Acta Cybernetica **23** (2018) 815–828.

# Partition-Crossing Hypergraphs*

Csilla Bujtás[a] and Zsolt Tuza[ab]

**Abstract**

For a finite set $X$, we say that a set $H \subseteq X$ crosses a partition $\mathcal{P} = (X_1, \ldots, X_k)$ of $X$ if $H$ intersects $\min(|H|, k)$ partition classes. If $|H| \geq k$, this means that $H$ meets all classes $X_i$, whilst for $|H| \leq k$ the elements of the crossing set $H$ belong to mutually distinct classes. A set system $\mathcal{H}$ crosses $\mathcal{P}$, if so does some $H \in \mathcal{H}$. The minimum number of $r$-element subsets, such that every $k$-partition of an $n$-element set $X$ is crossed by at least one of them, is denoted by $f(n, k, r)$.

The problem of determining these minimum values for $k = r$ was raised and studied by several authors, first by Sterboul in 1973 [*Proc. Colloq. Math. Soc. J. Bolyai*, Vol. 10, Keszthely 1973, North-Holland/American Elsevier, 1975, pp. 1387–1404]. The present authors determined asymptotically tight estimates on $f(n, k, k)$ for every fixed $k$ as $n \to \infty$ [*Graphs Combin.*, 25 (2009), 807–816]. Here we consider the more general problem for two parameters $k$ and $r$, and establish lower and upper bounds for $f(n, k, r)$. For various combinations of the three values $n, k, r$ we obtain asymptotically tight estimates, and also point out close connections of the function $f(n, k, r)$ to Turán-type extremal problems on graphs and hypergraphs, or to balanced incomplete block designs.

**Keywords:** partition, set system, crossing set, Turán-type problem, hypergraph, upper chromatic number

## 1 Introduction

Let $X$ be a finite set. By a *k-partition* of $X$ we mean a partition $\mathcal{P} = (X_1, \ldots, X_k)$ into *precisely k nonempty classes*. For a natural number $r \geq 2$, the family of all $r$-element subsets of $X$ — also termed *r-subsets*, for short (similarly, '$r$-set' may abbreviate '$r$-element set') — is denoted by $\binom{X}{r}$.A set system $\mathcal{H}$ over $X$ is *r-uniform* if $\mathcal{H} \subseteq \binom{X}{r}$.We shall use the term *hypergraph* for the pair $(X, \mathcal{H})$ — where $X$ is

*Research supported in part by the National Research, Development and Innovation Office – NKFIH under the grant SNN 116095.

[a]Faculty of Information Technology, University of Pannonia, H–8200 Veszprém, Egyetem u. 10, Hungary

[b]Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, H–1053 Budapest, Reáltanoda u. 13–15, Hungary

DOI: 10.14232/actacyb.23.3.2018.6

the set of *vertices* and $\mathcal{H}$ is the set of *edges* or *hyperedges* — and also for the set system $\mathcal{H}$ itself, when $X$ is understood. The number of vertices is called the *order* of $\mathcal{H}$, and will usually be denoted by $n$.

Given a $k$-partition $\mathcal{P} = (X_1, \dots, X_k)$ of $X$, we say that an $r$-set $H \subseteq X$ *crosses* $\mathcal{P}$ if $H$ intersects $\min(r, k)$ partition classes. If $r \geq k$, this means that all classes $X_i$ are intersected, whilst for $r \leq k$ the elements of the crossing set $H$ belong to mutually distinct classes. A hypergraph $\mathcal{H}$ is said to cross $\mathcal{P}$ if so does at least one of its edges $H \in \mathcal{H}$.

It is a very natural problem to ask for the minimum number $f(n, k, r)$ of $r$-subsets (minimum number of edges in an $r$-uniform hypergraph), by which every $k$-partition of the $n$-element set $X$ is crossed. The importance of this question is demonstrated by the fact that its variants have been raised by several authors independently in different contexts under various names: Sterboul in 1973 [11] (*cochromatic number*, also discussed by Berge [4, pp. 151–152], Arocha et al. in 1992 [1] (*heterochromatic number*), and Voloshin in 1995 [14, p. 43, Open problem 11] (*upper chromatic number*, also recalled in the monograph [15, Chapter 2.6, p. 43, Problem 2]. What is more, the formula

$$f(n, 2, 2) = n - 1$$

is equivalent to the basic fact that every connected graph has at least $n - 1$ edges and that this bound is tight for all $n \geq 2$.

**Further terminology and notation.** For a family $\mathfrak{F}$ of $r$-uniform hypergraphs (or graphs if $r = 2$), and for any natural number $n$, we denote by $\mathrm{ex}(n, \mathfrak{F})$ the corresponding *Turán number*; that is, the maximum number of edges in an $r$-uniform hypergraph of order $n$ that does not contain any subhypergraph isomorphic to any $\mathcal{F} \in \mathfrak{F}$. If $\mathfrak{F}$ consists of just one hypergraph $\mathcal{F}$, we simply write $\mathrm{ex}(n, \mathcal{F})$ instead of $\mathrm{ex}(n, \{\mathcal{F}\})$.

An $r$-uniform hypergraph $(X, \mathcal{H})$ is *$r$-partite* if it admits a vertex partition $X_1 \cup \dots \cup X_r = X$ such that $|H \cap X_i| = 1$ for all $H \in \mathcal{H}$ and all $1 \leq i \leq r$. If $\mathcal{H}$ consists of *all* $r$-sets meeting each $X_i$ in precisely one vertex, then we call it a *complete $r$-partite hypergraph.*

**Earlier results.** One can observe that a hypergraph crosses all 2-partitions of its vertex set if and only if it is connected. For this reason, beyond the equation $f(n, 2, 2) = n - 1$ mentioned above, we obtain that

$$f(n, 2, r) = \left\lceil \frac{n-1}{r-1} \right\rceil$$

because this is the minimum number of edges[1] in a connected $r$-uniform hypergraph of order $n$.

