Acta Cybernetica **26** (2023) 83–103.

B_π^R-Matrices, B-Matrices, and Doubly B-Matrices in the Interval Setting*

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Abstract

In this paper, we focus on generalizing B_π^R-matrices into the interval setting, including some results regarding this class. There are two possible ways to generalize B_π^R-matrices into the interval setting, but we prove that, in a sense, they are one. We derive mainly recognition methods for this interval matrix class, such as characterizations, necessary conditions, and sufficient ones.

Next, we also take a look at interval B-matrices and interval doubly B-matrices, which were introduced recently, and we present characterizations through reduction for them and for B_π^R-matrices.

Keywords: B_π^R-matrix, B-matrix, doubly B-matrix, interval analysis, interval matrix, P-matrix

1 Introduction

***P*-matrices.** An important class of matrices, in optimization as well as linear algebra and graph theory (see [7]), is the class of P-matrices. Recall that $A \in \mathbb{R}^{n\times n}$ is a *P-matrix* if all its principal minors (i.e. determinants of its principal submatrices) are positive.

The class of P-matrices has a close connection to the linear complementarity problem (which is more thoroughly described in [1]), which is one of the reasons the P-matrices are studied. A connection has even been found between P-matrices and the regularity of interval matrices, as shown in [5] or [16]. However, the task of verifying whether a given matrix is a P-matrix is co-NP-complete, as proved in [2].

***B*-matrices, Doubly *B*-matrices, B_π^R-matrices.** Testing P-matrix property is hard; it is important to identify such subclasses of P-matrices which are efficiently

*Supported by the Czech Science Foundation Grant P403-22-11117S.

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DOI: `10.14232/actacyb.294423`

recognizable. Besides positive definite matrices or M-matrices, those might be e.g., B-matrices (introduced in [14]), doubly B-matrices (introduced in [15]) or B_π^R-matrices (introduced in [12]); here we will focus mainly on the last mentioned. In addition to their usefulness as subclasses of the P-matrices, these matrix classes also appeared in the context of Markov chains and in localization of eigenvalues.

Interval analysis. Interval analysis was developed to deal with inaccuracy in data, rounding errors, or a certain form of uncertainty. A central concept of interval analysis is an interval matrix. We denote the set of all real intervals by $\mathbb{IR}$. Now, let us define an interval matrix.

Definition 1.1 (Interval matrix)**.** *An* interval matrix $\boldsymbol{A}$*, which we denote by* $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$*, is defined as*

$$\boldsymbol{A} = \left[\underline{A}, \overline{A}\right] = \left\{ A \in \mathbb{R}^{m\times n} \middle| \underline{A} \le A \le \overline{A} \right\},$$

where $\underline{A}, \overline{A}$ *are called the lower or upper bound matrices of* $\boldsymbol{A}$*, respectively, and* $\le$ *is understood entrywise.*

We can look at $\boldsymbol{A}$ *as a matrix with its entries from* $\mathbb{IR}$*, hence* $\forall i \in [m], \forall j \in [n] : \boldsymbol{a}_{ij} = \left[\underline{a}_{ij}, \overline{a}_{ij}\right]$*, where* $[m] = \{1, 2, \ldots, m\}$ *and analogously for* $[n]$*.*

Definition 1.2. *Let* $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$*. We say that* $\boldsymbol{A}$ *has positive row sums if the intervals of the row sums are positive. In other words, if* $\forall i \in [m] : \sum_{j=1}^{n} \underline{a}_{ij} > 0$.

We call an interval matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ an *interval P-matrix* if every $A \in \boldsymbol{A}$ is a P-matrix. Similarly other matrix classes might be defined, e.g., the class of Z-matrices, which are matrices with non-positive off-diagonal elements. We can also define some basic properties, such as regularity, which are studied more in the following works: [3], [6], [8], and [9], among many others.

Structure and contribution of the paper. In this work, we present some results based on [10], such as a generalization of B_π^R-matrices into the interval settings, and lay the foundations for recognizing the interval variants through characterization, or sufficient conditions and necessary ones. We then proceed to introduce characterizations through reduction of interval B-matrices, doubly B-matrices, and B_π^R-matrices.

As we show, these interval variants of our matrix classes are connected to the interval P-matrices in the same way the real variants are connected to the real P-matrices. Interval P-matrices are closely connected to the linear complementarity problem with uncertain data, which might be modeled by intervals. So again, it is useful to have easily recognizable subclasses of interval P-matrices.

2 B_π^R-matrices

2.1 Real B_π^R-matrices

Let us start by introducing real B_π^R-matrices and a few facts about them, which were introduced by Neumann, Peña, and Pryporova in [12] or by Araújo and Mendes-Gonçalves in [11], and which we will later transfer into the interval setting.

Definition 2.1 (B_π^R-matrix, [12])**.** *Let $A \in \mathbb{R}^{n\times n}$, let $\pi \in \mathbb{R}^n$ such that it fulfills*

$$0 < \sum_{j=1}^{n} \pi_j \leq 1, \tag{1}$$

*and let $R \in \mathbb{R}^n$ be the vector formed by the row sums of A (hence $\forall i \in [n]: \ R_i = \sum_{j=1}^{n} a_{ij}$). We say that A is a B_π^R-*matrix *if* $\forall i \in [n]:$

$$\begin{aligned} &a) \quad R_i > 0 \\ &b) \quad \forall k \in [n] \setminus \{i\} : \quad \pi_k \cdot R_i > a_{ik} \end{aligned}$$

The next proposition is introduced in [12] as Observation 3.2.

Proposition 2.1. *Let $A \in \mathbb{R}^{n\times n}$ have positive row sums, and let $R \in \mathbb{R}^n$ be the vector formed by the row sums of A. There exists a vector $\pi \in \mathbb{R}^n$ satisfying inequality* (1) *such that A is a B_π^R-matrix if and only if*

$$\sum_{j=1}^{n} \max\left\{ \frac{a_{ij}}{R_i} \,\middle|\, i \neq j \right\} < 1.$$

Remark 2.1. If for any matrix $A \in \mathbb{R}^{n\times n}$ the condition from Proposition 2.1 is satisfied, then we are able to construct a vector $\pi \in \mathbb{R}^n$ satisfying inequality (1) such that A is a B_π^R-matrix in the following manner:

1. We define $\epsilon \in \mathbb{R}$ as
$$\epsilon = 1 - \sum_{j=1}^{n} \max\left\{ \frac{a_{ij}}{R_i} \,\middle|\, i \neq j \right\},$$
2. and then for every $j \in [n]$ we define π_j as
$$\pi_j = \max\left\{ \frac{a_{ij}}{R_i} \,\middle|\, i \neq j \right\} + \frac{\epsilon}{n}.$$

Of course, instead of $\frac{\epsilon}{n}$ in the second step we may use any constant $0 < c \leq \frac{\epsilon}{n}$, or we might use a vector $\xi \in {\mathbb{R}^+}^n$ such that $0 < \sum_{j=1}^{n} \xi_j \leq \epsilon$, and define π_j as

$$\pi_j = \max\left\{ \frac{a_{ij}}{R_i} \,\middle|\, i \neq j \right\} + \xi_j.$$

(It is easy to verify that this holds from Definition 2.1, because so defined π meets condition b) for the above-mentioned definition, and also satisfies inequality (1).)

