; these are given by $x\underline{\perp} = \perp$, $x; \mathsf{id} = x$ for $x \in [0, \infty]_\perp$, and $\perp\overline{\top} = \perp$, $x\overline{\top} = \infty$ for $x \in [0, \infty]$.

Lemma 2. *With the ordering $\leq$, $\mathcal{E}$ is a complete lattice with bottom element $\underline{\perp}$ and top element $\overline{\top}$. The supremum on $\mathcal{E}$ is pointwise, i.e., $x(\sup_{i\in I} f_i) = \sup_{i\in I} xf_i$ for any set I, all $f_i \in \mathcal{E}$ and $x \in [0, \infty]_\perp$. Also, $h(\sup_{i\in I} f_i) = \sup_{i\in I}(hf_i)$ for all $h \in \mathcal{E}$.*

Proof. The pointwise supremum of any set of extended energy functions is an extended energy function. Indeed, if f_i, $i \in I$ are extended energy functions and $x < y$ in $\mathbb{R}_{\geq 0}$, then $yf_i \geq xf_i + y - x$ for all i. It follows that $\sup_{i\in I} yf_i \geq \sup_{i\in I} xf_i + y - x$. Also, since $\perp f_i = \perp$ for all $i \in I$, $\sup_{i\in I} \perp f_i = \perp$. Finally, if there is some i such that $\infty f_i = \infty$, then $\sup_{i\in I} \infty f_i = \infty$. Otherwise each function f_i is constant with value $\perp$.

The fact that $h(\sup_{i\in I} f_i) = \sup_{i\in I} hf_i$ is now clear, since the supremum is taken pointwise: For all x, $x(h(\sup_{i\in I} f_i)) = (xh)(\sup_{i\in I} f_i) = \sup_{i\in I} xhf_i = x(\sup_i hf_i)$. □

We denote binary suprema using the symbol $\vee$; hence $f \vee g$, for $f, g \in \mathcal{E}$, is the function $x(f \vee g) = \max(xf, xg)$.

Recall that an *idempotent semiring* [1, 22] $S = (S, \vee, \cdot, \perp, 1)$ consists of a commutative idempotent monoid $(S, \vee, \perp)$ and a monoid $(S, \cdot, 1)$ such that the distributive laws

$$x(y \vee z) = xy \vee xz$$
$$(y \vee z)x = yx \vee zx$$

and the zero laws

$$\perp \cdot x = \perp = x \cdot \perp$$

hold for all $x, y, z \in S$. It follows that the product operation distributes over all finite sums.

Each idempotent semiring S is partially ordered by its *natural order* relation $x \leq y$ iff $x \vee y = y$, and then sum and product preserve the partial order and $\perp$ is the least element. Moreover, for all $x, y \in S$, $x \vee y$ is the least upper bound of the set $\{x, y\}$.

Lemma 3. $(\mathcal{E}, \vee, ;, \underline{\perp}, \mathsf{id})$ *is an idempotent semiring with natural order $\leq$.*

Proof. It is clear that $(\mathcal{E}, \vee, \perp\!\!\!\perp)$ is a commutative idempotent monoid and that $(\mathcal{E}, ;, \mathsf{id})$ is a monoid. $\leq$ is the natural order on $\mathcal{E}$ because $\vee$ is given pointwise. It is also clear that $\perp\!\!\!\perp f = f \perp\!\!\!\perp = \perp\!\!\!\perp$ for all $f \in \mathcal{E}$.

To show distributivity, we have already shown that $x(h(f \vee g)) = x(hf \vee hg)$ in the proof of Lemma 2; using monotonicity of h, we also have

$$x((f \vee g)h) = x(f \vee g)h = (xf \vee xg)h = xfg \vee xgh = x(fh \vee gh)\,.$$

The proof is complete. $\square$

We will show in the second paper [15] of this series that $\mathcal{E}$ in fact forms a **-continuous Kleene algebra* [25], which will allow us to solve energy problems algebraically.

4 Continuous Kleene Algebras and Continuous Kleene ω-Algebras

We have already recalled the notion of idempotent semiring in the last section. A *homomorphism* of idempotent semirings $(S, \vee, \cdot, \perp, 1)$, $(S', \vee', \cdot', \perp', 1')$ is a function $h : S \to S'$ which respects the constants and operations, *i.e.,* such that $h(\perp) = \perp'$, $h(1) = 1'$, $h(x \vee y) = h(x) \vee' h(y)$, and $h(x \cdot y) = h(x) \cdot' h(y)$ for all $x, y \in S$.

A Kleene algebra [25] is an idempotent semiring $S = (S, \vee, \cdot, \perp, 1)$ equipped with a star operation $^* : S \to S$ such that for all $x, y \in S$, yx^* is the least solution of the fixed point equation $z = zx \vee y$ and x^*y is the least solution of the fixed point equation $z = xz \vee y$ with respect to the natural order. A Kleene algebra *homomorphism* is a semiring homomorphism h which respects the star: $h(x^*) = (h(x))^*$ for all $x \in S$.

Examples of Kleene algebras include the language semiring $P(A^*)$ over an alphabet A, whose elements are the subsets of the set A^* of all finite words over A, and whose operations are set union and concatenation, with the languages $\emptyset$ and $\{\varepsilon\}$ serving as $\perp$ and 1. Here, ε denotes the empty word. The star operation is the usual Kleene star: $X^* = \bigcup_{n \geq 0} X^n = \{u_1 \dots u_n : u_1, \dots, u_n \in X,\ n \geq 0\}$.

Another example is the Kleene algebra $P(A \times A)$ of binary relations over any set A, whose operations are union and relational composition (written in diagrammatic order), and where the empty relation $\emptyset$ and the identity relation id serve as the constants $\perp$ and 1. The star operation is the formation of the reflexive-transitive closure, so that $R^* = \bigcup_{n \geq 0} R^n$ for all $R \in P(A \times A)$.

The above examples are in fact *continuous Kleene algebras*, *i.e.,* idempotent semirings S such that equipped with the natural order, they are complete lattices (hence all suprema exist), and the product operation preserves arbitrary suprema in either argument:

$$y(\bigvee X) = \bigvee yX \quad \text{and} \quad (\bigvee X)y = \bigvee Xy$$

for all $X \subseteq S$ and $y \in S$. The star operation is given by $x^* = \bigvee_{n \geq 0} x^n$, so that x^* is the supremum of the set $\{x^n : n \geq 0\}$ of all powers of x.

Homomorphisms of continuous Kleene algebras S, S' are homomorphisms of idempotent semirings $h : S \to S'$ which respect arbitrary suprema: $h(\bigvee X) = \bigvee h(X) = \bigvee\{h(x) \mid x \in X\}$ for all $X \subseteq S$. To distinguish these from semiring homomorphisms, they are sometimes called *continuous homomorphisms*, but we will not do this here.

A larger class of models is given by the **-continuous Kleene algebras* [25]. By the definition of a *-continuous Kleene algebra $S = (S, \vee, \cdot, \bot, 1)$, all suprema of sets of the form $\{x^n \mid n \geq 0\}$ are required to exist, where x is any element of S, and x^* is given by this supremum. Moreover, product preserves such suprema in both arguments:

$$y(\bigvee_{n\geq 0} x^n) = \bigvee_{n\geq 0} yx^n \quad \text{and} \quad (\bigvee_{n\geq 0} x^n)y = \bigvee_{n\geq 0} x^n y\,.$$

Every *-continuous Kleene algebra is a Kleene algebra. For any alphabet A, the collection $R(A^*)$ of all regular languages over A is an example of a *-continuous Kleene algebra which is not continuous. There exist Kleene algebras which are not *-continuous, see [25]. For non-idempotent extensions of the notions of continuous Kleene algebras, *-continuous Kleene algebras and Kleene algebras, we refer to [17, 16]. *Homomorphisms* of *-continuous Kleene algebras are the Kleene algebra homomorphisms.

Recall that an *idempotent semiring-semimodule pair* [19, 2] (S, V) consists of an idempotent semiring $S = (S, \vee, \cdot, \bot, 1)$ and a commutative idempotent monoid $V = (V, \vee, \bot)$ which is equipped with a left S-action $S \times V \to V$, $(x, v) \mapsto xv$, satisfying

$$\begin{aligned} (x \vee x')v &= xv \vee x'v & \qquad x(v \vee v') &= xv \vee xv' \\ (xx')v &= x(x'v) & \bot v &= \bot \\ x\bot &= \bot & 1v &= v \end{aligned}$$

for all $x, x' \in S$ and $v \in V$. In that case, we also call V a *(left) S-semimodule*.