[1]It is well known that if $(X, \mathcal{H})$ is a *connected* hypergraph, then $\sum_{H \in \mathcal{H}} (|H| - 1) \geq |X| - 1$. The earliest source of this inequality that we have been able to find is Berge's classic book [3], where Proposition 4 on page 392 is stated more generally for a given number of connected components.

Let us observe further that the case of $r = 2$ simply means graphs with at most $k-1$ connected components, therefore

$$f(n,k,2) = n-k+1.$$

This strong relationship with connected components, however, does not extend to $r > 2$.

As far as we know, for $k \geq 3$ and $r \geq 3$ only the 'diagonal case' $k = r$ of $f(n,k,r)$ has been studied up to now. Below we quote the known results, using the simplified notation $f(n,k)$ for $f(n,k,k)$.

- $f(n,k) \geq \frac{2}{n-k+2}\binom{n}{k}$, for every $n \geq k \geq 3$ ([12]; later proved independently in [1], and also rediscovered in [8]).
- $f(n,3) = \lceil \frac{n(n-2)}{3} \rceil$, for every $n \geq 3$ ([7]; proved independently in a series of papers whose completing item is [2]; see also [13] for partial results).
- $f(n,n-2) = \binom{n}{2} - \mathrm{ex}(n,\{C_3,C_4\})$ holds[2] for every $n \geq 4$, where the last term is the Turán number for graphs of girth 5 ([12]).

Although the exact value of $f(n,k)$ is not known for any $k > 3$, its asymptotic behavior has been determined for quite a wide range of $k$.

**Theorem 1** ([5])**.** *Assume $n > k > 2$.*

(*i*) $f(n,k) \leq \frac{2}{n-1}\binom{n-1}{k} + \frac{n-1}{k-1}\left(\binom{n-2}{k-2} - \binom{n-k-1}{k-2}\right)$ *for all $n$ and $k$.*

(*ii*) $f(n,k) = (1+o(1))\,\frac{2}{k}\binom{n-2}{k-1}$ *for all $k = o(n^{1/3})$ as $n \to \infty$.*

**Structure of the paper.** In Section 2, we first prove several preliminary results, also including an inequality for non-uniform partition-crossing hypergraphs in terms of the edge sizes. Then, we turn to uniform set systems and study the function $f(n,k,r)$ separately under the conditions $k \leq r$ and $r \leq k$. We prove general lower and upper bounds for $f(n,k,r)$ in both cases. In Section 3, we assume that $n-k$ and $n-r$ are fixed while $n \to \infty$, and give asymptotically tight estimates for $f(n,k,r)$. It is worth noting that the latter problem can be reduced to Turán-type problems if $k \leq r$, while the same question leads us to the theory of balanced incomplete block designs if $r \leq k$ is assumed.

# 2 General estimates

Most of this section deals with uniform hypergraphs; but we shall also put comments on non-uniform ones which cross either all partitions or at least some large families of partitions. Nevertheless the uniform systems play a central role in partition crossing, what will turn out already in the next subsection.

[2]It was quoted with a misprint in the paper [5].

## 2.1 Monotonicity

**Proposition 2.** *For every three integers $r, k, k'$, if either*

(*i*) $2 \le r \le k \le k' \le n$, *or*

(*ii*) $2 \le k' \le k \le r \le n$

*holds, and an $r$-uniform hypergraph $\mathcal{H}$ crosses all $k$-partitions of the vertex set, then $\mathcal{H}$ crosses all $k'$-partitions, as well. As a consequence, for every four integers $n, k, k', r$ satisfying (i) or (ii) we have*

$$f(n, k, r) \ge f(n, k', r).$$

**Proof** Assume that an $r$-uniform hypergraph $\mathcal{H}$ crosses all $k$-partitions of the vertex set $X$. Consider a $k'$-partition $\mathcal{P}' = (X_1, \ldots, X_{k'})$ of $X$.

(*i*) If $r \le k \le k'$, take the union of the last $k' - k + 1$ partition classes of $\mathcal{P}'$. Due to our assumption, $\mathcal{H}$ crosses the $k$-partition $\mathcal{P} = (X_1, \ldots, X_{k-1}, \bigcup_{i=k}^{k'} X_i)$ obtained. Since $r \le k$, this means that there exists an $H \in \mathcal{H}$ which contains at most one element from each partition class of $\mathcal{P}$. Hence, the same $H$ and consequently, $\mathcal{H}$ as well, crosses the $k'$-partition $\mathcal{P}'$.

(*ii*) Next, assume that $k' \le k \le r$ holds. Since the statement clearly holds for $k' = k$, we may suppose $k' < k \le n$. Then, some of the $k'$ partition classes of $\mathcal{P}'$ can be split into nonempty parts such that a $k$-partition $\mathcal{P}$ is obtained. By assumption, some $H \in \mathcal{H}$ crosses $\mathcal{P}$. This means that the $r$-element $H$ contains at least one element from each partition class. By the construction of $\mathcal{P}$, $H$ contains at least one element from every partition class of $\mathcal{P}'$; that is, $\mathcal{H}$ crosses $\mathcal{P}'$.

Since the above arguments are valid for any $k'$-partition $\mathcal{P}'$, the statements follow. □

The analogous property is valid for the other parameter of $f(n, k, r)$ as well.