The following result is stated and proved in [13].

Proposition 2.2. *Every B_π^R-matrix with $\pi \geq 0$ is a P-matrix.*

Remark 2.2. We can show an example of a B_π^R-matrix with $\pi_i < 0$ for some $i \in [n]$, which is not a B_ψ^R-matrix for any $\psi \geq 0$. (To verify this fact, the reader may use the properties of B_π^R-matrices stated in the next proposition, more precisely, part 1).)

Example 2.1.

$$A = \begin{pmatrix} \frac{3}{2} & -1 \\ 2 & -\frac{1}{2} \end{pmatrix}$$

It is easy to check that A is a B_π^R-matrix for $\pi = (2, -1)^T$. (And it is clearly not a P-matrix.)

Hence, for the purpose of this work, we are interested only in such B_π^R-matrices that have $\pi \geq 0$, since only those ought to be P-matrices.

The next proposition is introduced in [11] as Proposition 2.1.

Proposition 2.3. *Let $\pi \in \mathbb{R}^n$ such that inequality* (1) *holds, and let $A \in \mathbb{R}^{n\times n}$ be a B_π^R-matrix, where $R \in \mathbb{R}^n$ is the vector of row sums of A. Then the following holds:*

1. $\forall i \in [n]: \quad a_{ii} > \pi_i \cdot R_i,$
2. $\forall (i,j) \in [n]^2, j \neq i: \quad \pi_i \geq \pi_j \;\Rightarrow\; a_{ii} > a_{ij},$
3. *let* $k = \operatorname{argmax}\{\pi_i \mid i \in [n]\}$, *then* $\forall j \neq k: \quad a_{kk} > a_{kj}$, *and*
4. $\forall (i,j) \in [n]^2, j \neq i: \quad \pi_j \leq 0 \;\Rightarrow\; a_{ij} < 0.$

The next proposition is introduced in [11] as Proposition 2.5.

Proposition 2.4. *Let $\pi \in \mathbb{R}^n$ such that condition* (1) *holds, and let $A \in \mathbb{R}^{n\times n}$ be a B_π^R-matrix. If $\alpha \in \mathbb{R}^n$ satisfies analogy of inequality* (1) *and $\alpha \geq \pi$, then A is a B_α^R-matrix.*

2.2 Interval $B_\pi^{\boldsymbol{R}}$-matrices

Next, we proceed to generalize the class of B_π^R-matrices into the interval setting. However, there are two ways to do so, differing in the order of quantifiers.

Definition 2.2 (Homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix)**.** *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, $\pi \in \mathbb{R}^n$ such that inequality* (1) *holds, and let $\boldsymbol{R} \in \mathbb{IR}^n$. We say that $\boldsymbol{A}$ is a* homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix *if $\forall A \in \boldsymbol{A}$: $\exists R \in \boldsymbol{R}$ such that A is a (real) B_π^R-matrix.*

Here, the $\boldsymbol{R}$ in the definition can be perceived as the vector whose entries correspond to the intervals of the row sums of matrices $A \in \boldsymbol{A}$, but in the interval setting it is more of a symbol than of any greater significance. This is because if we have two interval $B_\pi^{\boldsymbol{R}}$-matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, we cannot say that any two $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ are real B_π^R-matrices for the same R. Despite that, we decided to include it in the notation of the interval matrix class for compatibility with the real case definition.

Corollary 2.1. *Every homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix with $\pi \geq 0$ is an interval P-matrix.*

Proof. It holds for every instance, hence it holds for the whole interval matrix. □

Definition 2.3 (Heterogeneous interval $B_\Pi^{\boldsymbol{R}}$-matrix)**.** *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, and let $\boldsymbol{R} \in \mathbb{IR}^n$. We say that $\boldsymbol{A}$ is a* heterogeneous interval $B_\Pi^{\boldsymbol{R}}$-matrix *if $\forall A \in \boldsymbol{A}$: $\exists R \in \boldsymbol{R}, \exists \pi \in \mathbb{R}^n$ such that condition* (1) *holds and A is a (real) B_π^R-matrix.*

Here, the $\boldsymbol{R}$ in the definition again has the same meaning as in the case of homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrices. As for the Π, we may understand it as a set of all such vectors π satisfying condition (1) such that there exists $A \in \boldsymbol{A}$, for which it holds that A is a real B_π^R-matrix. However, again it can be perceived just as a symbol that distinguishes this interval matrix class, since the exact form or content of the set Π holds no real significance to us, and we have no way of deriving it yet.

Corollary 2.2. *Every homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix is a heterogeneous interval $B_\Pi^{\boldsymbol{R}}$-matrix.*

Proof. It trivially follows from the definitions. □

Let us start by stating a characterization that helps us with the recognition of the class of homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrices in finite time.

Theorem 2.1. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, let $\pi \in \mathbb{R}^n$ satisfy inequality* (1)*, and let $\boldsymbol{R} \in \mathbb{IR}^n$ be the vector of intervals of individual row sums in matrix $\boldsymbol{A}$. The matrix $\boldsymbol{A}$ is a homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix if and only if $\forall i \in [n]$ the following properties hold:*

$$
\begin{aligned}
&a)\quad \underline{R}_i > 0\\
&b)\quad \forall k \in [n] \setminus \{i\}:\\
&\qquad \left(\pi_k > 1 \quad\Rightarrow\quad \sum_{j\neq k} \underline{a}_{ij} \;>\; \left(\frac{1}{\pi_k} - 1\right) \cdot \underline{a}_{ik}\right)\\
&\qquad \wedge \quad \left(0 < \pi_k \leq 1 \quad\Rightarrow\quad \sum_{j\neq k} \underline{a}_{ij} \;>\; \left(\frac{1}{\pi_k} - 1\right) \cdot \overline{a}_{ik}\right)\\
&\qquad \wedge \quad \left(\pi_k = 0 \quad\Rightarrow\quad 0 \;>\; \overline{a}_{ik}\right)\\
&\qquad \wedge \quad \left(\pi_k < 0 \quad\Rightarrow\quad \sum_{j\neq k} \overline{a}_{ij} \;<\; \left(\frac{1}{\pi_k} - 1\right) \cdot \overline{a}_{ik}\right)
\end{aligned}
$$

Proof. Condition *a*) of Definition 2.1 evaluated for every $A \in \boldsymbol{A}$ is equivalent to $\underline{R}_i > 0$. As for the condition *b*) of the definition, it may be modified for every $k \neq i$ as follows (while noting that $\pi_k \cdot R_i = \pi_k \cdot \sum_{j=1}^{n} a_{ij}$):