A *homomorphism* of semiring-semimodule pairs (S, V) and (S', V') is a pair $h = (h_S, h_V)$ of functions $h_S : S \to S'$ and $h_V : V \to V'$ such that h_S is a semiring homomorphism, h_V is a monoid homomorphism, and h respects the action, *i.e.*, $h_V(xv) = h_S(x)h_V(v)$ for all $x \in S$ and $v \in V$.

Definition 4. *A* continuous Kleene ω-algebra *is an idempotent semiring-semimodule pair (S, V) in which S is a continuous Kleene algebra, V is a complete lattice with the natural order, and the action preserves all suprema in either argument. Additionally, there is an infinite product operation which is compatible with the action and associative in the sense that the following hold:*

1. *For all $x_0, x_1, \ldots \in S$, $\prod_{n\geq 0} x_n = x_0 \prod_{n\geq 0} x_{n+1}$.*
2. *Let $x_0, x_1, \ldots \in S$ and $0 = n_0 \leq n_1 \cdots$ be a sequence which increases without a bound. Let $y_k = x_{n_k} \cdots x_{n_{k+1}-1}$ for all $k \geq 0$. Then $\prod_{n\geq 0} x_n = \prod_{k\geq 0} y_k$.*

Moreover, the infinite product operation preserves all suprema:

3. $\prod_{n\geq 0}(\bigvee X_n) = \bigvee\{\prod_{n\geq 0} x_n : x_n \in X_n,\ n \geq 0\}$, *for all* $X_0, X_1, \ldots \subseteq S$.

The above notion of continuous Kleene ω-algebra may be seen as a special case of the not necessarily idempotent complete semiring-semimodule pairs of [19]. A *homomorphism* of continuous Kleene ω-algebras is a semiring-semimodule homomorphism $h = (h_S, h_V)$ such that h_S is a homomorphism of continuous Kleene algebras, h_V preserves all suprema, and h respects infinite products: for all $x_0, x_1, \ldots \in S$, $h_V(\prod_{n\geq 0} x_n) = \prod_{n\geq 0} h_S(x_n)$.

One of our aims in this paper is to provide an extension of the notion of continuous Kleene ω-algebras to **-continuous Kleene ω-algebras*, which are semiring-semimodule pairs (S, V) consisting of a *-continuous Kleene algebra S acting on a necessarily idempotent semimodule V, such that the action preserves certain suprema in its first argument, and which are equipped with an infinite product operation satisfying the above compatibility and associativity conditions and some weaker forms of the last axiom.

5 Free Continuous Kleene ω-Algebras

In this section, we offer descriptions of the free continuous Kleene ω-algebras and the free continuous Kleene ω-algebras satisfying the identity $1^\omega = \bot$. We recall the following folklore result.

Theorem 1. *For each set A, the language semiring $(P(A^*), \vee, \cdot, \bot, 1)$ is the free continuous Kleene algebra on A.*

In more detail, if S is a continuous Kleene algebra and $h : A \to S$ is any function, then there is a unique homomorphism $h^\sharp : P(A^*) \to S$ of continuous Kleene algebras which extends h.

In view of Theorem 1, it is not surprising that the free continuous Kleene ω-algebras can be described using languages of finite and infinite words. Suppose that A is a set. Let A^ω denote the set of all ω-words over A and $A^\infty = A^* \cup A^\omega$. Let $P(A^*)$ denote the language semiring over A and $P(A^\infty)$ the semimodule of all subsets of A^∞ equipped with the action of $P(A^*)$ defined by $XY = \{xy : x \in X, y \in Y\}$ for all $X \subseteq A^*$ and $Y \subseteq A^\infty$. We also define an infinite product by $\prod_{n\geq 0} X_n = \{u_0 u_1 \ldots : u_n \in X_n\}$. It is clear that $(P(A^*), P(A^\infty))$ is a continuous Kleene ω-algebra.

Theorem 2. *For each set A, $(P(A^*), P(A^\infty))$ is the free continuous Kleene ω-algebra on A.*

Proof. Suppose that (S, V) is any continuous Kleene ω-algebra an let $h : A \to S$ be a mapping. We want to show that there is a unique extension of h to a homomorphism $(h_S^\sharp, h_V^\sharp)$ from $(P(A^*), P(A^\infty))$ to (S, V).

For each $u = a_0 \ldots a_{n-1}$ in A^*, define $h_S(u) = h(a_0) \cdots h(a_{n-1})$ and $h_V(u) = h(a_0) \cdots h(a_{n-1}) 1^\omega = \prod_{k\geq 0} b_k$, where $b_k = a_k$ for all $k < n$ and $b_k = 1$ for all

$k \geq n$. When $u = a_0 a_1 \ldots \in A^\omega$, define $h_V(u) = \prod_{k\geq 0} h(a_k)$. Note that we have $h_S(uv) = h_S(u)h_S(v)$ for all $u, v \in A^*$ and $h_S(\varepsilon) = 1$. Also, $h_V(uv) = h_S(u)h_V(v)$ for all $u \in A^*$ and $v \in A^\infty$. Thus, $h_V(XY) = h_S(X)h_V(Y)$ for all $X \subseteq A^*$ and $Y \subseteq A^\infty$. Moreover, for all $u_0, u_1, \ldots$ in A^*, if $u_i \neq \varepsilon$ for infinitely many i, then $h_V(u_0u_1\ldots) = \prod_{k\geq 0} h_S(u_k)$. If on the other hand, $u_k = \varepsilon$ for all $k \geq n$, then $h_V(u_0u_1\ldots) = h_S(u_0)\cdots h_S(u_{n-1})1^\omega$. In either case, if $X_0, X_1, \ldots \subseteq A^*$, then $h_V(\prod_{n\geq 0} X_n) = \prod_{n\geq 0} h_S(X_n)$.

Suppose now that $X \subseteq A^*$ and $Y \subseteq A^\infty$. We define $h^\sharp_S(X) = \bigvee h_S(X)$ and $h^\sharp_V(Y) = \bigvee h_V(Y)$. It is well-known that $h^\sharp_S$ is a continuous semiring morphism $P(A^*) \to S$. Also, $h^\sharp_V$ preserves arbitrary suprema, since $h^\sharp_V(\bigcup_{i\in I} Y_i) = \bigvee h_V(\bigcup_{i\in I} Y_i) = \bigvee \bigcup_{i\in I} h_V(Y_i) = \bigvee_{i\in I} \bigvee h_V(Y_i) = \bigvee_{i\in I} h^\sharp_V(Y_i)$.

We prove that the action is preserved. Let $X \subseteq A^*$ and $Y \subseteq A^\infty$. Then $h^\sharp_V(XY) = \bigvee h_V(XY) = \bigvee h_S(X)h_V(Y) = \bigvee h_S(X) \bigvee h_V(Y) = h^\sharp_S(X)h^\sharp_V(Y)$.

Finally, we prove that the infinite product is preserved. Let $X_0, X_1, \ldots \subseteq A^*$. Then $h^\sharp_V(\prod_{n\geq 0} X_n) = \bigvee h_V(\prod_{n\geq 0} X_n) = \bigvee \prod_{n\geq 0} h_S(X_n) = \prod_{n\geq 0} \bigvee h_S(X_n) = \prod_{n\geq 0} h^\sharp_S(X_n)$.

It is clear that h_S extends h, and that (h_S, h_V) is unique. □

Consider now $(P(A^*), P(A^\omega))$ with infinite product defined, as a restriction of the above infinite product, by $\prod_{n\geq 0} X_n = \{u_0u_1\ldots \in A^\omega : u_n \in X_n,\ n \geq 0\}$. It is also a continuous Kleene ω-algebra. Moreover, it satisfies $1^\omega = \bot$.

Lemma 4. *$(P(A^*), P(A^\omega))$ is a quotient of $(P(A^*), P(A^\infty))$ under the homomorphism (φ_S, φ_V) such that φ_S is the identity on $P(A^*)$ and φ_V maps $Y \subseteq A^\infty$ to $Y \cap A^\omega$.*

Proof. Suppose that $Y_i \subseteq A^\infty$ for all $i \in I$. It holds that $\varphi_V(\bigcup_{i\in I} Y_i) = A^\omega \cap \bigcup_{i\in I} Y_i = \bigcup_{i\in I}(A^\omega \cap Y_i) = \bigcup_{i\in I} \varphi_V(Y_i)$.

Let $X \subseteq A^*$ and $Y \subseteq A^\infty$. Then $h_V(XY) = XY \cap A^\omega = X(Y \cap A^\omega) = \varphi_S(X)\varphi_V(Y)$.