**Proposition 3.** *For every four integers $n, k, r, r'$, if*

(*i*) $2 \le r' \le r \le k \le n$, *or*

(*ii*) $2 \le k \le r \le r' \le n$ *holds, then*

$$f(n, k, r) \ge f(n, k, r').$$

**Proof** Consider an $r$-uniform hypegraph $(X, \mathcal{H})$ of size $f(n, k, r)$ which crosses all $k$-partitions of the $n$-element vertex set $X$.

(*i*) If $r' \le r \le k$, then for each $H \in \mathcal{H}$ choose an $r'$-element subset $H'$ and define the $r'$-uniform set system $\mathcal{H}' = \{H' \mid H \in \mathcal{H}\}$. Since for every $k$-partition $\mathcal{P}$ there exists an $H \in \mathcal{H}$ which contains at most one element from each partition class, the same is true for the corresponding $H' \in \mathcal{H}'$. Hence, $\mathcal{H}'$ crosses all $k$-partitions and has at most $f(n, k, r)$ elements. This proves that $f(n, k, r) \ge f(n, k, r')$.

(*ii*) In the other case we have $k \le r \le r'$. Let each $H \in \mathcal{H}$ be extended to an arbitrary $r'$-element $H'$. We observe that the $r'$-uniform set system $\mathcal{H}' = \{H' \mid$

$H \in \mathcal{H}\}$ has at most $f(n,k,r)$ elements and crosses all $k$-partitions. Indeed, for every $k$-partition $\mathcal{P}$, there exists some $H \in \mathcal{H}$ intersecting each partition class of $\mathcal{P}$, and hence the same is true for the corresponding $H' \in \mathcal{H}'$. This yields again that $f(n,k,r) \geq f(n,k,r')$ is valid. □

The following corollaries show the central role of the 'symmetric' case $k = r$:

**Corollary 4.** *If an $r$-uniform hypergraph $\mathcal{H}$ crosses all $r$-partitions of the vertex set $X$, then $\mathcal{H}$ crosses all partitions of $X$.*

Numerically, we have obtained that the function $f_{n,r}(x) = f(n,x,r)$ (where $x$ is an integer in the range $2 \leq x \leq n$) has its maximum value when $x = r$; and the situation is similar if $n$ and $k$ are fixed and $r$ is variable; that is, the function $f_{n,k}(x) = f(n,k,x)$ attains its maximum at $x = k$.

**Corollary 5.** *For every three integers $n \geq k, r \geq 2$,*

$$f(n,k,r) \leq f(n,k,k).$$

**Corollary 6.** *For every three integers $n \geq k, r \geq 2$,*

$$f(n,k,r) \leq f(n,r,r).$$

## 2.2 Lower bound for non-uniform systems

For hypergraphs without very small edges, we prove the following general inequality.

**Theorem 7.** *Let $k \geq 2$ be an integer, and let $(X, \mathcal{H})$ be a hypergraph of order $n$, which contains no edge $H \in \mathcal{H}$ of cardinality smaller than $k$. If $\mathcal{H}$ crosses all $k$-partitions of $X$, then*

$$\sum_{H \in \mathcal{H}} \binom{|H|}{k} \frac{1}{|H| - k + 2} \geq \binom{n}{k} \frac{1}{n - k + 2}.$$

**Proof** Since $|H| \geq k$ holds for every $H \in \mathcal{H}$, a $k$-partition $\mathcal{P}$ of $X$ is crossed by $\mathcal{H}$ if, and only if, there exists an edge in $\mathcal{H}$ which intersects all the $k$ partition classes of $\mathcal{P}$. For every $(k-2)$-element subset $Y = \{x_1, \ldots, x_{k-2}\}$ of $X$, define

$$\mathcal{H}_Y^- = \{A \mid A \subseteq (X \setminus Y) \ \wedge \ (A \cup Y) \in \mathcal{H}\}.$$

We claim that $\mathcal{H}_Y^-$ is connected on $X \setminus Y$. Assume for a contradiction that it is not, and denote one of its components by $Z$. Consider the $k$-partition

$$\{x_1\}, \ \ldots, \ \{x_{k-2}\}, \ Z, \ X \setminus (Y \cup Z)$$

This is not crossed by $\mathcal{H}$ since a crossing set $H$ would contain all of $x_1, \ldots, x_{k-2}$, moreover at least one element from each of the last two partition classes, what contradicts to our assumption on disconnectivity.

Therefore, $\mathcal{H}_Y^-$ must be connected on the $(n-k+2)$-element $X \setminus Y$, and hence

$$\sum_{A \in \mathcal{H}_Y^-} (|A| - 1) \geq (n-k+2) - 1.$$

The corresponding inequality holds for every $Y \in \binom{X}{k-2}$. Moreover, for each edge $H \in \mathcal{H}$, every $(|H|-k+2)$-element subset of $H$ is counted in exactly one of these $\binom{n}{k-2}$ inequalities. Hence, we have

$$\sum_{H \in \mathcal{H}} \binom{|H|}{k-2} (|H| - k + 1) \geq \binom{n}{k-2} (n-k+1),$$

which is equivalent to the assertion. □

Beside the rather trivial hypergraph with vertex set $X$ and edge set $\mathcal{H} = \{X\}$, which crosses every partition of $X$, the following construction also shows that Theorem 7 is tight.

**Example 8.** Let $n = |X| = 2m$ be even. Let the edge set of $\mathcal{H}$ consist of one $m$-subset $H$ of $X$ together with $m$ mutually disjoint 2-element sets, each of which has precisely one vertex in $H$ and one in $X \setminus H$. This hypergraph crosses all partitions of $X$. Indeed, if none of the $m$ selected 2-sets crosses a partition $\mathcal{P}$, then each class of $\mathcal{P}$ meets $H$. For this $\mathcal{H}$, both sides of the inequality in Theorem 7 equal $\frac{n-1}{2}$ for $k = 2$. (We necessarily have $k = 2$, due to the conditions in the theorem.)