1. $\pi_k > 1$:
$$\pi_k \cdot \sum_{j=1}^{n} a_{ij} > a_{ik} \quad \Leftrightarrow \quad \sum_{j=1}^{n} a_{ij} > \frac{1}{\pi_k} \cdot a_{ik} \quad \Leftrightarrow \quad \sum_{j \neq k} a_{ij} > \left(\frac{1}{\pi_k} - 1 \right) \cdot a_{ik} \tag{2}$$
We observe that the highest value of $\left(\frac{1}{\pi_k} - 1 \right) \cdot a_{ik}$ is attained at the lower bound on the $\boldsymbol{a}_{ik}$, because when $\pi_k > 1$, we have $\left(\frac{1}{\pi_k} - 1 \right) < 0$. Whence, the condition above holds for every $A \in \boldsymbol{A}$ if and only if the following condition holds:
$$\sum_{j \neq k} \underline{a}_{ij} > \left(\frac{1}{\pi_k} - 1 \right) \cdot \underline{a}_{ik}$$
2. $0 < \pi_k \leq 1$: Using the chain of equivalences (2) from the previous part, we observe that the highest value of $\left(\frac{1}{\pi_k} - 1 \right) \cdot a_{ik}$ is obtained by the upper bound on the $\boldsymbol{a}_{ik}$, because when $0 < \pi_k \leq 1$, then $\left(\frac{1}{\pi_k} - 1 \right) \geq 0$. Thus, condition *b*) of Definition 2.1 holds for every $A \in \boldsymbol{A}$ if and only if the following condition holds:
$$\sum_{j \neq k} \underline{a}_{ij} > \left(\frac{1}{\pi_k} - 1 \right) \cdot \overline{a}_{ik}$$
3. $\pi_k = 0 : \pi_k \cdot \sum_{j=1}^{n} a_{ij} > a_{ik} \quad \Leftrightarrow \quad 0 > a_{ik}$
The condition above holds for every $A \in \boldsymbol{A}$ if and only if $0 > \overline{a}_{ik}$
4. $\pi_k < 0$:
$$\pi_k \cdot \sum_{j=1}^{n} a_{ij} > a_{ik} \quad \Leftrightarrow \quad \sum_{j=1}^{n} a_{ij} < \frac{1}{\pi_k} \cdot a_{ik} \quad \Leftrightarrow \quad \sum_{j \neq k} a_{ij} < \left(\frac{1}{\pi_k} - 1 \right) \cdot a_{ik}$$
We observe that the smallest value of $\left(\frac{1}{\pi_k} - 1 \right) \cdot a_{ik}$ is obtained by the upper bound on the $\boldsymbol{a}_{ik}$, because when $\pi_k < 0$, then $\left(\frac{1}{\pi_k} - 1 \right) < 0$. From that we have that the condition above holds for every $A \in \boldsymbol{A}$ if and only if the following condition holds:
$$\sum_{j \neq k} \overline{a}_{ij} < \left(\frac{1}{\pi_k} - 1 \right) \cdot \overline{a}_{ik}$$
□

Remark 2.3. This characterization has time complexity $O(n^2)$, which is, surprisingly, the same as a characterization from the definition of the real case, Definition

2.1 (although the interval case has undoubtedly higher implementational complexity).

Let us now introduce an analogy of Proposition 2.1 for homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrices.

Theorem 2.2. *If $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ has positive row sums, then there exists a vector $\pi \in \mathbb{R}^n$ satisfying inequality* (1) *such that $\boldsymbol{A}$ is a homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix if and only if*

$$\sum_{j=1}^{n} \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \,\middle|\, i \neq j \right\} < 1. \tag{3}$$

Proof. "$\Rightarrow$": $\boldsymbol{A}$ is a $B_\pi^{\boldsymbol{R}}$-matrix for some π satisfying the property (1), hence every $A \in \boldsymbol{A}$ is a B_π^R-matrix, thus, in particular, matrices $A_j \in \boldsymbol{A}$ for every $j \in [n]$ defined as follows:

$$A_j = (a'_{m_1m_2});$$
$$a'_{m_1m_2} = \begin{cases} \overline{a}_{m_1m_2} & \text{if } m_2 = j \ \wedge\ \frac{\overline{a}_{m_1j}}{\overline{a}_{m_1j}+\sum\limits_{m\neq j}\underline{a}_{m_1m}} > \frac{\underline{a}_{m_1j}}{\underline{a}_{m_1j}+\sum\limits_{m\neq j}\underline{a}_{m_1m}}, \\ \underline{a}_{m_1m_2} & \text{otherwise.} \end{cases} \tag{4}$$

Therefore, (if we denote R^j the vector of row sums of A_j) we have

$$\forall j \in [n]:$$
$$\max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \,\middle|\, i \neq j \right\} = \max\left\{ \frac{a'_{ij}}{R_i^j} \,\middle|\, i \neq j \right\} < \pi_j. \tag{5}$$

But then

$$\sum_{j=1}^{n} \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \,\middle|\, i \neq j \right\} < \sum_{j=1}^{n} \pi_j \leq 1. \tag{6}$$

"$\Leftarrow$": Let

$$\epsilon = 1 - \sum_{j=1}^{n} \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \,\middle|\, i \neq j \right\} > 0,$$

and for every $j \in [n]$ set the $\pi_j = \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij}+\sum\limits_{m\neq j}\underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij}+\sum\limits_{m\neq j}\underline{a}_{im}} \,\middle|\, i \neq j \right\} + \frac{\epsilon}{n}$. Then

$\boldsymbol{A}$ is a homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix. That is because for any $A \in \boldsymbol{A}$

$$\max\left\{\frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \,\middle|\, i \neq j\right\}$$
$$\geq \quad \max\left\{\frac{a_{ij}}{a_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \,\middle|\, i \neq j\right\} \quad \geq \quad \max\left\{\frac{a_{ij}}{\sum\limits_{m=1}^{n} a_{im}} \,\middle|\, i \neq j\right\}.$$

Thus, for every $A \in \boldsymbol{A}$ and for every $(k, j) \in [n]^2, j \neq k$, it holds that

$$\frac{a_{kj}}{R_k} \quad = \quad \frac{a_{kj}}{\sum\limits_{m=1}^{n} a_{km}} \quad \leq \quad \max\left\{\frac{a_{ij}}{\sum\limits_{m=1}^{n} a_{im}} \,\middle|\, i \neq j\right\}$$
$$\leq \quad \max\left\{\frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \,\middle|\, i \neq j\right\}$$
$$< \quad \max\left\{\frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \,\middle|\, i \neq j\right\} + \frac{\epsilon}{n} \quad = \quad \pi_j,$$

ergo $\pi_j \cdot R_k > a_{kj}$. Therefore, every $A \in \boldsymbol{A}$ is a B_π^R-matrix. □

Remark 2.4. If any matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ satisfies the condition from Theorem 2.2, we can construct a vector $\pi \in \mathbb{R}^n$ satisfying condition (1) such that $\boldsymbol{A}$ is a homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix in an analogous way to what we did in Remark 2.1.

Next, let us introduce one interesting fact about the class of heterogeneous interval B_Π^R-matrices that helps us to characterize it. For that, we first need to state a few auxiliary propositions.

Proposition 2.5. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. The matrix $\boldsymbol{A}$ is a heterogeneous interval $B_\Pi^{\boldsymbol{R}}$-matrix only if*

$$\sum_{j=1}^{n} \max\left\{\frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \,\middle|\, i \neq j\right\} < 1.$$

Proof. $\boldsymbol{A}$ is a heterogeneous $B_\Pi^{\boldsymbol{R}}$-matrix, hence every $A \in \boldsymbol{A}$ is a B_π^R-matrix for some $\pi = (\pi_1, \ldots, \pi_n)$ satisfying the property (1), thus in particular the matrices $A_j \in \boldsymbol{A}$ for every $j \in [n]$ defined as in the proof of Theorem 2.2 in expression (4) are B_π^R-matrices.