Finally, let $X_0, X_1, \ldots \subseteq A^*$. Then $h_V(\prod_{n\geq 0} X_n) = \{u_0u_1\ldots \in A^\omega : u_n \in X_n\} = \prod_{n\geq 0} h_S(X_n)$. □

Lemma 5. *Suppose that (S, V) is a continuous Kleene ω-algebra satisfying $1^\omega = \bot$. Let (h_S, h_V) be a homomorphism $(P(A^*), P(A^\infty)) \to (S, V)$. Then (h_S, h_V) factors through (φ_S, φ_V).*

Proof. Define $h'_S = h_S$ and $h'_V : P(A^\omega) \to V$ by $h'_V(Y) = h_V(Y)$, for all $Y \subseteq A^\omega$. Then clearly $h_S = h'_S \circ \varphi_S$. Moreover, $h_V = h'_V \circ \varphi_V$, since for all $Y \subseteq A^\infty$, $h'_V(\varphi_V(Y)) = h_V(Y \cap A^\omega) = h_V(Y \cap A^\omega) \vee h_S(Y \cap A^*)1^\omega = h_V(Y \cap A^\omega) \vee h_V((Y \cap A^*)1^\omega) = h_V((Y \cap A^\omega) \cup (Y \cap A^*)1^\omega) = h_V(Y)$.

Since (φ_S, φ_V) and (h_S, h_V) are homomorphisms, so is (h'_S, h'_V). It is clear that h'_V preserves all suprema. □

Theorem 3. *For each set A, $(P(A^*), P(A^\omega))$ is the free continuous Kleene ω-algebra on A satisfying $1^\omega = \bot$.*

Proof. Suppose that (S, V) is a continuous Kleene ω-algebra satisfying $1^\omega = \bot$. Let $h : A \to S$. By Theorem 2, there is a unique homomorphism $(h_S, h_V) : (P(A^*), P(A^\infty)) \to (S, V)$ extending h. By Lemma 5, h_S and h_V factor as $h_S = h'_S \circ \varphi_S$ and $h_V = h'_V \circ \varphi_V$, where (h'_S, h'_V) is a homomorphism $(P(A^*), P(A^\omega)) \to (S, V)$. This homomorphism (h'_S, h'_V) is the required extension of h to a homomorphism $(P(A^*), P(A^\omega)) \to (S, V)$. Since the factorization is unique, so is this extension. $\square$

6 *-Continuous Kleene ω-Algebras

In this section, we define **-continuous Kleene ω-algebras* and *finitary *-continuous Kleene ω-algebras* as an extension of the *-continuous Kleene algebras of [24]. We establish several basic properties of these structures, including the existence of the suprema of certain subsets corresponding to regular ω-languages.

Definition 5. *A* generalized *-continuous Kleene algebra *is a semiring-semimodule pair (S, V) in which S is a *-continuous Kleene algebra (hence S and V are idempotent), subject to the usual laws of unitary action as well as the following axiom*

Ax0: For all $x, y \in S$ and $v \in V$, $xy^*v = \bigvee_{n \ge 0} xy^n v$.

Definition 6. *A* *-continuous Kleene ω-algebra *is a generalized *-continuous Kleene algebra (S, V) together with an infinite product operation $S^\omega \to V$ which maps every ω-word $x_0 x_1 \dots$ over S to an element $\prod_{n \ge 0} x_n$ of V, subject to the following axioms:*

Ax1: For all $x_0, x_1, \ldots \in S$, $\prod_{n \ge 0} x_n = x_0 \prod_{n \ge 0} x_{n+1}$.

Ax2: Let $x_0, x_1, \ldots \in S$ and $0 = n_0 \le n_1 \cdots$ be a sequence which increases without a bound. Let $y_k = x_{n_k} \cdots x_{n_{k+1}-1}$ for all $k \ge 0$. Then $\prod_{n \ge 0} x_n = \prod_{k \ge 0} y_k$.

Ax3: For all $x_0, x_1, \ldots$ and y, z in S, $\prod_{n \ge 0} (x_n (y \vee z)) = \bigvee_{x'_n \in \{y,z\}} \prod_{n \ge 0} x_n x'_n$.

Ax4: For all $x, y_0, y_1, \ldots \in S$, $\prod_{n \ge 0} x^* y_n = \bigvee_{k_n \ge 0} \prod_{n \ge 0} x^{k_n} y_n$.

The first two axioms are the same as for continuous Kleene ω-algebras. The last two are weaker forms of the complete preservation of suprema of continuous Kleene ω-algebras. It is clear that every continuous Kleene ω-algebra is *-continuous.

A *homomorphism* of *-continuous Kleene ω-algebras is a semiring-semimodule homomorphism $h = (h_S, h_V) : (S, V) \to (S', V')$ such that h_S is a *-continuous Kleene algebra homomorphism and h respects infinite products: for all $x_0, x_1, \ldots \in S$, $h_V(\prod_{n \ge 0} x_n) = \prod_{n \ge 0} h_S(x_n)$.

Some of our results will also hold for weaker structures. We define a *finitary *-continuous Kleene ω-algebra* as a structure (S, V) as above, equipped with a star operation and an infinite product $\prod_{n \ge 0} x_n$ restricted to *finitary ω-words* over S, *i.e.,* to sequences $x_0, x_1, \ldots$ such that there is a finite subset F of S such that each x_n is a finite product of elements of F. (Note that F is not fixed and may depend on the sequence $x_0, x_1, \ldots$) It is required that the axioms hold whenever the involved ω-words are finitary.

The above axioms have a number of consequences. For example, if $x_0, x_1, \ldots \in S$ and $x_i = \bot$ for some i, then $\prod_{n\geq 0} x_n = \bot$. Indeed, if $x_i = \bot$, then $\prod_{n\geq 0} x_n = x_0 \cdots x_i \prod_{n\geq i+1} x_n = \bot \prod_{n\geq i+1} x_n = \bot$. By Ax1 and Ax2, each *-continuous Kleene ω-algebra is an ω-semigroup [26].

Suppose that (S, V) is a *-continuous Kleene ω-algebra. For each word $w \in S^*$ there is a corresponding element $\overline{w}$ of S which is the product of the letters of w in the semiring S. Similarly, when $w \in S^*V$, there is an element $\overline{w}$ of V corresponding to w, and when $X \subseteq S^*$ or $X \subseteq S^*V$, then we can associate with X the set $\overline{X} = \{\overline{w} : w \in X\}$, which is a subset of S or V. Below we will denote $\overline{w}$ and $\overline{X}$ by just w and X, respectively.

The following two lemmas are well-known and follow from the fact that the semirings of regular languages are the free *-continuous Kleene algebras [24] (and also the free Kleene algebras [25]).

Lemma 6. *Suppose that S is a *-continuous Kleene algebra. If $R \subseteq S^*$ is regular, then $\bigvee R$ exists. Moreover, for all $x, y \in S$, $x(\bigvee R)y = \bigvee xRy$.*

Lemma 7. *Let S be a *-continuous Kleene algebra. Suppose that R, R_1 and R_2 are regular subsets of S^*. Then*

$$\begin{aligned}\bigvee(R_1 \cup R_2) &= \bigvee R_1 \vee \bigvee R_2\\ \bigvee(R_1 R_2) &= (\bigvee R_1)(\bigvee R_2)\\ \bigvee(R^*) &= (\bigvee R)^*.\end{aligned}$$

In a similar way, we can prove:

Lemma 8. *Let (S, V) be a generalized *-continuous Kleene algebra. If $R \subseteq S^*$ is regular, $x \in S$ and $v \in V$, then $x(\bigvee R)v = \bigvee xRv$.*