## 2.3 Estimates for $k \leq r$

The following lower bound follows immediately from Theorem 7.

**Corollary 9.** *For every three integers $n \geq r \geq k \geq 2$ the inequality*

$$f(n,k,r) \geq \frac{\binom{n}{k}}{\binom{r}{k}} \cdot \frac{r-k+2}{n-k+2}$$

*holds.*

Next, we prove a general asymptotic upper bound.

**Proposition 10.** *For every two fixed integers $r \geq k \geq 2$ the inequality*

$$f(n,k,r) \leq \frac{\binom{n}{k}}{\binom{r}{k}} \cdot \frac{r}{n} + o(n^{k-1})$$

*holds as $n \to \infty$.*

**Proof** If $k = r$, then the inequality holds also without the error term, and as a matter of fact, an even better upper bound on $f(n,r,r)$ is guaranteed by Theorem 1($i$). Hence, we may suppose $r > k$.

Consider an $n$-element vertex set $X = X' \cup \{z\}$ and an $(r-1)$-uniform hypergraph $\mathcal{H}'$ over $X'$ such that every $(k-1)$-subset of $X'$ is covered by at least one $H' \in \mathcal{H}'$. By Rödl's theorem [10], such hypergraphs $\mathcal{H}'$ of size

$$|\mathcal{H}'| = \frac{\binom{n-1}{k-1}}{\binom{r-1}{k-1}} + o(n^{k-1})$$

exist as $n \to \infty$.

Consider now the $r$-uniform hypergraph

$$\mathcal{H} = \{H' \cup \{z\} \mid H' \in \mathcal{H}'\}.$$

For every $k$-partition $\mathcal{P}$ we can choose a $k$-element crossing set $A$ with $z \in A$, by picking any vertex from each of those classes of $\mathcal{P}$ which do not contain $z$. Since $A \setminus \{z\} \subset H'$ for some $H' \in \mathcal{H}'$, it follows that $\mathcal{H}$ crosses $\mathcal{P}$. □

We note that, beyond tight asymptotics, the above construction can be applied also to derive exact results for some restricted combinations of the parameters.

Next, we establish recursive relations to get lower bounds on $f(n,k,r)$. Although they do not improve earlier bounds automatically, such inequalities may raise the possibility to propagate better estimates for larger values of the parameters when they are available for smaller ones.

**Proposition 11.** *If $n \geq r \geq r' \geq k \geq 2$, then*

$$f(n,k,r) \geq \frac{f(n,k,r')}{f(r,k,r')}.$$

**Proof** Given an $n$-element vertex set $X$, consider an $r$-uniform hypergraph $\mathcal{H}$ of size $f(n,k,r)$ which crosses all $k$-partitions. Then, for each $H_j \in \mathcal{H}$ construct an $r'$-uniform hypergraph $\mathcal{H}'_j$ crossing all $k$-partitions of the set $H_j$. This can be done such that $|\mathcal{H}'_j| = f(r,k,r')$, hence the $r'$-uniform $\mathcal{R} = \bigcup_{j=1}^{f(n,k,r)} \mathcal{H}'_j$ contains at most $f(n,k,r) \cdot f(r,k,r')$ sets.

For every partition $\mathcal{P} = (X_1, \ldots, X_k)$, there exists some $H_j \in \mathcal{H}$ with $|X_i \cap H_j| \geq 1$ for every $1 \leq i \leq k$. Moreover, for this $j$, the system $\mathcal{H}'_j$ crosses also the $k$-partition $X_1 \cap H_j, \ldots, X_k \cap H_j$. Consequently, there exists an $R \in \mathcal{H}'_j \subseteq \mathcal{R}$ which intersects every class of $\mathcal{P}$. Thus, $\mathcal{R}$ crosses all $k$-partitions of $X$, therefore

$$f(n,k,r) \cdot f(r,k,r') \geq f(n,k,r')$$

holds and the theorem follows. □

Particularly, if $r'$ is chosen to be equal to $k$, we obtain that

$$f(n,k,r) \geq \frac{f(n,k)}{f(r,k)}.$$

Since $f(k+1,k)=k$, then

$$f(n,k,k+1) \geq \frac{f(n,k)}{k}.$$

More generally, applying Proposition 11 repeatedly, and using the fact $f(i,k,i-1)=k$ that is valid for all $i>k$ (cf. Proposition 19 below), we obtain the following lower bound.

**Corollary 12.** *If $n \geq r \geq k \geq 2$, then*

$$f(n,k,r) \geq \frac{f(n,k)}{\prod_{i=k+1}^{r} f(i,k,i-1)} = \frac{f(n,k)}{k^{r-k}}.$$

## 2.4 Estimates for $k \geq r$

**Proposition 13.** *For every three integers $n \geq k \geq r \geq 2$ the inequality*

$$f(n,k,r) \geq \frac{\binom{n}{r-1}}{\binom{k-2}{r-2}} \cdot \frac{n-k+2}{r(n-r+2)}$$

*holds.*

**Proof** Consider an $r$-uniform hypergraph $\mathcal{H}$ on the $n$-element vetex set $X$, such that $\mathcal{H}$ crosses all $k$-partitions. We claim that every $(k-1)$-subset of $X$ shares at least $r-1$ vertices with some $H \in \mathcal{H}$. Suppose for a contradiction that a set $A \in \binom{X}{k-1}$ intersects no $H \in \mathcal{H}$ in more than $r-2$ elements. Then every $H \in \mathcal{H}$ has at least two vertices in $X \setminus A$. Now, consider the $k$-partition whose first partition class is $X \setminus A$ and the others are singletons. This partition is not crossed by $\mathcal{H}$, which is a contradiction.