Therefore, (if we denote R^j the vector of row sums of A_j) again just as in the proof of Theorem 2.2, the expression (5) holds $\forall j \in [n]$. From that we also get that expression (6) from the proof holds, which is exactly what we wanted to prove here. $\square$

Corollary 2.3. *Every heterogeneous interval $B_\Pi^{\boldsymbol{R}}$-matrix is a homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix for some π fulfilling inequality* (1)*.*

Proof. Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ have positive row sums. From Proposition 2.5, we get the following implication:

$\boldsymbol{A}$ is a heterogeneous interval $B_\Pi^{\boldsymbol{R}}$-matrix $\Rightarrow$ Inequality (3) holds.

From the equivalence from Theorem 2.2 we use the following implication:

Inequality (3) holds $\Rightarrow$ $\exists\pi : \pi$ satisfies condition (1) $\wedge$ $\boldsymbol{A}$ is a homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix.

Ergo we compose these two implications (because from Definition 2.3 we can easily observe that if $\boldsymbol{A}$ is a heterogeneous interval $B_\Pi^{\boldsymbol{R}}$-matrix, then it has positive row sums, therefore fulfilling the assumptions of Theorem 2.2), and thus obtain the desired implication. $\square$

What we obtained is the second inclusion we need to show the equality among our two interval matrix classes, the class of homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrices and that of the heterogeneous interval $B_\Pi^{\boldsymbol{R}}$-matrices.

Theorem 2.3. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ have positive row sums. We have that $\boldsymbol{A}$ is a heterogeneous interval $B_\Pi^{\boldsymbol{R}}$-matrix if and only if $\exists\pi \in \mathbb{R}^n$ such that condition* (1) *holds and that $\boldsymbol{A}$ is a homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix.*

Proof. Follows from Corollaries 2.2 and 2.3. $\square$

We proved that the two classes we have defined at the beginning of this subsection are the same, hence it does not make any sense to differentiate the two. Thus, from now on we refer to them as *interval $B_\pi^{\boldsymbol{R}}$-matrices.*

Definition 2.4 (Interval $B_\pi^{\boldsymbol{R}}$-matrix)**.** *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, and let $\pi \in \mathbb{R}^n$ satisfy inequality* (1)*. We say that $\boldsymbol{A}$ is an* interval $B_\pi^{\boldsymbol{R}}$-matrix *if it is a homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix.*

Remark 2.5. Because of this definition, we can use the same characterizations we use to characterize the homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrices (Theorem 2.1, Theorem 2.2) to characterize the interval $B_\pi^{\boldsymbol{R}}$-matrices (and because of Theorem 2.3 also the $B_\Pi^{\boldsymbol{R}}$-matrices).

Now, let us generalize some properties of real B_π^R-matrices to the interval $B_\pi^{\boldsymbol{R}}$-matrices. The first is a direct consequence of the definition.

Corollary 2.4. *Every interval $B_\pi^{\boldsymbol{R}}$-matrix with $\pi \geq 0$ is an interval P-matrix.*

Proposition 2.6. *Let $\pi \in \mathbb{R}^n$ such that inequality* (1) *is fulfilled, and let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval $B_\pi^{\boldsymbol{R}}$-matrix. The following holds:*

1. $\forall i \in [n]: \quad \underline{a}_{ii} > \max\left\{\pi_i \cdot \left(\underline{a}_{ii} + \sum_{j\neq i} \underline{a}_{ij}\right), \pi_i \cdot \left(\underline{a}_{ii} + \sum_{j\neq i} \overline{a}_{ij}\right)\right\}$,

2. $\forall (i,j) \in [n]^2, j \neq i: \quad \pi_i \geq \pi_j \Rightarrow \underline{a}_{ii} > \overline{a}_{ij}$,

3. *if* $k = \operatorname{argmax}\{\pi_i \mid i \in [n]\}$, *then* $\forall j \neq k: \quad \underline{a}_{kk} > \overline{a}_{kj}$, *and*

4. $\forall (i,j) \in [n]^2, j \neq i: \quad \pi_j \leq 0 \Rightarrow \overline{a}_{ij} < 0$.

Proof. Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval $B_\pi^{\boldsymbol{R}}$-matrix for some $\pi \in \mathbb{R}^n$ fulfilling inequality (1).

1. Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be defined as follows:

$$A_1 = \underline{A}$$

$$A_2 = (a_{m_1m_2}); \quad a_{m_1m_2} = \begin{cases} \underline{a}_{ii} & \text{if } m_1 = m_2 = i, \\ \overline{a}_{m_1m_2} & \text{otherwise.} \end{cases}$$

   Because $A_1, A_2 \in \boldsymbol{A}$, they are both B_π^R-matrices, thus from Proposition 2.3, part 1) we get that this point holds.

2. Let $A' \in \mathbb{R}^{n\times n}$ be defined as $A' = A_2$, where A_2 is defined in the previous part of this proof. Because $A' \in \boldsymbol{A}$, it is a B_π^R-matrix, thus from Proposition 2.3, part 2) we get that this point holds.

3. Direct consequence of the previous point.

4. Because $\overline{A} \in \boldsymbol{A}$, it is a B_π^R-matrix, thus from Proposition 2.3, part 4) we get that this point holds. □

Proposition 2.7. *Let* $\pi \in \mathbb{R}^n$ *fulfill inequality* (1), *and let* $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ *be an interval* $B_\pi^{\boldsymbol{R}}$*-matrix. If* $\alpha \in \mathbb{R}^n$ *satisfies the analogy of inequality* (1) *and* $\alpha \geq \pi$, *then* $\boldsymbol{A}$ *is an interval* $B_\alpha^{\boldsymbol{R}}$*-matrix.*

Proof. It holds for every instance of the interval matrix (see Proposition 2.4), thus it holds for the whole interval matrix. □

3 Characterizations through reduction

Here, in this section, we take a closer look at how we may characterize B_π^R-matrices, B-matrices, and doubly B-matrices through reduction. By that we mean testing an interval matrix for the property of being an interval B_π^R-matrix, B-matrix or doubly B-matrix, respectively, using only a finite subset of instances of the interval matrix, and testing them on being a member of the corresponding real matrix class. Reductions for other matrix classes were surveyed, e.g., by Garloff et al. in [4].

Both the class of interval B-matrices and the one of interval doubly B-matrices were introduced in [10], and we use the characterizations stated and proved there in our proofs. However, everything we use is also stated here as well.

3.1 $\boldsymbol{B_\pi^R}$-matrices

Let us begin with the interval $B_\pi^{\boldsymbol{R}}$-matrices we introduced in section 2.