Proof. Suppose that $R = \emptyset$. Then $x(\bigvee R)v = \bot = \bigvee xRv$. If R is a singleton set $\{y\}$, then $x(\bigvee R)v = xyv = \bigvee xRv$. Suppose now that $R = R_1 \cup R_2$ or $R = R_1 R_2$, where R_1, R_2 are regular, and suppose that our claim holds for R_1 and R_2. Then, if $R = R_1 \cup R_2$,

$$\begin{aligned}x(\bigvee R)v &= x(\bigvee R_1 \vee \bigvee R_2)v \quad \text{(by Lemma 7)}\\ &= x(\bigvee R_1)v \vee x(\bigvee R_2)v\\ &= \bigvee xR_1v \vee \bigvee xR_2v\\ &= \bigvee x(R_1 \cup R_2)v = \bigvee xRv,\end{aligned}$$

where the third equality uses the induction hypothesis. If $R = R_1R_2$, then

$$\begin{aligned} x(\bigvee R)v &= x(\bigvee R_1)(\bigvee R_2)v \quad \text{(by Lemma 7)} \\ &= \bigvee(xR_1(\bigvee R_2)v) \\ &= \bigvee\{y(\bigvee R_2)v : y \in xR_1\} \\ &= \bigvee\{\bigvee yR_2v : y \in xR_1\} \\ &= \bigvee xR_1R_2v = \bigvee xRv, \end{aligned}$$

where the second equality uses the induction hypothesis for R_1 and the fourth the one for R_2. Suppose last that $R = R_0^*$, where R_0 is regular and our claim holds for R_0. Then, using the previous case, it follows by induction that

$$x(\bigvee R_0^n)v = \bigvee xR_0^n v \tag{1}$$

for all $n \geq 0$. Using this and $\mathsf{Ax0}$, it follows now that

$$\begin{aligned} x(\bigvee R)v = x(\bigvee R_0^*)y &= x(\bigvee_{n\geq 0} \bigvee R_0^n)v \\ &= x(\bigvee_{n\geq 0} (\bigvee R_0)^n)v \quad \text{(by Lemma 7)} \\ &= \bigvee_{n\geq 0} x(\bigvee R_0)^n v \quad \text{(by}\mathsf{Ax0}\text{)} \\ &= \bigvee_{n\geq 0} x(\bigvee R_0^n)v \quad \text{(by Lemma 7)} \\ &= \bigvee_{n\geq 0} \bigvee xR_0^n v \quad \text{(by (1))} \\ &= \bigvee xR_0^* v = \bigvee xRv. \end{aligned}$$

The proof is complete. □

Lemma 9. *Let (S, V) be a *-continuous Kleene ω-algebra. Suppose that the languages $R_0, R_1, \ldots \subseteq S^*$ are regular and that $\mathcal{R} = \{R_0, R_1, \ldots\}$ is a finite set. Moreover, let $x_0, x_1, \ldots \in S$. Then*

$$\prod_{n\geq 0} x_n(\bigvee R_n) = \bigvee \prod_{n\geq 0} x_n R_n.$$

Proof. If one of the R_i is empty, our claim is clear since both sides are equal to $\perp$, so we suppose they are all nonempty.

Below we will suppose that each regular language comes with a fixed decomposition having a minimal number of operations needed to obtain the language from the empty set and singleton sets. For a regular language R, let $|R|$ denote the minimum number of operations needed to construct it. When $\mathcal{R}$ is a finite set of regular languages, let $\mathcal{R}_{\rm ns}$ denote the set of non-singleton languages in it. Let $|\mathcal{R}| = \sum_{R\in\mathcal{R}_{\rm ns}} 3^{|R|}$. Our definition ensures that if $\mathcal{R} = \{R, R_1, \dots, R_n\}$ and $R = R' \cup R''$ or $R = R'R''$ according to the fixed minimal decomposition of R, and if $\mathcal{R}' = \{R', R'', R_1, \dots, R_n\}$, then $|\mathcal{R}'| < |\mathcal{R}|$. Similarly, if $R = R_0^*$ by the fixed minimal decomposition and $\mathcal{R}' = \{R_0, R_1, \dots, R_n\}$, then $|\mathcal{R}'| < |\mathcal{R}|$.

We will argue by induction on $|\mathcal{R}|$.

When $|\mathcal{R}| = 0$, then $\mathcal{R}$ consists of singleton languages and our claim follows from Ax3. Suppose that $|\mathcal{R}| > 0$. Let R be a non-singleton language appearing in $\mathcal{R}$. If R appears only a finite number of times among the R_n, then there is some m such that R_n is different from R for all $n \geq m$. Then,

$$\begin{aligned}
\prod_{n\geq 0} x_n(\bigvee R_n) &= \prod_{i<m} x_i(\bigvee R_i) \prod_{n\geq m} x_n(\bigvee R_n) \quad \text{(by Ax1)}\\
&= (\bigvee x_0R_0 \cdots x_{n-1}R_{n-1}) \prod_{n\geq m} x_n(\bigvee R_n) \quad \text{(by Lemma 7)}\\
&= \bigvee (x_0R_0 \cdots x_{n-1}R_{n-1} \prod_{n\geq m} x_n(\bigvee R_n)) \quad \text{(by Lemma 8)}\\
&= \bigvee \{y \prod_{n\geq m} x_n(\bigvee R_n) : y \in x_0R_0 \cdots x_{n-1}R_{n-1}\}\\
&= \bigvee \{\bigvee y \prod_{n\geq m} x_nR_n : y \in x_0R_0 \cdots x_{n-1}R_{n-1}\}\\
&= \bigvee \prod_{n\geq 0} x_nR_n,
\end{aligned}$$

where the passage from the 4th line to the 5th uses induction hypothesis and Ax1.

Suppose now that R appears an infinite number of times among the R_n. Let $R_{i_1}, R_{i_2}, \dots$ be all the occurrences of R among the R_n. Define

$$\begin{aligned}
y_0 &= x_0(\bigvee R_0) \cdots (\bigvee R_{i_1-1})x_{i_1}\\
y_j &= x_{i_j+1}(\bigvee R_{i_j+1}) \cdots (\bigvee R_{i_{j+1}-1})x_{i_{j+1}},
\end{aligned}$$

for $j \geq 1$. Similarly, define

$$\begin{aligned}
Y_0 &= x_0R_0 \cdots R_{i_1-1}x_{i_1}\\
Y_j &= x_{i_j+1}R_{i_j+1} \cdots R_{i_{j+1}-1}x_{i_{j+1}},
\end{aligned}$$

for all $j \geq 1$. It follows from Lemma 7 that

$$y_j = \bigvee Y_j$$

for all $j \geq 0$. Then

$$\prod_{n\geq 0} x_n(\bigvee R_n) = \prod_{j\geq 0} y_j(\bigvee R), \tag{2}$$

by $\mathsf{Ax2}$, and

$$\prod_{n\geq 0} x_n R_n = \prod_{j\geq 0} Y_j R.$$

If $R = R' \cup R''$, then:

$$\begin{aligned}
\prod_{n\geq 0} x_n(\bigvee R_n) &= \prod_{j\geq 0} y_j(\bigvee (R' \cup R'')) \quad \text{(by (2))}\\
&= \prod_{j\geq 0} y_j(\bigvee R' \vee \bigvee R'') \quad \text{(by Lemma 7)}\\
&= \bigvee_{z_j \in \{\bigvee R', \bigvee R''\}} \prod_{j\geq 0} y_j z_j \quad \text{(by } \mathsf{Ax3})\\
&= \bigvee_{z_j \in \{\bigvee R', \bigvee R''\}} \bigvee \prod_{j\geq 0} Y_j z_j\\
&= \bigvee_{Z_j \in \{R', R''\}} \bigvee \prod_{j\geq 0} Y_j Z_j\\
&= \bigvee \prod_{n\geq 0} x_n(R' \cup R'') = \bigvee \prod_{n\geq 0} x_n R,
\end{aligned}$$

where the 4th and 5th equalities hold by the induction hypothesis and $\mathsf{Ax2}$.

Suppose now that $R = R'R''$. Then, applying the induction hypothesis almost directly,

$$\begin{aligned}
\prod_{n\geq 0} x_n(\bigvee R_n) &= \prod_{j\geq 0} y_j(\bigvee R'R'')\\
&= \prod_{j\geq 0} y_j(\bigvee R')(\bigvee R'') \quad \text{(by Lemma 7)}\\
&= \bigvee \prod_{j\geq 0} Y_j(\bigvee R')(\bigvee R'')\\
&= \bigvee \prod_{j\geq 0} Y_j R'R''\\
&= \bigvee \prod_{n\geq 0} x_n R'R'' = \bigvee \prod_{n\geq 0} x_n R,
\end{aligned}$$

where the third and fourth equalities come from the induction hypothesis and $\mathsf{Ax2}$.

The last case to consider is when $R = T^*$, where T is regular. We argue as follows:

$$\begin{aligned}
\prod_{n\geq 0} x_n(\bigvee R_n) &= \prod_{j\geq 0} y_j(\bigvee T^*) \\
&= \prod_{j\geq 0} y_j(\bigvee T)^* \quad \text{(by Lemma 7)} \\
&= \bigvee_{k_0,k_1,\dots} \prod_{j\geq 0} y_j(\bigvee T)^{k_j} \quad \text{(by } \mathsf{Ax1} \text{ and } \mathsf{Ax4}) \\
&= \bigvee_{k_0,k_1,\dots} \bigvee \prod_{j\geq 0} Y_j(\bigvee T)^{k_j} \\
&= \bigvee_{k_0,k_1,\dots} \bigvee \prod_{j\geq 0} Y_j T^{k_j} \\
&= \bigvee_{j\geq 0} Y_j T^* = \bigvee_{j\geq 0} Y_j R_j = \bigvee_{n\geq 0} x_n R_n,
\end{aligned}$$

where the 4th and 5th equalities follow from the induction hypothesis and Ax2. The proof is complete. □

By the same proof, we have the following version of Lemma 9 for the finitary case:

Lemma 10. *Let (S, V) be a finitary *-continuous Kleene ω-algebra. Suppose that the languages $R_0, R_1, \ldots \subseteq S^*$ are regular and that $\mathcal{R} = \{R_0, R_1, \ldots\}$ is a finite set. Moreover, let $x_0, x_1, \ldots$ be a finitary sequence of elements of S. Then*

$$\prod_{n\geq 0} x_n(\bigvee R_n) = \bigvee \prod_{n\geq 0} x_n R_n.$$

Note that each sequence $x_0, y_0, x_1, y_1, \ldots$ with $y_n \in R_n$ is finitary.