Consequently, every $(k-1)$-element subset of $X$ must contain an $(r-1)$-element subset of some $H \in \mathcal{H}$. Hence, for the 'shadow' system

$$\partial_{r-1} = \left\{ B \mid \exists H \in \mathcal{H} \quad \text{s.t.} \ B \in \binom{H}{r-1} \right\},$$

the independence number must be smaller than $k-1$. Taking into consideration the lower bound on the complementary Turán number $T(n,k-1,r-1) = \binom{n}{r-1} - \mathrm{ex}\,(n, \mathcal{K}_{k-1}^{(r-1)})$ of complete uniform hypergraphs, as proved in [6],

$$r \cdot |\mathcal{H}| \geq |\partial_{r-1}| \geq T(n,k-1,r-1) \geq \frac{\binom{n}{r-1}}{\binom{k-2}{r-2}} \cdot \frac{n-k+2}{n-r+2}$$

is obtained, from which the statement follows. □

For $k$ and $r$ fixed, the lower bound gives the right order $O(n^{r-1})$, as shown by the following construction.

**Theorem 14.** *Let $k \geq 3$, and assume that $k-2$ is divisible by $r-2$. If $n \to \infty$, then*
$$f(n,k,r) \leq \frac{2(r-2)^{r-2}}{r(k-2)^{r-2}} \binom{n}{r-1} + o(n^{r-1}).$$

**Proof** Let $|X| = n$, denote $q = (k-2)/(r-2)$, and write $n' = \lceil (n-1)/q \rceil + 1$. We fix a special element $z \in X$, and partition the remaining $(n-1)$-element set $X \setminus \{z\}$ into $q$ nearly equal parts, the largest one having $n'-1$ vertices:
$$X = Y_1 \cup \cdots \cup Y_q \cup \{z\}, \qquad |Y_i| = \left\lfloor \frac{n+i-2}{q} \right\rfloor \qquad \text{for all } \ 1 \leq i \leq q.$$

For every set $Y_i \cup \{z\}$ we take an optimal $r$-uniform hypergraph $\mathcal{H}_i$ crossing all $r$-partitions. By Theorem 1, we have
$$|\mathcal{H}_i| \leq f(n',r) \leq (1+o(1))\, \frac{2}{r} \binom{n'-2}{r-1}.$$

Here $n'-2 < n/q = \frac{r-2}{k-2}\, n$, hence the binomial coefficient on the right-hand side is smaller than $\left(\frac{r-2}{k-2}\right)^{r-1} \binom{n}{r-1}$ Let $\mathcal{H} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_q$. By the estimates above, we have
$$|\mathcal{H}| \leq \frac{2(r-2)^{r-2}}{r(k-2)^{r-2}} \binom{n}{r-1} + o(n^{r-1})$$
as $n \to \infty$. To complete the proof, it suffices to show that $\mathcal{H}$ crosses all $k$-partitions of $X$.

Let $\mathcal{P}$ be any partition into $k = 1 + q(r-2) + 1$ classes. One of the classes contains $z$. By the pigeonhole principle, there is an index $i$ ($1 \leq i \leq q$) such that, among the other $k-1$ classes of $\mathcal{P}$ there exist at least $r-1$ which have at least one vertex in $Y_i$. Hence we have a partition $\mathcal{P}_i$ induced on $Y_i \cup \{z\}$, with some number $r'$ of classes, where $r' \geq r$. Since the $r$-uniform $\mathcal{H}_i$ crosses all $r$-partitions of $Y_i \cup \{z\}$, Corollary 4 implies that $\mathcal{H}_i$ crosses $\mathcal{P}_i$, too. That is, an $r$-set $H_i \in \mathcal{H}_i$ has all its vertices in mutually distinct classes of $\mathcal{P}_i$, which are then in distinct classes of $\mathcal{P}$ as well. Thus, $\mathcal{H}$ crosses $\mathcal{P}$. $\square$

The idea behind the construction of the above proof also yields the following additive upper bound.

**Proposition 15.** *Suppose that all the following conditions hold:*

- $n \geq k \geq r$,
- $n \leq 1 - p + \sum_{i=1}^{p} n_i$,
- $k \leq 2 - 2p + \sum_{i=1}^{p} k_i$,
- $n_i \geq k_i \geq r$ *for every* $1 \leq i \leq p$.

*Then*
$$f(n,k,r) \leq \sum_{i=1}^{p} f(n_i,k_i,r).$$

# 3 Asymptotics for large $k$ and $r$

In this section we prove asymptotically tight estimates for $f(n,k,r)$, under the assumptions that the differences $s = n-k$ and $t = n-r$ are fixed and $n \to \infty$. For this purpose, we need to consider two types of complementation — one from the viewpoint of set theory, the other one analogously to graph theory.

- Given a hypergraph $(X, \mathcal{H})$, let $(X, \mathcal{H}^c)$ denote the hypergraph of the complements of the edges. That is, $\mathcal{H}^c = \{X \setminus H \mid H \in \mathcal{H}\}$.
- Given an $r$-*uniform* hypergraph $(X, \mathcal{H})$, its complement $\overline{\mathcal{H}}$ contains all the $r$-element subsets of $X$ which are missing from $\mathcal{H}$. Formally, $\overline{\mathcal{H}} = \binom{X}{r} \setminus \mathcal{H}$.