Proposition 3.1. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, let $\pi \in \mathbb{R}^n$ satisfy inequality* (1)*, and let $\boldsymbol{R} \in \mathbb{IR}^n$ be the vector of intervals of the individual row sums in matrix $\boldsymbol{A}$. Let $\forall i \in [n] : A_i \in \mathbb{R}^{n\times n}$ be defined as follows:*

1. *if $\pi_i > 1$, then:*
   $A_i = \underline{A}$
2. *else if $0 \leq \pi_i \leq 1$, then:*
   $$A_i = (a_{m_1m_2}); \quad a_{m_1m_2} = \begin{cases} \overline{a}_{m_1m_2} & \text{if } m_1 \neq i, m_2 = i, \\ \underline{a}_{m_1m_2} & \text{otherwise.} \end{cases}$$
3. *else if $\pi_i < 0$, then:*
   $$A_i = (a_{m_1m_2}); \quad a_{m_1m_2} = \begin{cases} \underline{a}_{m_1m_2} & \text{if } m_1 = i, \\ \overline{a}_{m_1m_2} & \text{otherwise.} \end{cases}$$

It holds that $\boldsymbol{A}$ is an interval $B_\pi^{\boldsymbol{R}}$-matrix if and only if $\forall i \in [n]:\ A_i$ is a B_π^R-matrix, where $R \in \mathbb{R}^n$ is the vector of values corresponding to the row sums of A_i.

Proof. "⇒" This holds, because $\forall i \in [n] : A_i \in \boldsymbol{A}$ (and the corresponding $R \in \boldsymbol{R}$).

"⇐"

a)$\forall i \in [n] : \underline{R}_i > 0$, because A_i is a B_π^R-matrix, and $(A_i)_{i,*} = (\underline{A})_{i,*}$, the entries of $\underline{R}$ are positive.

b)$\forall i \in [n]\ \forall k \neq i :\quad A_k$ is a $B_\pi^{\boldsymbol{R}}$-matrix and so, from Definition 2.1:

1. $\pi_k > 1$:
$$\pi_k \cdot \sum_{j=1}^{n} (A_k)_{ij} > (A_k)_{ik} \quad \Leftrightarrow \quad \pi_k \cdot \sum_{j=1}^{n} \underline{a}_{ij} > \underline{a}_{ik}$$
$$\Leftrightarrow \quad \sum_{j\neq k} \underline{a}_{ij} > \left(\frac{1}{\pi_k} - 1\right) \cdot \underline{a}_{ik}$$
2. $0 < \pi_k \leq 1$: $\quad \pi_k \cdot \sum_{j=1}^{n} (A_k)_{ij} > (A_k)_{ik} \quad \Leftrightarrow \quad \sum_{j\neq k} \underline{a}_{ij} > \left(\frac{1}{\pi_k} - 1\right) \cdot \overline{a}_{ik}$
3. $\pi_k = 0$: $\quad \pi_k \cdot \sum_{j=1}^{n} (A_k)_{ij} > (A_k)_{ik} \quad \Leftrightarrow \quad 0 > \overline{a}_{ik}$
4. $\pi_k < 0$: $\quad \pi_k \cdot \sum_{j=1}^{n} (A_k)_{ij} > (A_k)_{ik} \quad \Leftrightarrow \quad \sum_{j\neq k} \overline{a}_{ij} < \left(\frac{1}{\pi_k} - 1\right) \cdot \overline{a}_{ik}$

Thus, $\boldsymbol{A}$ fulfills the conditions of Theorem 2.1, and so it is an interval B_π^R-matrix. □

Proposition 3.2. *The characterization of the interval $B_\pi^{\boldsymbol{R}}$-matrices through the reduction given by Proposition 3.1 is for $\pi \geq 0$ minimal with respect to inclusion.*

Proof. First, we notice that from the condition (1) on π, it follows that $\forall j \in [n]$: $0 \leq \pi_j \leq 1$, so every matrix from the reduction has the form given by point 2).

If we skip any A_i for arbitrary $i \in [n]$, then we could construct a counterexample, e.g., a unit matrix with interval $[0, \frac{\pi_i}{1-\pi_i}]$ at position (j, i) for arbitrary $j \neq i$. Then $\forall k \neq i : A_k = I_n$, which surely is a B_π^R-matrix. But A_i does not fulfill condition *b)* from Definition 2.1 in the j-th row. That is because the sum of the j-th row is equal to $1 + \frac{\pi_i}{1-\pi_i}$ and $(A_i)_{ji} = \frac{\pi_i}{1-\pi_i}$, so we get

$$\pi_i \cdot R_j = \pi_i \cdot \left(1 + \frac{\pi_i}{1-\pi_i}\right) = \pi_i \cdot \left(\frac{1-\pi_i+\pi_i}{1-\pi_i}\right) = \frac{\pi_i}{1-\pi_i} = (A_i)_{ji}\,,$$

which violates the condition, and so the A_i is not a B_π^R-matrix. □

Remark 3.1. In Proposition 3.2, the assumption that $\pi \geq 0$ is present both because such a π is what we are interested in in this work, and, more importantly, because for the general case we might have such a π that two entries of the vector are larger than 1. However, then the two matrices A_i corresponding to those entries are the same and equal to $\underline{A}$, and so we may remove one of the two matrices from the reduction, and it still works. As for the case of almost general π, where we only want that there is at most one entry larger than one, we have not managed to prove or disprove the statement yet.

Remark 3.2. This reduction reduces the problem of verifying whether any given interval matrix is an interval $B_\pi^{\boldsymbol{R}}$-matrix, into testing whether n matrices are real B_π^R-matrices.

Example 3.1. Here we show an example of an interval B_π^R-matrix, and use it to point out some things. Let us have a vector π, such that $\pi = (0.36, 0.28, 0.36)$, and let us define an interval B_π^R-matrix $\boldsymbol{A} \in \mathbb{IR}^{3\times 3}$ as follows:

$$\boldsymbol{A} = \begin{pmatrix} [7.95, 8.05] & [-7.05, -6.95] & [-0.05, 0.05] \\ [0.95, 1.05] & [0.95, 1.05] & [0.95, 1.05] \\ [8.95, 9.05] & [10.95, 11.05] & [19.95, 20.05] \end{pmatrix}$$

It is easy to verify that $\boldsymbol{A}$ belongs to the class of B_π^R-matrices for some π satisfying the condition (1) by using Theorem 2.2 or to verify whether it is a B_π^R-matrix for our value of π using Theorem 2.1.

What is quite interesting and important is the fact that this matrix is not positive definite (it is not symmetric), it is not an interval M-matrix (it is not a Z-matrix), nor is it an interval H-matrix (e.g., the central matrix is not an H-matrix). This shows that for this matrix other usual conditions of P-matrices fail, while we might recognize it as a P-matrix due to it being a B_π^R-matrix. This shows a reason for studying this matrix class.

Now, let us conclude this illustration by showing the characterization through reduction on this example. The three instances from the reduction from Proposition 3.1 are:

$$A_1 = \begin{pmatrix} 7.95 & -7.05 & -0.05 \\ 1.05 & 0.95 & 0.95 \\ 9.05 & 10.95 & 19.95 \end{pmatrix}, \; A_2 = \begin{pmatrix} 7.95 & -6.95 & -0.05 \\ 0.95 & 0.95 & 0.95 \\ 8.95 & 11.05 & 19.95 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 7.95 & -7.05 & 0.05 \\ 0.95 & 0.95 & 1.05 \\ 8.95 & 10.95 & 19.95 \end{pmatrix}.$$

3.2 B-matrices

As written at the beginning of this section, we need to use a characterization of interval B-matrices introduced in [10] plus a definition of real B-matrices and one of their characterizations introduced by Peña in [14], so let us state them here.