Corollary 1. *Let (S, V) be a finitary *-continuous Kleene ω-algebra. Suppose that $R_0, R_1, \ldots \subseteq S^*$ are regular and that $\mathcal{R} = \{R_0, R_1, \ldots\}$ is a finite set. Then $\bigvee \prod_{n\geq 0} R_n$ exists and is equal to $\prod_{n\geq 0} \bigvee R_n$.*

Using our earlier convention that ω-words $v = x_0 x_1 \ldots \in S^\omega$ over S determine elements $\prod_{n\geq 0} x_n$ of V and subsets $X \subseteq S^\omega$ determine subsets of V, Lemma 9 may be rephrased as follows.

For any *-continuous Kleene ω-algebra (S, V), $x_0, x_1, \ldots \in S$ and regular sets $R_0, R_1, \ldots \subseteq S^*$ for which $\mathcal{R} = \{R_0, R_1, \ldots\}$ is a finite set, it holds that

$$\prod_{n\geq 0} x_n(\bigvee R_n) = \bigvee X,$$

where $X \subseteq S^\omega$ is the set of all ω-words $x_0 y_0 x_1 y_1 \ldots$ with $y_i \in R_i$ for all $i \geq 0$, *i.e.*, $X = x_0 R_0 x_1 R_1 \ldots$

Similarly, Corollary 1 asserts that if a subset of V corresponds to an infinite product over a finite collection of ordinary regular languages in S^*, then the supremum of this set exists.

In any (finitary or non-finitary) *-continuous Kleene ω-algebra (S, V), we define an *ω-power operation* $S \to V$ by $x^\omega = \prod_{n \geq 0} x$ for all $x \in S$. From the axioms we immediately have:

Corollary 2. *Suppose that (S, V) is a (finitary or non-finitary) *-continuous Kleene ω-algebra. Then the following hold for all $x, y \in S$:*

$$\begin{aligned} x^\omega &= x x^\omega \\ (xy)^\omega &= x(yx)^\omega \\ x^\omega &= (x^n)^\omega, \quad n \geq 2. \end{aligned}$$

Thus, each *-continuous Kleene ω-algebra gives rise to a Wilke algebra [29].

Lemma 11. *Let (S, V) be a (finitary or non-finitary) *-continuous Kleene ω-algebra. Suppose that $R \subseteq S^\omega$ is ω-regular. Then $\bigvee R$ exists in V.*

Proof. It is well-known that R can be written as a finite union of sets of the form $R_0(R_1)^\omega$ where $R_0, R_1 \subseteq S^*$ are regular, moreover, R_1 does not contain the empty word. It suffices to show that $\bigvee R_0(R_1)^\omega$ exists. But this holds by Corollary 1. □

Lemma 12. *Let (S, V) be a (finitary or non-finitary) *-continuous Kleene ω-algebra. For all ω-regular sets $R_1, R_2 \subseteq S^\omega$ and regular sets $R \subseteq S^*$ it holds that*

$$\begin{aligned} \bigvee (R_1 \cup R_2) &= \bigvee R_1 \vee \bigvee R_2 \\ \bigvee (RR_1) &= (\bigvee R)(\bigvee R_1). \end{aligned}$$

And if R does not contain the empty word, then

$$\bigvee R^\omega = (\bigvee R)^\omega.$$

Proof. The first claim is clear. The second follows from Lemma 8. For the last, see the proof of Lemma 11. □

7 Free Finitary *-Continuous Kleene ω-Algebras

Recall that for a set A, $R(A^*)$ denotes the collection of all regular languages in A^*. It is well-known that $R(A^*)$, equipped with the usual operations, is a *-continuous Kleene algebra on A. Actually, $R(A^*)$ is characterized up to isomorphism by the following universal property.

Theorem 4 ([25])**.** *For each set A, $R(A^*)$ is the free *-continuous Kleene algebra on A.*

Thus, if S is any *-continuous Kleene algebra and $h : A \to S$ is any mapping from any set A into S, then h has a unique extension to a *-continuous Kleene algebra homomorphism $h^\sharp : R(A^*) \to S$.

Now let $R'(A^\infty)$ denote the collection of all subsets of A^∞ which are finite unions of finitary infinite products of regular languages, that is, finite unions of sets of the form $\prod_{n\geq 0} R_n$, where each $R_n \subseteq A^*$ is regular, and the set $\{R_0, R_1, \ldots\}$ is finite. Note that $R'(A^\infty)$ contains the empty set and is closed under finite unions. Moreover, when $Y \in R'(A^\infty)$ and $u = a_0a_1 \ldots \in Y \cap A^\omega$, then the alphabet of u is finite, *i.e.,* the set $\{a_n : n \geq 0\}$ is finite. Also, $R'(A^\infty)$ is closed under the action of $R(A^*)$ inherited from $(P(A^*), P(A^\infty))$. The infinite product of a sequence of regular languages in $R(A^*)$ is not necessarily contained in $R'(A^\infty)$, but by definition $R'(A^\infty)$ contains all infinite products of finitary sequences over $R(A^*)$.

Example 3. Let $A = \{a, b\}$ and consider the set $X = \{aba^2b \ldots a^nb \ldots\} \in P(A^\infty)$ containing a single ω-word. X can be written as an infinite product of subsets of A^*, but it cannot be written as an infinite product $R_0R_1 \ldots$ of regular languages in A^* such that the set $\{R_0, R_1, \ldots\}$ is finite. Hence $X \notin R'(A^\infty)$.

Theorem 5. *For each set A, $(R(A^*), R'(A^\infty))$ is the free finitary *-continuous Kleene ω-algebra on A.*

Proof. Our proof is modeled after the proof of Theorem 2. First, it is clear from the fact that $(P(A^*), P(A^\infty))$ is a continuous Kleene ω-algebra, and that $R(A^*)$ is a *-continuous semiring, that $(R(A^*), R'(A^\infty))$ is indeed a finitary *-continuous Kleene ω-algebra.

Suppose that (S, V) is any finitary *-continuous Kleene ω-algebra and let $h : A \to S$ be a mapping. For each $u = a_0 \ldots a_{n-1}$ in A^*, let $h_S(u) = h(a_0) \cdots h(a_{n-1})$ and $h_V(u) = h(a_0) \cdots h(a_{n-1})1^\omega = \prod_{k\geq 0} b_k$, where $b_k = a_k$ for all $k < n$ and $b_k = 1$ for all $k \geq n$. When $u = a_0a_1 \ldots \in A^\omega$ whose alphabet is finite, define $h_V(u) = \prod_{k\geq 0} h(a_k)$. This infinite product exists in $R'(A^\infty)$.

Note that we have $h_S(uv) = h_S(u)h_S(v)$ for all $u, v \in A^*$, and $h_S(\varepsilon) = 1$. And if $u \in A^*$ and $v \in A^\infty$ such that the alphabet of v is finite, then $h_V(uv) = h_S(u)h_V(v)$. Also, $h_V(XY) = h_S(X)h_V(Y)$ for all $X \subseteq A^*$ in $R(A^*)$ and $Y \subseteq A^\infty$ in $R'(A^\infty)$.

Moreover, for all $u_0, u_1, \ldots$ in A^*, if $u_i \neq \varepsilon$ for infinitely many i, such that the alphabet of $u_0u_1 \ldots$ is finite, then $h_V(u_0u_1 \ldots) = \prod_{k\geq 0} h_S(u_k)$. If on the other hand, $u_k = \varepsilon$ for all $k \geq n$, then $h_V(u_0u_1 \ldots) = h_S(u_0) \cdots h_S(u_{n-1})1^\omega$. In either case, if $X_0, X_1, \ldots \subseteq A^*$ are regular and form a finitary sequence, then the sequence $h_S(X_0), h_S(X_1), \ldots$ is also finitary as is each infinite word in $\prod_{n\geq 0} X_n$, and $h_V(\prod_{n\geq 0} X_n) = \prod_{n\geq 0} h_S(X_n)$.