**Theorem 16.** *Let $s$ and $t$ be fixed, with $s \le t$, and $n \to \infty$. Then*

$$f(n, n-s, n-t) = (1+o(1))\frac{\binom{n}{s}}{\binom{t}{s}}.$$

**Proof** First we prove the lower bound $f(n, n-s, n-t) \ge (1-o(1))\binom{n}{s}/\binom{t}{s}$. Suppose for a contradiction that there exists a constant $\epsilon > 0$ and an infinite sequence of $r$-uniform hypergraphs $(X, \mathcal{H})$ with $n$ vertices and $m$ edges, edge size $r = n-t$, such that $\mathcal{H}$ crosses all $(n-s)$-partitions of its $n$-element vertex set $X$, but $m \le \frac{\binom{n}{s}}{\binom{t}{s}} - \epsilon n^s$. We consider the $t$-uniform hypergraph $\mathcal{H}^c$ whose edges are the complements of the edges of $\mathcal{H}$. Since it has $m$ edges, there are at least $\epsilon\binom{t}{s}n^s \ge Cn^s$ distinct $s$-tuples of $X$ not covered by the edges of $\mathcal{H}^c$. Note that $C$ can be chosen as a positive absolute constant, valid for all possible values of $n$, once we fix the triplet $s, t, \epsilon$. We let $\mathcal{F}$ to be the collection of $s$-tuples not contained in any of the edges of $\mathcal{H}^c$. Hence $|\mathcal{F}| \ge Cn^s$.

Consider now the complete $s$-partite hypergraph $\mathcal{F}_s$ on $2s$ vertices, each partite set having just 2 vertices. That is, the vertex set of $\mathcal{F}_s$ is $V_1 \cup \cdots \cup V_s$, with $|V_i| = 2$ for all $1 \le i \le s$, and an $s$-element set $F$ is an edge in $\mathcal{F}_s$ if and only $|F \cap V_i| = 1$ for every $i$. It is well known that the Turán number of $\mathcal{F}_s$ satisfies

$$\mathrm{ex}\,(n, \mathcal{F}_s) = o(n^s)$$

for any fixed $s$, as $n \to \infty$. Thus, if $n$ is chosen to be sufficiently large, $\mathcal{F}$ contains a subhypergraph $\mathcal{F}'$ isomorphic to $\mathcal{F}_s$.

We now consider the partition $\mathcal{P}$ of $X$ into $k = n-s$ classes in which the $s$ partite sets of $\mathcal{F}'$ are 2-element classes, and the other $n-2s$ classes are singletons. By assumption, $\mathcal{H}$ crosses $\mathcal{P}$. It means that there exists an edge $H \in \mathcal{H}$ that meets each of the 2-element classes in at most one vertex. Let $x_i$ be a vertex in $V_i \setminus H$ for $i = 1, \ldots, s$. Then $\{x_1, \ldots, x_s\} \notin \mathcal{F}$, which is a contradiction to $\{x_1, \ldots, x_s\} \in \mathcal{F}_s \subset \mathcal{F}$, hence completing the proof of the lower bound.

Next, we prove the upper bound $f(n, n-s, n-t) \le (1+o(1))\binom{n}{s}/\binom{t}{s}$. For every $n$, consider a $t$-uniform hypergraph $\mathcal{H}_n^0$ on the $n$-element vertex set $X$, such that each $s$-subset of $X$ is contained in a $t$-set $H \in \mathcal{H}_n^0$. By Rödl's theorem [10], if $s$ and $t$ are fixed and $n \to \infty$, then $\mathcal{H}_n^0$ can be chosen such that $|\mathcal{H}_n^0| = \binom{n}{s}/\binom{t}{s} + o(n^s)$.

Starting with such a system $\mathcal{H}_n^0$, we consider the hypergraph $\mathcal{H}_n = (\mathcal{H}_n^0)^c$ whose edge set is $\{X \setminus H \mid H \in \mathcal{H}_n^0\}$. By the complementation, for $k = n-s$ and $r = n-t$, each $k$-element subset of $X$ contains some $r$-element set $H \in \mathcal{H}_n$. Then, for any $k$-partition $\mathcal{P}$ of $X$, we can pick one vertex from each partition class, and this $k$-element set has to contain an edge $H \in \mathcal{H}_n$. Hence, $\mathcal{H}_n$ crosses all $k$-partitions of the vertex set, moreover we have $|\mathcal{H}_n| = |\mathcal{H}_n^0|$. This yields the claimed upper bound on $f(n, n-s, n-t)$. □

In particular, for $s = t$ we have the following consequence. We formulate it for $s \geq 2$, because the case of $f(n, n, n) = 1$ is trivial and the exact formula of $f(n, n-1, n-1) = n-1$ is a particular case of Proposition 19 below.

**Corollary 17.** *For every $s \geq 2$, as $n \to \infty$*

$$f(n, n-s, n-s) = \binom{n}{s} + o(n^s).$$

To study the other range for $f(n, n-s, n-t)$, namely $s > t$, first we will make a simple but useful observation. We say that a set $T$ is a *transversal* of a partition[3] $\mathcal{P} = (X_1, \ldots, X_k)$ if $|T \cap X_i| \geq 1$ holds for every $i$. The complement $S = X \setminus T$ of a transversal $T$ is called an *independent set* for $\mathcal{P}$. This means that $|S \cap X_i| < |X_i|$ holds for every partition class. Let $\mathcal{I}_t(\mathcal{P})$ denote the set system containing all $t$-element independent sets for the partition $\mathcal{P}$.