Definition 3.1 (B-matrix, [14])**.** *Let $A \in \mathbb{R}^{n\times n}$. We say that A is a* B-matrix *if $\forall i \in [n]$ the following holds:*

$$a) \quad \sum_{j=1}^{n} a_{ij} > 0$$
$$b) \quad \forall k \in [n] \setminus \{i\} : \quad \frac{1}{n}\sum_{j=1}^{n} a_{ij} > a_{ik}$$

Remark 3.3. We can see that from Definition 3.1 we have that B-matrices are B-$^R_\pi$matrices for $\pi = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$. Therefore, the B-$^R_\pi$matrices might be seen as a generalization of the B-matrices.

Proposition 3.3. *If $A \in \mathbb{R}^{n\times n}$, then A is a B-matrix if and only if $\forall i \in [n]$ the following holds:*

$$\sum_{j=1}^{n} a_{ij} > n \cdot r_i^+,$$

where $r_i^+ = \max\{0, a_{ij} | \; j \neq i\}$.

Definition 3.2 (Interval B-matrix)**.** *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. We say that $\boldsymbol{A}$ is an* interval B-matrix *if $\forall A \in \boldsymbol{A}$: A is a (real) B-matrix.*

Proposition 3.4. *If $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, then $\boldsymbol{A}$ is an interval B-matrix if and only if $\forall i \in [n]$ the following two properties hold:*

$$a) \quad \sum_{j=1}^{n} \underline{a}_{ij} > 0$$
$$b) \quad \forall k \in [n] \setminus \{i\} : \quad \sum_{j\neq k} \underline{a}_{ij} > (n-1) \cdot \overline{a}_{ik}$$

Now, let us introduce the reduction.

Proposition 3.5. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, and let $\forall i \in [n] : A_i$ be matrices defined as follows:*

$$A_i = (a_{m_1 m_2}); \quad a_{m_1 m_2} = \begin{cases} \overline{a}_{m_1 m_2} & \text{if } m_1 \neq i, m_2 = i, \\ \underline{a}_{m_1 m_2} & \text{otherwise.} \end{cases}$$

It follows that $\boldsymbol{A}$ is an interval B-matrix if and only if $\forall i \in [n] : A_i$ is a B-matrix.

Proof. "$\Rightarrow$" This holds trivially, because $\forall i \in [n] : A_i \in \boldsymbol{A}$

"$\Leftarrow$"

a)$\forall i \in [n] : \sum_{j=1}^{n} \underline{a}_{ij} > 0$, because A_i is a B-matrix, and $(A_i)_{i,*} = (\underline{A})_{i,*}$, so the row sums of $\underline{A}$ are positive.

b)$\forall i \in [n]\ \forall k \neq i :\quad A_k$ is a B-matrix $\Rightarrow$ (From Proposition 3.3:)

$$\overline{a}_{ik} + \sum_{j \neq k} \underline{a}_{ij} \quad = \quad \sum_{j=1}^{n} (A_k)_{ij} \quad > \quad n \cdot r_i^+ \quad \geq \quad n \cdot (A_k)_{ik} \quad = \quad n \cdot \overline{a}_{ik}$$

$\Rightarrow$

$$\sum_{j \neq k} \underline{a}_{ij} \quad > \quad (n-1) \cdot \overline{a}_{ik}$$

Whence it follows that $\boldsymbol{A}$ fulfills the conditions of Proposition 3.4, and so is an interval B-matrix. □

Proposition 3.6. *The characterization of interval B-matrices through the reduction given by Proposition 3.5 is minimal with respect to inclusion.*

Proof. If we skip any A_i for arbitrary $i \in [n]$, then we would be able to construct a counterexample, e.g., a unit matrix with an additional interval $[0, 1]$ on position (j, i) for arbitrary $j \neq i$. Then $\forall k \neq i : A_k = I_n$, which surely is a B-matrix, but A_i does not fulfill condition $b)$ from Definition 3.1 in the j-th row. (The sum of the j-th row is equal to 2, so we get $2/n > 1 = (A_i)_{ji}$, which does not hold for $n \geq 2$.) □

Remark 3.4. This reduction reduces the problem of verifying whether any given interval matrix is an interval B-matrix, into testing whether n matrices are real B-matrices.

3.3 Doubly *B*-matrices

As written at the beginning of this section, we need to use a characterization of interval doubly B-matrices introduced in [10] and a definition of real doubly B-matrices introduced by Peña in [15], so let us state them here.

Definition 3.3 (Doubly B-matrix, [15]). *Let $A \in \mathbb{R}^{n\times n}$. We say that A is a* doubly B-matrix *if $\forall i \in [n]$ the following holds:*

a) $a_{ii} > r_i^+$

b) $\forall j \in [n] \setminus \{i\} : \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) > \left(\sum_{k \neq i} \left(r_i^+ - a_{ik}\right)\right) \left(\sum_{k \neq j} \left(r_j^+ - a_{jk}\right)\right)$

Remark 3.5. We can rearrange the inequality from Proposition 3.3 and hence obtain the following characterization of B-matrices:

$$\forall i \in [n] : \left(a_{ii} - r_i^+\right) > \sum_{k \neq i} \left(r_i^+ - a_{ik}\right)$$

This shows that doubly B-matrices are another generalization of B-matrices. Is there then any difference between the two generalizations, between doubly B-matrices and B^R_π-matrices? Yes, there is. The two matrix classes indeed have a nonempty intersection with B-matrices in it, however, as we will see in the following example, the intersection is just a proper subset of each of those two classes.

Example 3.2. Let us show two examples of matrices that demonstrate the difference between the class of doubly B-matrices and that of B^R_π-matrices.

$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
✓ Doubly B-matrix	× Doubly B-matrix
× B$^R_\pi$-matrix (for no π)	✓ B$^R_\pi$-matrix (e.g. $\pi = (\frac{1}{3}, \frac{2}{3})$)
It does not have a positive row sum.	The diagonal element is not the largest.

Definition 3.4 (Interval doubly B-matrix)**.** *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. We say that $\boldsymbol{A}$ is an* interval doubly B-matrix *if $\forall A \in \boldsymbol{A}$: A is a (real) doubly B-matrix.*

Proposition 3.7. *If $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, then $\boldsymbol{A}$ is an interval doubly B-matrix if and only if the following two properties hold:*

a) $\forall i \in [n] : \quad \underline{a}_{ii} > \max\{0, \overline{a}_{ij} | j \neq i\}$*, and*

b) $\forall (i,j) \in [n]^2, j \neq i, \forall (k,l) \in [n]^2, k \neq i, l \neq j :$

I. $$\left(\underline{a}_{ii} - \overline{a}_{ik}\right)\left(\underline{a}_{jj} - \overline{a}_{jl}\right) > \left(\max\left\{0, \sum_{\substack{m\neq i\\ m\neq k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right)\right\}\right)\left(\max\left\{0, \sum_{\substack{m\neq j\\ m\neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right)$$

II. $$\underline{a}_{ii}\left(\underline{a}_{jj} - \overline{a}_{jl}\right) > \left(\max\left\{0, -\sum_{m\neq i} \underline{a}_{im}\right\}\right)\left(\max\left\{0, \sum_{\substack{m\neq j\\ m\neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right)$$

III. $$\underline{a}_{ii} \cdot \underline{a}_{jj} > \left(\max\left\{0, -\sum_{m\neq i} \underline{a}_{im}\right\}\right)\left(\max\left\{0, -\sum_{m\neq j} \underline{a}_{jm}\right\}\right)$$

Now, let us present the reductions.