Suppose now that $X \subseteq A^*$ is regular and $Y \subseteq A^\infty$ is in $R'(A^\infty)$. We define $h^\sharp_S(X) = \bigvee h_S(X)$ and $h^\sharp_V(Y) = \bigvee h_V(Y)$. It is well-known that $h^\sharp_S$ is a *-continuous Kleene algebra morphism $R(A^*) \to S$. Also, $h^\sharp_V$ preserves finite suprema, since when I is finite, $h^\sharp_V(\bigcup_{i\in I} Y_i) = \bigvee h_V(\bigcup_{i\in I} Y_i) = \bigvee \bigcup_{i\in I} h_V(Y_i) = \bigvee_{i\in I} \bigvee h_V(Y_i) = \bigvee_{i\in I} h^\sharp_V(Y_i)$.

We prove that the action is preserved. Let $X \in R(A^*)$ and $Y \in R'(A^\infty)$. Then $h^\sharp_V(XY) = \bigvee h_V(XY) = \bigvee h_S(X)h_V(Y) = \bigvee h_S(X) \bigvee h_V(Y) = h^\sharp_S(X)h^\sharp_V(Y)$.

Finally, we prove that infinite product of finitary sequences is preserved. Let $X_0, X_1, \dots$ be a finitary sequence of regular languages in $R(A^*)$. Then, using Corollary 1, $h_V^\sharp(\prod_{n\geq 0} X_n) = \bigvee h_V(\prod_{n\geq 0} X_n) = \bigvee \prod_{n\geq 0} h_S(X_n) = \prod_{n\geq 0} \bigvee h_S(X_n) = \prod_{n\geq 0} h_S^\sharp(X_n)$.

It is clear that h_S extends h, and that (h_S, h_V) is unique. $\square$

Consider now $(R(A^*), R'(A^\omega))$ equipped with the infinite product operation $\prod_{n\geq 0} X_n = \{u_0 u_1 \in A^\omega : u_n \in X_n,\ n \geq 0\}$, defined on finitary sequences $X_0, X_1, \dots$ of languages in $R(A^*)$.

Lemma 13. *Suppose that (S, V) is a finitary *-continuous Kleene ω-algebra satisfying $1^\omega = \bot$. Let (h_S, h_V) be a homomorphism $(R(A^*), R'(A^\infty)) \to (S, V)$. Then (h_S, h_V) factors through (φ_S, φ_V).*

Proof. Similar to the proof of Lemma 5. $\square$

Theorem 6. *For each set A, $(R(A^*), R'(A^\omega))$ is the free finitary *-continuous Kleene ω-algebra satisfying $1^\omega = \bot$ on A.*

Proof. This follows from Theorem 5 using Lemma 13. $\square$

8 *-Continuous Kleene ω-Algebras Are Iteration Semiring-Semimodule Pairs

In this section, we will show that every (finitary or non-finitary) *-continuous Kleene ω-algebra is an iteration semiring-semimodule pair.

Some definitions are in order. Suppose that $S = (S, \vee, \cdot, \bot, 1)$ is an idempotent semiring. Following [2], we call S a *Conway semiring* if S is equipped with a star operation ${}^* : S \to S$ satisfying, for all $x, y \in S$,

$$\begin{aligned}(x \vee y)^* &= (x^* y)^* x^* \\ (xy)^* &= 1 \vee x(yx)^* y\,.\end{aligned}$$

(Note that in [2], also non-idempotent Conway semirings have been considered, but we stick to the idempotent case here.)

It is known [2] that if S is a Conway semiring, then for each $n \geq 1$, so is the semiring $S^{n\times n}$ of all $n \times n$-matrices over S with the usual sum and product operations and the star operation defined by induction on n so that if $n > 1$ and $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$, where a and d are square matrices of dimension $< n$, then

$$M^* = \begin{pmatrix} (a \vee bd^*c)^* & (a \vee bd^*c)^* bd^* \\ (d \vee ca^*b)^* ca^* & (d \vee ca^*b)^* \end{pmatrix}.$$

The above definition does not depend on how M is split into submatrices.

Suppose that S is a Conway semiring and $G = \{g_1, \dots, g_n\}$ is a finite group of order n. For each $x_{g_1}, \dots, x_{g_n} \in S$, consider the $n \times n$ matrix $M_G = M_G(x_{g_1}, \dots, x_{g_n})$

whose ith row is $(x_{g_i^{-1}g_1}, \ldots, x_{g_i^{-1}g_n})$, for $i = 1, \ldots, n$, so that each row (and column) is a permutation of the first. We say that the group identity [8] associated with G holds in S if for each $x_{g_1}, \ldots, x_{g_n}$, the first (and then any) row sum of M_G^* is $(x_{g_1} \vee \cdots \vee x_{g_n})^*$. Finally, we call S an *iteration semiring* [2, 11] if the group identities hold in S for all finite groups of order n.

Classes of examples of (idempotent) iteration semirings are given by the continuous and the *-continuous Kleene algebras defined in the introduction. As mentioned above, the language semirings $P(A^*)$ and the semirings $P(A \times A)$ of binary relations are continuous and hence also *-continuous Kleene algebras, and the semirings $R(A^*)$ of regular languages are *-continuous Kleene algebras.

When S is a *-continuous Kleene algebra and n is a nonnegative integer, then the matrix semiring $S^{n\times n}$ is also a *-continuous Kleene algebra and hence an iteration semiring, cf. [24]. The star operation is defined by

$$M^*_{i,j} = \bigvee_{m\geq 0,\ 1\leq k_1,\ldots,k_m\leq n} M_{i,k_1} M_{k_1,k_2} \cdots M_{k_m,j},$$

for all $M \in S^{n\times n}$ and $1 \leq i, j \leq n$. It is not trivial to prove that the above supremum exists. The fact that M^* is well-defined can be established by induction on n together with the well-known matrix star formula mentioned above.

An idempotent semiring-semimodule pair (S, V) is a *Conway semiring-semimodule pair* if it is equipped with a star operation $^* : S \to S$ and an omega operation $^\omega : S \to V$ such that S is a Conway semiring acting on the semimodule $V = (V, \vee, \perp)$ and the following hold for all $x, y \in S$:

$$(x \vee y)^\omega = (x^*y)^* x^\omega \vee (x^*y)^\omega$$
$$(xy)^\omega = x(yx)^\omega.$$

It is known [2] that when (S, V) is a Conway semiring-semimodule pair, then so is $(S^{n\times n}, V^n)$ for each n, where V^n denotes the $S^{n\times n}$-semimodule of all n-dimensional (column) vectors over V with the action of $S^{n\times n}$ defined similarly to matrix-vector product, and where the omega operation is defined by induction so that when $n > 1$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where a and d are square matrices of dimension $< n$, then

$$M^\omega = \begin{pmatrix} (a \vee bd^*c)^\omega \ \vee \ (a \vee bd^*c)^* bd^\omega \\ (d \vee ca^*b)^\omega \ \vee \ (d \vee ca^*b)^* ca^\omega \end{pmatrix}. \tag{3}$$

We also define *iteration semiring-semimodule pairs* [2, 19] as those Conway semiring-semimodule pairs such that S is an iteration semiring and the omega operation satisfies the following condition: let $M_G = M_G(x_{g_1}, \ldots, x_{g_n})$ like above, with $x_{g_1}, \ldots, x_{g_n} \in S$ for a finite group $G = \{g_1, \ldots, g_n\}$ of order n, then the first (and hence any) entry of M_G^ω is equal to $(x_{g_1} \vee \cdots \vee x_{g_n})^\omega$.

Examples of (idempotent) iteration semiring-semimodule pairs include the semiring-semimodule pairs $(P(A^*), P(A^\omega))$ of languages and ω-languages over an alphabet A mentioned earlier. The omega operation is defined by $X^\omega = \prod_{n\geq 0} X$. More generally, it is known that every continuous Kleene ω-algebra gives rise to an iteration semiring-semimodule pair. The omega operation is defined as for languages: $x^\omega = \prod_{n \geq 0} x_n$ with $x_n = x$ for all $n \geq 0$.

Other not necessarily idempotent examples include the *complete* and the *(symmetric) bi-inductive semiring-semimodule pairs* of [18, 19].

Suppose now that (S, V) is a *-continuous Kleene ω-algebra. Then for each $n \geq 1$, $(S^{n\times n}, V^n)$ is a semiring-semimodule pair. The action of $S^{n\times n}$ on V^n is defined similarly to matrix-vector product (viewing the elements of V^n as column vectors). It is easy to see that $(S^{n\times n}, V^n)$ is a generalized *-continuous Kleene algebra for each $n \geq 1$.