**Proposition 18.** *Let $(X, \mathcal{H})$ be an $r$-uniform hypergraph with $|X| = n$, and assume that $k \leq r$. Then, $\mathcal{H}$ crosses all $k$-partitions of the vertex set $X$ if and only if for every $k$-partition $\mathcal{P}$ of $X$ we have $\mathcal{I}_{n-r}(\mathcal{P}) \not\subseteq \overline{\mathcal{H}^c}$.*

**Proof** For a given $k$-partition $\mathcal{P}$, $\mathcal{H}$ is crossing if and only if it contains a transversal $T$ for $\mathcal{P}$; that is, if $\mathcal{H}^c$ contains an $(n-r)$-element independent set for $\mathcal{P}$. This equivalently means that $\overline{\mathcal{H}^c}$ does not contain all elements of $\mathcal{I}_{n-r}(\mathcal{P})$. Consequently, $\mathcal{H}$ crosses all $k$-partitions if and only if for every $k$-partition $\mathcal{P}$, $\overline{\mathcal{H}^c}$ does not contain $\mathcal{I}_{n-r}(\mathcal{P})$ as a subsystem. □

Concerning $f(n, n-s, n-t)$ the case of $t = 1$ is very simple. Certainly $s = 0$ means that all partition classes are singletons, hence $f(n, n, r) = 1$ for all values of $r \leq n$, also including $r = n-1$. The situation for smaller $k$ is different.

**Proposition 19.** *For every $n > k \geq 1$, we have $f(n, k, n-1) = k$.*

**Proof** For $X = \{x_1, \ldots, x_n\}$ define $\mathcal{H} = \{X \setminus \{x_i\} \mid 1 \leq i \leq k\}$. Consider any $k$-partition $\mathcal{P}$. It either has a class with at least two vertices $x_i, x_j$ in the range

[3] In fact this is the same as a transversal (also called vertex cover or hitting set) of the hypergraph $(X, \{X_1, \ldots, X_k\})$ in which the classes $X_i$ of the partition are viewed as edges. This also justifies the term 'independent set' for the complementary notion.

$1 \le i < j \le k$, or a class containing both $x_n$ and some $x_i$ with $1 \le i \le k$. Then we can choose $X \setminus \{x_i\} \in \mathcal{H}$, which crosses $\mathcal{P}$. Consequently, $f(n,k,n-1) \le k$.

To see the reverse inequality $f(n,k,n-1) \ge k$, without loss of generality we may restrict our attention to the $(n-1)$-uniform hypergraph $\mathcal{H}^- = \{X \setminus \{x_i\} \mid 1 \le i \le k-1\}$ which represents all $(n-1)$-uniform ones with $k-1$ edges up to isomorphism. Then the partition

$$\{x_1\},\ \ldots,\ \{x_{k-1}\},\ \{x_k, x_{k+1}, \ldots, x_n\}$$

is not crossed by any $H \in \mathcal{H}^-$, thus $k-1$ edges are not enough. $\square$

The problem becomes more complicated for $t > 1$. First we consider the case of $r = n-2$, and then a general estimate for $k = n-s \le n-t = r$ will be given under the assumption that $s$ and $t$ are fixed.

**Proposition 20.** *For every fixed $s \ge 2$,*

(*i*) $f(n, n-s, n-2) = \binom{n}{2} - \mathrm{ex}(n, \{K_{s+1}, K_{2s} - sK_2\})$;

(*ii*) $f(n, n-s, n-2) = \frac{1}{2s-2}\, n^2 + o(n^2)$, *if* $n \to \infty$.

**Proof** Consider a graph $G = (V, F)$ of order $n$, which contains neither a complete graph $K_{s+1}$ of order $s+1$, nor a complete graph minus a perfect matching $K_{2s} - sK_2$ on $2s$ vertices. By the double complementation we obtain the $(n-2)$-uniform hypergraph $(V, \mathcal{H}) = (\overline{G})^c$ with vertex set $V$ and edge set

$$\mathcal{H} = \{V \setminus e \mid e \in \binom{V}{2}\ \wedge\ e \notin F\}.$$

We claim that $\mathcal{H}$ crosses all $(n-s)$-partitions of $V$.

First, consider a partition $\mathcal{P} = (X_1, X_2, \ldots, X_{n-s})$ with at least one partition class $|X_i| \ge 3$. We can assume without loss of generality that $|X_1| \ge 3$. We also consider the partition $\mathcal{P}'$, obtained by removing all but one vertex from each of $X_2, \ldots, X_{n-s}$ and putting these vertices into $X_1$. This $\mathcal{P}'$ has an $(s+1)$-element class $X_1'$ and further $n-s-1$ singleton classes. Since the class $X_1$ in $\mathcal{P}$ has more than two vertices, every edge of $\mathcal{H}$ meets $X_1$. Hence, the hypergraph $\mathcal{H}$ does not cross $\mathcal{P}$ if and only if each of its edges is disjoint from at least one of the classes $X_2, \ldots, X_{n-s}$. But then every edge is also disjoint from at least one of the singleton classes of $\mathcal{P}'$, and so $\mathcal{H}$ does not cross $\mathcal{P}'$ either.

Consequently, it is sufficient to ensure that $\mathcal{H}$ crosses all $(n-s)$-partitions with classes of cardinalities $(s+1, 1, \ldots, 1)$ and $(2, \ldots, 2, 1, \ldots 1)$, and this will imply that $\mathcal{P}$ crosses all $(n-s)$-partitions.

An $(n-2)$-uniform hypergraph $\mathcal{H}$ crosses every partition of type $(s+1, 1, \ldots, 1)$ if, and only if, for every $(s+1)$-element subset $S$ of $V$, there exits an edge $H \in \mathcal{H}$ with $|H \cap S| = s-1$; that is, $\mathcal{H}^c$ has an edge inside $S$, and equivalently, $G = \overline{\mathcal{H}^c}$ contains no complete subgraph $K_{s+1}$. For the other case, $\mathcal{H}$ crosses every partition of type $(2, \ldots, 2, 1, \ldots 1)$, if and only if for every $s$ disjoint pairs of vertices there exists an edge $H$ whose complement $\overline{H}$ contains two vertices from different pairs. This exactly means that $G = \overline{\mathcal{H}^c}$ does not contain a subgraph $K_{2s} - sK_2$.