Proposition 3.8. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ for $n \geq 4$, and let us define $A_{(i,k),(j,l)} \in \mathbb{R}^{n\times n}$ as follows:*

$$A_{(i,k),(j,l)} = (a_{m_1m_2})\,; \quad a_{m_1m_2} = \begin{cases} \overline{a}_{ik} & \text{if } (m_1,m_2) = (i,k), \\ \overline{a}_{jl} & \text{if } (m_1,m_2) = (j,l), \\ \underline{a}_{m_1m_2} & \text{otherwise.} \end{cases}$$

It holds that $\boldsymbol{A}$ is an interval doubly B-matrix if and only if $\forall (i,j) \in [n]^2, j > i, \forall (k,l) \in [n]^2, k \neq i, l \neq j : A_{(i,k),(j,l)}$ is a doubly B-matrix.

Proof. "$\Rightarrow$" Trivial, for all such matrices: $A_{(i,k),(j,l)} \in \boldsymbol{A}$.

"$\Leftarrow$" We prove that the conditions of Proposition 3.7 hold:

a) $\forall i \in [n], \forall k \neq i : \quad \underline{a}_{ii} > \max\{0, \overline{a}_{ik}\}$, because for any arbitrary j, l the matrix $A_{(i,k),(j,l)}$ is a doubly B-matrix. Hence, $\forall i \in [n] : \quad \underline{a}_{ii} > \max\{0, \overline{a}_{ik} | k \neq i\}$.

b) Let us fix arbitrary $(i,j) \in [n]^2, j \neq i$ and arbitrary $(k,l) \in [n]^2, k \neq i, l \neq j$. Without loss of generality suppose $j > i$. (If $j < i$, we swap their values and we also swap the values of k and l, too.) Let us define $A = A_{(i,k),(j,l)}$ to simplify notation. Then:

I.

$$\begin{aligned} \left(\underline{a}_{ii} - \overline{a}_{ik}\right)\left(\underline{a}_{jj} - \overline{a}_{jl}\right) &\geq \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) \\ > &\left(\sum_{m\neq i}\left(r_i^+ - a_{im}\right)\right)\left(\sum_{m\neq j}\left(r_j^+ - a_{jm}\right)\right) \\ \geq &\left(\max\left\{0, \sum_{\substack{m\neq i\\ m\neq k}}\left(\overline{a}_{ik} - \underline{a}_{im}\right)\right\}\right)\left(\max\left\{0, \sum_{\substack{m\neq j\\ m\neq l}}\left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right) \end{aligned}$$

The second inequality holds, because A is a doubly B-matrix.

II.

$$\begin{aligned} \underline{a}_{ii}\left(\underline{a}_{jj} - \overline{a}_{jl}\right) &\geq \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) \\ > &\left(\sum_{m\neq i}\left(r_i^+ - a_{im}\right)\right)\left(\sum_{m\neq j}\left(r_j^+ - a_{jm}\right)\right) \\ \geq &\left(\max\left\{0, -\sum_{m\neq i}\underline{a}_{im}\right\}\right)\left(\max\left\{0, \sum_{\substack{m\neq j\\ m\neq l}}\left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right) \end{aligned}$$

The second inequality holds because of the fact that $A_{(x,y),(j,l)}$ for any $x \neq i$ and $y \neq x$ is a doubly B-matrix.

III.

$$
\begin{aligned}
\underline{a}_{ii} \cdot \underline{a}_{jj} &\geq \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) \\
&> \left(\sum_{m \neq i} \left(r_i^+ - a_{im}\right)\right)\left(\sum_{m \neq j} \left(r_j^+ - a_{jm}\right)\right) \\
&\geq \left(\max\left\{0, -\sum_{m \neq i} \underline{a}_{im}\right\}\right)\left(\max\left\{0, -\sum_{m \neq j} \underline{a}_{jm}\right\}\right)
\end{aligned}
$$

The second inequality holds because of the fact that $A_{(x,y),(u,v)}$ for any x, y, u, v, such that $x \neq i, x \neq j, y \neq x, u \neq i, u \neq j, u \neq x$, and $v \neq u$ is a doubly B-matrix and $n \geq 4$.

Thus, as we have shown, the $\boldsymbol{A}$ fulfills both the conditions of Proposition 3.7, therefore it is an interval doubly B-matrix. □

Remark 3.6. Proposition 3.8 could also work for $n \geq 3$, but we would have to add a requirement that $\underline{A}$ is a doubly B-matrix, too. Or it could work even for $n \geq 2$, but again we would have to add requirements that $\underline{A}$ is a doubly B-matrix and $\forall j \in [n], l \neq j : A_{(j,l)}$ is a doubly B-matrix, where

$$
A_{(j,l)} = (a_{m_1 m_2}); \quad \begin{cases} \overline{a}_{jl} & \text{if } (m_1, m_2) = (j, l), \\ \underline{a}_{m_1 m_2} & \text{otherwise.} \end{cases}
$$

These requirements are needed for proof of parts "*II*." and "*III*." of condition *b*) of the second (right-to-left) implication. However, we can show an example that they are not just formal requirements:

Example 3.3. Let $\boldsymbol{A} \in \mathbb{IR}^{3\times 3}$, such that $\boldsymbol{A}_{ij} = \begin{cases} [1,1] = 1 & \text{if } i = j, \\ [-\frac{1}{2}, 0] & \text{otherwise.} \end{cases}$

Then $\forall A_{(i,k),(j,l)} : \quad \forall z, z' \in [3], z' \neq z : r_z^+ = r_{z'}^+ = 0$, so:

$$
\left(a_{zz} - r_z^+\right)\left(a_{z'z'} - r_{z'}^+\right) = 1 \cdot 1 = 1,
$$

and

$$
\left(\sum_{m \neq z} \left(r_z^+ - a_{zm}\right)\right)\left(\sum_{m \neq z'} \left(r_{z'}^+ - a_{z'm}\right)\right) \leq \frac{1}{2} \cdot 1 = \frac{1}{2}.
$$

Thus, every $A_{(i,k),(j,l)}$ is a doubly B-matrix.

However, for $\underline{A}: \quad \forall z, z' \in [3], z' \neq z :$

$$
\left(a_{zz} - r_z^+\right)\left(a_{z'z'} - r_{z'}^+\right) = 1 \cdot 1 = 1,
$$

and

$$
\left(\sum_{m \neq z} \left(r_z^+ - a_{zm}\right)\right)\left(\sum_{m \neq z'} \left(r_{z'}^+ - a_{z'm}\right)\right) = \left(\frac{1}{2} + \frac{1}{2}\right)^2 = 1^2 = 1.
$$

Therefore, $\underline{A}$ is not a doubly B-matrix, and hence $\boldsymbol{A}$ cannot be an interval doubly B-matrix.