Suppose that $n \geq 2$. We would like to define an infinite product operation $(S^{n\times n})^\omega \to V^n$ on matrices in $S^{n\times n}$ by

$$(\prod_{m \geq 0} M_m)_i = \bigvee_{1 \leq i_1, i_2, \ldots \leq n} (M_0)_{i,i_1}(M_1)_{i_1,i_2} \cdots$$

for all $1 \leq i \leq n$. However, unlike in the case of complete semiring-semimodule pairs [19], the supremum on the right-hand side may not exist. Nevertheless it is possible to define an omega operation $S^{n\times n} \to V^n$ and to turn $(S^{n\times n}, V^n)$ into an iteration semiring-semimodule pair.

Lemma 14. *Let (S, V) be a (finitary or non-finitary) *-continuous Kleene ω-algebra. Suppose that $M \in S^{n\times n}$, where $n \geq 2$. Then for every $1 \leq i \leq n$,*

$$(\prod_{m \geq 0} M)_i = \bigvee_{1 \leq i_1, i_2, \ldots \leq n} M_{i,i_1} M_{i_1,i_2} \cdots$$

exists, so that we define *M^ω by the above equality. Moreover, when $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where a and d are square matrices of dimension $< n$, then* (3) *holds.*

Proof. Suppose that $n = 2$. Then by Corollary 1, $(a \vee bd^*c)^\omega$ is the supremum of the set of all infinite products $A_{1,i_1}A_{i_1,i_2}\cdots$ containing a or c infinitely often, and $(a \vee bd^*c)^*bd^\omega$ is the supremum of the set of all infinite products $A_{1,i_1}A_{i_1,i_2}\cdots$ containing a and c only finitely often. Thus, $(a \vee bd^*c)^\omega \vee (a \vee bd^*c)^*bd^\omega$ is the supremum of the set of all infinite products $A_{1,i_1}A_{i_1,i_2}\cdots$. Similarly, $(d \vee ca^*b)^\omega \vee (d \vee ca^*b)^*ca^\omega$ is the supremum of the set of all infinite products $A_{2,i_1}A_{i_1,i_2}\cdots$.

The proof of the induction step is similar. Suppose that $n > 2$, and let a be $k \times k$. Then by induction hypothesis, for every i with $1 \leq i \leq k$, the ith component of $(a \vee bd^*c)^\omega$ is the supremum of the set of all infinite products $A_{i,i_1}A_{i_1,i_2}\cdots$ containing an entry of a or c infinitely often, whereas the ith component of $(a \vee bd^*c)^*bd^\omega$ is the supremum of all infinite products $A_{i,i_1}A_{i_1,i_2}\cdots$ containing entries of a and c only finitely often. Thus, the ith component of $(a \vee bd^*c)^\omega \vee (a \vee bd^*c)^*bd^\omega$ is the supremum of the set of all infinite products $A_{i,i_1}A_{i_1,i_2}\cdots$. A similar fact holds for $(d \vee ca^*b)^\omega \vee (d \vee ca^*b)^*ca^\omega$. The proof is complete. $\square$

Theorem 7. *Every (finitary or non-finitary) *-continuous Kleene ω-algebra is an iteration semiring-semimodule pair.*

Proof. Suppose that (S, V) is a finitary *-continuous Kleene ω-algebra. Then

$$(x \vee y)^\omega = (x^*y)^\omega \vee (x^*y)^*x^\omega,$$

since by Lemma 7 and Lemma 12, $(x^*y)^\omega$ is the supremum of the set of all infinite products over $\{x, y\}$ containing y infinitely often, and $(x^*y)^*x^\omega$ is the supremum of the set of infinite products over $\{x, y\}$ containing y finitely often. Thus, $(x^*y)^\omega \vee (x^*y)^*x^\omega$ is equal to $(x \vee y)^\omega$, which by Ax3 is the supremum of all infinite products over $\{x, y\}$. As noted above, also

$$(xy)^\omega = x(yx)^\omega$$

for all $x, y \in S$. Thus, (S, V) is a Conway semiring-semimodule pair and hence so is each $(S^{n \times n}, V^n)$.

To complete the proof of the fact that (S, V) is an iteration semiring-semimodule pair, suppose that $x_1, \dots, x_n \in S$, and let $x = x_1 \vee \dots \vee x_n$. Let A be an $n \times n$ matrix whose rows are permutations of the $x_1, \dots, x_n$. We need to prove that each component of A^ω is x^ω. We use Lemma 14 and Ax3 to show that both are equal to the supremum of the set of all infinite products over the set $X = \{x_1, \dots, x_n\}$.

By Lemma 14, for each $i_0 = 1, \dots, n$, the i_0th row of A^ω is $\bigvee_{i_1, i_2, \dots} a_{i_0,i_1} a_{i_1,i_2} \cdots$. It is clear that each infinite product $a_{i_0,i_1} a_{i_1,i_2} \cdots$ is an infinite product over X. Suppose now that $x_{j_0} x_{j_1} \cdots$ is an infinite product over X. We define by induction on $k \geq 0$ an index i_{k+1} such that $a_{i_k, i_{k+1}} = x_{j_k}$. Suppose that $k = 0$. Then let i_1 be such that $a_{i_0,i_1} = x_{j_0}$. Since x_{j_0} appears in the i_0th row, there is such an i_1. Suppose that $k > 0$ and that i_k has already been defined. Since x_{j_k} appears in the i_kth row, there is some i_{k+1} with $a_{i_k,i_{k+1}} = x_{j_k}$. We have completed the proof of the fact that the i_0th entry of A^ω is the supremum of the set of all infinite products over the set $X = \{x_1, \dots, x_n\}$.

Consider now $x^\omega = xx \cdots$. We use induction on n to prove that x^ω is also the supremum of the set of all infinite products over the set $X = \{x_1, \dots, x_n\}$. When $n = 1$ this is clear. Suppose now that $n > 1$ and that the claim is true for $n - 1$. Let $y = x_1 \vee \dots \vee x_{n-1}$ so that $x = y \vee x_n$. We have:

$$\begin{aligned} x^\omega &= (y \vee x_n)^\omega \\ &= (x_n^* y)^* x_n^\omega \vee (x_n^* y)^\omega \\ &= (x_n^* y)^* x_n^\omega \vee (x_n^* x_1 \vee \dots \vee x_n^* x_{n-1})^\omega. \end{aligned}$$

Now

$$(x_n^* y)^* x_n^\omega = \bigvee_{k, m_1, \dots, m_k \geq 0} x_n^{m_1} y \cdots x_n^{m_k} y x_n^\omega$$

by Lemma 8, which is the supremum of all infinite products over X containing $x_1, \dots, x_{n-1}$ only a finite number of times.

Also, using the induction hypothesis and Ax4,

$$\begin{aligned}(x_n^* x_1 \vee \cdots \vee x_n^* x_{n-1})^\omega &= \bigvee_{1 \le i_1, i_2, \ldots \le n-1} x_n^* x_{i_1} x_n^* x_{i_2} \cdots \\ &= \bigvee_{1 \le i_1, i_2, \ldots \le n-1} \ \bigvee_{k_0, k_1, \ldots} x_n^{k_0} x_{i_1} x_n^{k_1} x_{i_2} \cdots\end{aligned}$$

which is the supremum of all infinite products over X containing one of $x_1, \ldots, x_{n-1}$ an infinite number of times. Thus, x^ω is the supremum of all infinite products over X as claimed. □

9 Kleene ω-Algebras

Recall that when S is a *-continuous Kleene algebra, then S is a Kleene algebra [24]. Thus, for all $x, y \in S$, x^*y is the least pre-fixed point (and thus the least fixed point) of the function $S \to S$ defined by $z \mapsto xz \vee y$ for all $z \in S$. Moreover, yx^* is the least pre-fixed point and the least fixed point of the function $S \to S$ defined by $z \mapsto zx \vee y$, for all $z \in S$. Similarly, when (S, V) is a generalized *-continuous Kleene algebra, then for all $x \in S$ and $v \in V$, x^*v is the least pre-fixed point and the least fixed point of the function $V \to V$ defined by $z \mapsto xz \vee v$.

As a natural analogy to Kleene algebras in semiring-semimodule pairs, we propose a notion of *Kleene ω-algebra*.