Consequently, an $(n-2)$-uniform $\mathcal{H}$ crosses all $(n-s)$-partitions if and only if $G$ is $(K_{s+1}, K_{2s} - sK_2)$-free. Applying the Erdős–Stone Theorem [9], for $s \geq 3$ this yields

$$f(n, n-s, n-2) = \binom{n}{2} - \mathrm{ex}(n, \{K_{s+1}, K_{2s} - sK_2\})$$

$$= \binom{n}{2} - (1 + o(1)) \cdot \mathrm{ex}(n, K_s) = \frac{1}{2s-2}\, n^2 + o(n^2).$$

In fact the asymptotic formula is valid also for $s = 2$ because then the exclusion of $K_{2,2} \cong C_4$ implies that $\mathrm{ex}(n, \{K_{s+1}, K_{2s} - sK_2\}) = o(n^2)$. □

**Theorem 21.** *Let $s$ and $t$ be fixed, with $s > t \geq 2$, and $n \to \infty$. Then,*

$$f(n, n-s, n-t) \leq (1-c)\binom{n}{t}$$

*for some constant $c = c(s,t) > 0$.*

**Proof** Let $\mathcal{H}_t$ be the complete $t$-partite hypergraph with vertex set $X_1 \cup \cdots \cup X_t$ such that each partite class has cardinality $|X_i| = \lfloor n/t \rfloor$ or $|X_i| = \lceil n/t \rceil$. We have $|\mathcal{H}_t| = (1 - o(1))\,(n/t)^t$ as $n \to \infty$, hence there exists a universal constant $c = c(t) > 0$ such that $|\mathcal{H}_t| \geq c\binom{n}{t}$ for all $n > t$. Let $\mathcal{H} = \left(\overline{\mathcal{H}_t}\right)^c$. Then $|\mathcal{H}| \leq (1-c)\binom{n}{t}$.

We claim that $\mathcal{H}$ crosses all $(n-s)$-partitions whenever $s > t$. Indeed, let $\mathcal{P}$ be any $(n-s)$-partition of $X$. Consider an $s$-set $S$ obtained by deleting precisely one vertex from each class of $\mathcal{P}$. Since $s > t$, this $S$ contains two vertices from the same class of $\mathcal{H}_t$, say $x', x'' \in X_i$. Therefore we can take a $t$-subset $T \subset S$ containing both $x'$ and $x''$, consequently $T \notin \mathcal{H}_t$. Thus, $X \setminus T \in \mathcal{H}$ holds, and this $X \setminus T$ meets all classes of $\mathcal{P}$ because it contains all elements of $X \setminus S$. It follows that $\mathcal{H}$ crosses every $\mathcal{P}$, hence

$$f(n, n-s, n-t) \leq |\mathcal{H}| \leq (1-c)\binom{n}{t}.$$

□

# References

[1] J. L. Arocha, J. Bracho, and V. Neumann-Lara, On the minimum size of tight hypergraphs. *J. Graph Theory* **16** (1992) 319–326.

[2] J. L. Arocha and J. Tey, The size of minimum 3-trees. *J. Graph Theory* **54** (2007) 103–114.

[3] C. Berge, *Graphs and Hypergraphs.* (North-Holland, 1973)

[4] C. Berge, *Hypergraphs.* (North-Holland, 1989)

[5] Cs. Bujtás and Zs. Tuza, Smallest set-transversals of $k$-partitions. *Graphs Combin.* **25** (2009) 807–816.

[6] D. de Caen, Extension of a theorem of Moon and Moser on complete subgraphs, *Ars Combinatoria* **16** (1983) 5–10.

[7] K. Diao, G. Liu, D. Rautenbach, and P. Zhao, A note on the least number of edges of 3-uniform hypergraphs with upper chromatic number 2. *Discrete Math.* **306** (2006) 670–672.

[8] K. Diao, P. Zhao, and H. Zhou, About the upper chromatic number of a $C$-hypergraph. *Discrete Math.* **220** (2000) 67–73.

[9] P. Erdős and A. H. Stone, On the structure of linear graphs. *Bull. Amer. Math. Soc.* **52** (1946) 1087–1091.

[10] V. Rödl, On a packing and covering problem. *Europ. J. Combin.* **5** (1985) 69–78.

[11] F. Sterboul, A new combinatorial parameter. In: *Infinite and Finite Sets* (A. Hajnal et al., eds.), Colloq. Math. Soc. J. Bolyai, **10**, Vol. III, Keszthely 1973 (North-Holland/American Elsevier, 1975) 1387–1404.

[12] F. Sterboul, Un problème extrémal pour les graphes et les hypergraphes. *Discrete Math.* **11** (1975) 71–78.

[13] F. Sterboul, A problem in constructive combinatorics and related questions. In: *Combinatorics* (A. Hajnal and V. T. Sós, eds.), Colloq. Math. Soc. J. Bolyai, **18**, Vol. II, Keszthely 1976 (North-Holland, 1978) 1049–1064.

[14] V. Voloshin, On the upper chromatic number of a hypergraph. *Australasian J. Combin.* **11** (1995) 25–45.

[15] V. I. Voloshin, *Coloring Mixed Hypergraphs: Theory, Algorithms and Applications.* Fields Institute Monographs **17** (AMS, 2002)


*Received 27th February 2018*