Proposition 3.9. *The characterization of interval doubly B-matrices through the reduction given by Proposition 3.8 is minimal with respect to inclusion.*

Proof. If we skip $A_{(i,k),(j,l)}$ for any arbitrary $(i,j,k,l) \in [n]^4, j \neq i, k \neq i, l \neq j$, then we would be able to construct a counterexample, e.g., a unit matrix with an additional interval $[0, \frac{1}{2}]$ at positions (i,k) and (j,l). Then $\forall (x,y,u,v) \in [n]^4, u \neq x, y \neq x, v \neq u$, such that $(x,y,u,v) \neq (i,k,j,l) : A_{(x,y),(u,v)}$ is a doubly B-matrix. That holds because $\forall (z,z') \in [n]^2, z' \neq z$:

$$\left(a_{zz} - r_z^+\right)\left(a_{z'z'} - r_{z'}^+\right) \geq \frac{1}{2},$$

and

$$\left(\sum_{m \neq z} \left(r_z^+ - a_{zm}\right)\right)\left(\sum_{m \neq z'} \left(r_{z'}^+ - a_{z'm}\right)\right) = 0.$$

However, $A_{(i,k),(j,l)}$ is not a doubly B-matrix, because

$$\left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4},$$

and

$$\left(\sum_{m \neq i} \left(r_i^+ - a_{im}\right)\right)\left(\sum_{m \neq j} \left(r_j^+ - a_{jm}\right)\right)$$
$$= \left(\left(\frac{1}{2} - \frac{1}{2}\right) + (n-2) \cdot \left(\frac{1}{2} - 0\right)\right)^2 = \left(\frac{n-2}{2}\right)^2,$$

and for $n \geq 3$ it does not hold that $\frac{1}{4} > \left(\frac{n-2}{2}\right)^2$. (Plus in Proposition 3.8 we assume $n \geq 4$.)

Hence, the whole interval matrix cannot be an interval doubly B-matrix. □

Whereas the previous reduction stated in Proposition 3.8 reduces the problem of verifying an interval matrix on being an interval doubly B-matrix to $O(n^4)$ matrices (more precisely, for its basic version for $n \geq 4$ it reduces the problem to $\binom{n}{2} \cdot (n-1)^2$ real instances), the following uses a bit different approach and achieves to reduce the definition to $O(n^3)$ (more precisely to $n^2 \cdot (n-1) + n^2 = n^3$) matrices.

Proposition 3.10. *Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$, and let us define $A_{(i,k),(*,l)}$ and ${}_iA_{(*,l)} \in \mathbb{R}^{n \times n}$ as follows:*

$$A_{(i,k),(*,l)} = \left(a_{m_1m_2}\right); \quad a_{m_1m_2} = \begin{cases} \overline{a}_{ik} & \text{if } (m_1,m_2) = (i,k), \\ \overline{a}_{m_1l} & \text{if } m_2 = l \wedge m_1 \neq i \wedge m_1 \neq l, \\ \underline{a}_{m_1m_2} & \text{otherwise.} \end{cases}$$

and

$${}_iA_{(*,l)} = \left(a'_{m_1m_2}\right); \quad a'_{m_1m_2} = \begin{cases} \overline{a}_{m_1l} & \text{if } m_2 = l \wedge m_1 \neq i \wedge m_1 \neq l, \\ \underline{a}_{m_1m_2} & \text{otherwise.} \end{cases}$$

The matrix $\boldsymbol{A}$ *is an interval doubly B-matrix if and only if* $\forall (i,l) \in [n]^2 : ({}_iA_{(*,l)}$ *is a doubly B-matrix* $\quad\wedge\quad \forall k \in [n] \setminus \{i\} : A_{(i,k),(*,l)}$ *is a doubly B-matrix*$)$.

Proof. "$\Rightarrow$" Trivial, for all such matrices are in $\boldsymbol{A}$.

"$\Leftarrow$" We prove that the conditions of Proposition 3.7 hold:

$a)$ $\forall i \in [n], \forall k \neq i : \quad \underline{a}_{ii} > \max\{0, \overline{a}_{ik}\}$, because for any arbitrary l the matrix $A_{(i,k),(*,l)}$ is a doubly B-matrix. Therefore, $\forall i \in [n] : \quad \underline{a}_{ii} > \max\{0, \overline{a}_{ik} | k \neq i\}$.

$b)$ Let us fix arbitrary $(i,j) \in [n]^2, j \neq i$ and arbitrary $(k,l) \in [n]^2, k \neq i, l \neq j$.

I. Let us take $A = A_{(i,k),(*,l)}$. Then:

$$\begin{aligned}
\left(\underline{a}_{ii} - \overline{a}_{ik}\right)\left(\underline{a}_{jj} - \overline{a}_{jl}\right) \quad &\geq \quad \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) \\
> \quad &\left(\sum_{m\neq i}\left(r_i^+ - a_{im}\right)\right)\left(\sum_{m\neq j}\left(r_j^+ - a_{jm}\right)\right) \\
\geq \quad &\left(\max\left\{0, \sum_{\substack{m\neq i\\ m\neq k}}\left(\overline{a}_{ik} - \underline{a}_{im}\right)\right\}\right)\left(\max\left\{0, \sum_{\substack{m\neq j\\ m\neq l}}\left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right)
\end{aligned}$$

II. Let us take $A = {}_iA_{(*,l)}$. Then:

$$\begin{aligned}
\underline{a}_{ii}\left(\underline{a}_{jj} - \overline{a}_{jl}\right) \quad &\geq \quad \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) \\
> \quad &\left(\sum_{m\neq i}\left(r_i^+ - a_{im}\right)\right)\left(\sum_{m\neq j}\left(r_j^+ - a_{jm}\right)\right) \\
\geq \quad &\left(\max\left\{0, -\sum_{m\neq i}\underline{a}_{im}\right\}\right)\left(\max\left\{0, \sum_{\substack{m\neq j\\ m\neq l}}\left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right)
\end{aligned}$$

III. Let us take $A = {}_iA_{(*,j)}$. Then:

$$\begin{aligned}
\underline{a}_{ii} \cdot \underline{a}_{jj} \quad &\geq \quad \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) \\
> \quad &\left(\sum_{m\neq i}\left(r_i^+ - a_{im}\right)\right)\left(\sum_{m\neq j}\left(r_j^+ - a_{jm}\right)\right) \\
\geq \quad &\left(\max\left\{0, -\sum_{m\neq i}\underline{a}_{im}\right\}\right)\left(\max\left\{0, -\sum_{m\neq j}\underline{a}_{jm}\right\}\right)
\end{aligned}$$

Therefore, as we have proved, the $\boldsymbol{A}$ fulfills both the conditions of characterization stated in Proposition 3.7, thus it is an interval doubly B-matrix. $\square$

4 Conclusion and future work

There are several ways in which the current results might be extended. One possibility is to generalize our three classes even further, into parametric matrices, otherwise known as linearly dependent, addressed, for example, in [17]. Another direction is to generalize another subclass of P-matrices. Those might be, for example, so-called mimes, which stands for "M-matrix and Inverse M-matrix Extension", as they were introduced in [18]. Or it still remains unresolved whether the reductions presented in this paper are optimal with respect to the number of real instances used, or whether there exists some other reduction achieving to characterize one of the interval matrix classes using fewer instances. For the reduction from Proposition 3.10, the minimality with respect to inclusion is still undecided.

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