Definition 7. *A* Kleene ω-algebra *is a semiring-semimodule pair (S, V) in which S is a Kleene algebra and equipped with an omega operation $^\omega : S \to V$ such that the following hold for all $x, y \in S$ and $v \in V$:*

- *x^*v is the least pre-fixed point of the function $V \to V$ defined by $z \mapsto xz \vee v$,*
- *$x^\omega \vee x^*v$ is the greatest post-fixed point of the function $V \to V$ defined by $z \mapsto xz \vee v$.*

It is clear that any Kleene ω-algebra is a bi-inductive semiring-semimodule pair in the sense of [19]. By the above remarks we have:

Lemma 15. *Suppose that (S, V) is a (finitary or non-finitary) *-continuous Kleene ω-algebra. When for all $x \in S$ and $v \in V$, $x^\omega \vee x^*v$ is the greatest post-fixed point of the function $V \to V$ defined by $z \mapsto xz \vee v$, then (S, V) is a Kleene ω-algebra.*

We remark that the precondition of the lemma is indeed necessary, and it is not the case that any *-continuous Kleene ω-algebra is a Kleene ω-algebra. As an example, note that the above property implies that 1^ω is the greatest fixed point of the mapping $z \mapsto z$; but we have seen in Theorem 6 that there are finitary *-continuous Kleene ω-algebras with $1^\omega = \bot$.

10 Conclusion

Motivated by an application to energy problems, we have introduced continuous and *-continuous Kleene ω-algebras and exposed some of their basic properties. Continuous Kleene ω-algebras are idempotent complete semiring-semimodule pairs, and conceptually, *-continuous Kleene ω-algebras are a generalization of continuous Kleene ω-algebras in much the same way as *-continuous Kleene algebras are of continuous Kleene algebras: In *-continuous Kleene algebras, suprema of finite sets and of sets of powers are required to exist and to be preserved by the product; in *-continuous Kleene ω-algebras these suprema are also required to be preserved by the infinite product.

We have seen that the sets of finite and infinite languages over an alphabet are the free continuous Kleene ω-algebras, and that the free finitary *-continuous Kleene ω-algebras are given by the sets of regular languages and of finite unions of finitary infinite products of regular languages. A characterization of the free (non-finitary) *-continuous Kleene ω-algebras (and whether they even exist) is left open.

We have seen that every *-continuous Kleene ω-algebra is an iteration semiring-semimodule pair, hence also matrix-vector semiring-semimodule pairs over *-continuous Kleene ω-algebras are iteration semiring-semimodule pairs. In the second paper of the series [15], we will apply the algebraic setting developed here in order to solve energy problems.

References

[1] Jean Berstel and Christophe Reutenauer. *Noncommutative Rational Series With Applications.* Cambridge Univ. Press, 2010.

[2] Stephen L. Bloom and Zoltán Ésik. *Iteration Theories: The Equational Logic of Iterative Processes.* EATCS monographs on theoretical computer science. Springer-Verlag, 1993.

[3] Patricia Bouyer, Uli Fahrenberg, Kim G. Larsen, and Nicolas Markey. Timed automata with observers under energy constraints. In Karl Henrik Johansson and Wang Yi, editors, *HSCC*, pages 61–70. ACM, 2010.

[4] Patricia Bouyer, Uli Fahrenberg, Kim G. Larsen, Nicolas Markey, and Jiří Srba. Infinite runs in weighted timed automata with energy constraints. In Franck Cassez and Claude Jard, editors, *FORMATS*, volume 5215 of *Lect. Notes Comput. Sci.*, pages 33–47. Springer-Verlag, 2008.

[5] Patricia Bouyer, Kim G. Larsen, and Nicolas Markey. Lower-bound-constrained runs in weighted timed automata. *Perform. Eval.*, 73:91–109, 2014.

[6] Romain Brenguier, Franck Cassez, and Jean-François Raskin. Energy and mean-payoff timed games. In Martin Fränzle and John Lygeros, editors, *HSCC*, pages 283–292. ACM, 2014.

[7] Krishnendu Chatterjee and Laurent Doyen. Energy parity games. *Theor. Comput. Sci.*, 458:49–60, 2012.

[8] John H. Conway. *Regular Algebra and Finite Machines.* Chapman and Hall, 1971.

[9] Aldric Degorre, Laurent Doyen, Raffaella Gentilini, Jean-François Raskin, and Szymon Toruńczyk. Energy and mean-payoff games with imperfect information. In Anuj Dawar and Helmut Veith, editors, *CSL*, volume 6247 of *Lect. Notes Comput. Sci.*, pages 260–274. Springer-Verlag, 2010.

[10] Manfred Droste, Werner Kuich, and Heiko Vogler. *Handbook of Weighted Automata*. EATCS Monographs in Theoretical Computer Science. Springer-Verlag, 2009.

[11] Zoltán Ésik. Iteration semirings. In Masami Ito and Masafumi Toyama, editors, *DLT*, volume 5257 of *Lect. Notes Comput. Sci.*, pages 1–20. Springer-Verlag, 2008.

[12] Zoltán Ésik, Uli Fahrenberg, and Axel Legay. *-continuous Kleene ω-algebras. In Igor Potapov, editor, *DLT*, volume 9168 of *Lect. Notes Comput. Sci.*, pages 240–251. Springer-Verlag, 2015.

[13] Zoltán Ésik, Uli Fahrenberg, and Axel Legay. *-continuous Kleene ω-algebras for energy problems. In Ralph Matthes and Matteo Mio, editors, *FICS*, volume 191 of *Electr. Proc. Theor. Comput. Sci.*, pages 48–59, 2015.

[14] Zoltán Ésik, Uli Fahrenberg, Axel Legay, and Karin Quaas. Kleene algebras and semimodules for energy problems. In Dang Van Hung and Mizuhito Ogawa, editors, *ATVA*, volume 8172 of *Lect. Notes Comput. Sci.*, pages 102–117. Springer-Verlag, 2013.

[15] Zoltán Ésik, Uli Fahrenberg, Axel Legay, and Karin Quaas. An algebraic approach to energy problems II: The algebra of energy functions. *Acta Cyb.*, 2017. In this issue.

[16] Zoltán Ésik and Werner Kuich. Rationally additive semirings. *J. Univ. Comput. Sci.*, 8(2):173–183, 2002.

[17] Zoltán Ésik and Werner Kuich. Inductive star-semirings. *Theor. Comput. Sci.*, 324(1):3–33, 2004.

[18] Zoltán Ésik and Werner Kuich. A semiring-semimodule generalization of ω-regular languages, Parts 1 and 2. *J. Aut. Lang. Comb.*, 10:203–264, 2005.

[19] Zoltán Ésik and Werner Kuich. On iteration semiring-semimodule pairs. *Semigroup Forum*, 75:129–159, 2007.

[20] Uli Fahrenberg, Line Juhl, Kim G. Larsen, and Jiří Srba. Energy games in multiweighted automata. In Antonio Cerone and Pekka Pihlajasaari, editors, *ICTAC*, volume 6916 of *Lect. Notes Comput. Sci.*, pages 95–115. Springer-Verlag, 2011.

[21] Uli Fahrenberg, Axel Legay, and Karin Quaas. Büchi conditions for generalized energy automata. In Manfred Droste and Heiko Vogler, editors, *WATA*, page 47, 2012.

[22] Jonathan S. Golan. *Semirings and their Applications*. Springer-Verlag, 1999.

[23] Line Juhl, Kim G. Larsen, and Jean-François Raskin. Optimal bounds for multiweighted and parametrised energy games. In Zhiming Liu, Jim Woodcock, and Huibiao Zhu, editors, *Theories of Programming and Formal Methods*, volume 8051 of *Lect. Notes Comput. Sci.*, pages 244–255. Springer-Verlag, 2013.

[24] Dexter Kozen. On Kleene algebras and closed semirings. In Branislav Rovan, editor, *MFCS*, volume 452 of *Lect. Notes Comput. Sci.*, pages 26–47. Springer-Verlag, 1990.

[25] Dexter Kozen. A completeness theorem for Kleene algebras and the algebra of regular events. *Inf. Comput.*, 110(2):366–390, 1994.

[26] Dominique Perrin and Jean-Éric Pin. *Infinite Words: Automata, Semigroups, Logic and Games*. Academic Press, 2004.

[27] Karin Quaas. On the interval-bound problem for weighted timed automata. In Adrian Horia Dediu, Shunsuke Inenaga, and Carlos Martín-Vide, editors, *LATA*, volume 6638 of *Lect. Notes Comput. Sci.*, pages 452–464. Springer-Verlag, 2011.

[28] Yaron Velner, Krishnendu Chatterjee, Laurent Doyen, Thomas A. Henzinger, Alexander Moshe Rabinovich, and Jean-François Raskin. The complexity of multi-mean-payoff and multi-energy games. *Inf. Comput.*, 241:177–196, 2015.

[29] Thomas Wilke. An Eilenberg theorem for infinity-languages. In Javier Leach Albert, Burkhard Monien, and Mario Rodríguez-Artalejo, editors, *ICALP*, volume 510 of *Lect. Notes Comput. Sci.*, pages 588–599. Springer-Verlag, 1991.