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On Some Convergence Properties for Finite Element Approximations to the Inverse of Linear Elliptic Operators*

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Abstract

This paper deals with convergence theorems of the Galerkin finite element approximation for the second-order elliptic boundary value problems. Under some quite general settings, we show not only the pointwise convergence but also prove that the norm of approximate operator converges to the corresponding norm for the inverse of a linear elliptic operator. Since the approximate norm estimates of linearized inverse operator play an essential role in the numerical verification method of solutions for non-linear elliptic problems, our result is also important in terms of guaranteeing its validity. Furthermore, the present method can also be applied to more general elliptic problems, e.g., biharmonic problems and so on.

Keywords: linear elliptic problems, finite element approximation, norm estimation of the inverse operator, convergence theorem

1 Introduction

In this section, we describe the background of the present study with notations of related function spaces, including finite elements, and the formulation of the problem. We will also mention the previous results that motivated this article.

1.1 Notations

We now introduce some function spaces necessary to consider the concerned problems.

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Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal or polyhedral domain where $d \in \{1,2,3\}$. For a non-negative integer m, let $H^m(\Omega)$ be the real L^2 Sobolev space with order m on Ω. We define

$$H_0^1(\Omega) := \left\{ u \in H^1(\Omega) \,\middle|\, u = 0 \text{ on } \partial\Omega \right\}$$

then $H_0^1(\Omega)$ is a Hilbert space with respect to the inner product $(u,v)_{H_0^1(\Omega)} := (\nabla u, \nabla v)_{L^2(\Omega)^d}$ and its norm is given by $\|u\|_{H_0^1(\Omega)} := \sqrt{(u,u)_{H_0^1(\Omega)}}$ where $(\,\cdot\,,\,\cdot\,)_{L^2}$ is the usual L^2 inner product on Ω. Let $H^{-1}(\Omega)$ be the dual space of $H_0^1(\Omega)$.

For a given non-linear function $f : H_0^1(\Omega) \to H^{-1}(\Omega)$ with certain properties, we often consider the existence and local uniqueness of the solution u satisfying the following non-linear elliptic boundary value problem of the form (e.g. [6] etc.):

$$\begin{cases} -\triangle u = f(u) & \text{in } \Omega \qquad (1a) \\ \quad u = 0 & \text{on } \partial\Omega. \qquad (1b) \end{cases}$$

To prove the existence of the solution of (1a)-(1b), the information on the linearized operator $\mathscr{L} := -\triangle - f'(u_k) : H_0^1(\Omega) \to H^{-1}(\Omega)$ and its inverse play important roles where u_k is a suitable approximation of u and $f'(u_k)$ is the Fréchet derivative of f at u_k. Moreover, we assume that $f'(u_k) \in \mathcal{L}\big(H_0^1(\Omega), L^2(\Omega)\big)$ for u_k with suitable regularities and the weak Laplace operator $-\triangle \in \mathcal{L}\big(H_0^1(\Omega), H^{-1}(\Omega)\big)$ where $\mathcal{L}(X,Y)$ is the linear space of all bounded linear operators from X to Y. As well known, by the Riesz representation lemma, the Poisson equation with homogeneous Dirichlet boundary condition is uniquely solvable. Namely, there exists a bounded inverse operator of $-\triangle$ such that $(-\triangle)^{-1} \in \mathcal{L}\big(H^{-1}(\Omega), H_0^1(\Omega)\big)$. Then, $\mathscr{L}$ can be represented as $\mathscr{L} = (-\triangle)\big(I - (-\triangle)^{-1} f'(u_k)\big)$ where I is the identity map on $H_0^1(\Omega)$. We denote $A := (-\triangle)^{-1} f'(u_k) \in \mathcal{L}\big(H_0^1(\Omega)\big)$. Note that A is a compact operator on $H_0^1(\Omega)$.

For an arbitrary $w \in H_0^1(\Omega)$, we set $u := Aw \in H_0^1(\Omega)$. Then, u satisfies the following variational equation:

$$(\nabla u, \nabla v)_{L^2(\Omega)^d} = \big(I_e g\big)(v) \quad \forall v \in H_0^1(\Omega) \qquad (2)$$

where $I_e : L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ is an embedding operator and $g := f'(u_k)w \in L^2(\Omega)$. By some standard arguments using the Riesz representation theorem, we can rewrite (2) simply as

$$(\nabla u, \nabla v)_{L^2(\Omega)^d} = (g, v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega). \qquad (3)$$

In general, the regularity of the solution (3) is smoother than $H_0^1(\Omega)$. Particularly, $u \in H\big(\triangle; L^2(\Omega)\big)$ holds where $H\big(\triangle; L^2(\Omega)\big) := \left\{ u \in H_0^1(\Omega) \,\middle|\, \triangle u \in L^2(\Omega) \right\}$.

Note that, if there exists a bounded inverse of $I - A$, then $\mathscr{L}$ also has an inverse: $\mathscr{L}^{-1} = (I - A)^{-1}(-\triangle)^{-1}$, and that $\left\|\mathscr{L}^{-1}\right\|_{\mathcal{L}\left(H^{-1}(\Omega), H_0^1(\Omega)\right)} = \left\|(I - A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}$ holds (also see [4, Remark 1.3]).

Nakao et al. [5, 7] proposed numerical verification approaches for computing upper bounds of $\left\|\mathscr{L}^{-1}\right\|_{\mathcal{L}\left(H^{-1}(\Omega),H_0^1(\Omega)\right)}$ (*cf.* [9, 10, 12, 3]).

Now, in order to define the approximation to the inverse operator $\mathscr{L}^{-1}$, we introduce the finite element space in the most general way possible. Let $S_h(\Omega)$ be a finite-dimensional subspace of $H_0^1(\Omega)$ depending on the discretization parameter $h > 0$ corresponding to the mesh size. We define the H_0^1-projection P_h from $H_0^1(\Omega)$ to $S_h(\Omega)$ such that

$$(u - P_h u, v_h)_{H_0^1(\Omega)} = 0 \quad \forall v_h \in S_h(\Omega). \tag{4}$$

Let $\{\phi_i\}_{i=1}^n \subset H_0^1(\Omega)$ be the set of basis functions in $S_h(\Omega)$ where $n := \dim S_h(\Omega)$. Let D_ϕ and G_ϕ be n-by-n matrices whose (i,j) elements are defined by

$$\begin{aligned} D_{\phi,i,j} &= (\nabla\phi_j, \nabla\phi_i)_{L^2(\Omega)^d}\,, \\ G_{\phi,i,j} &= (\nabla\phi_j, \nabla\phi_i)_{L^2(\Omega)^d} - (f'(u_k)\phi_j, \phi_i)_{L^2(\Omega)}\,, \end{aligned}$$

where matrix G_ϕ is the corresponding representation to the Galerkin approximation of operator $\mathscr{L}$. Since D_ϕ is a positive definite matrix, it can be Cholesky decomposed as $D_\phi = E_\phi E_\phi^T$ where E_ϕ is a lower triangular matrix and E_ϕ^T is the transposed matrix of E_ϕ. We define the Galerkin approximation of $I - A$ by $[I-A]_h := P_h(I-A)|_{S_h(\Omega)} : S_h(\Omega) \to S_h(\Omega)$ where $(I-A)|_{S_h(\Omega)}$ is the restriction of $I - A$ on $S_h(\Omega)$ and let $[I-A]_h^{-1} := \left(P_h(I-A)|_{S_h(\Omega)}\right)^{-1}$, if the inverse exists. Then, $\left\|[I-A]_h^{-1}\right\|_{\mathcal{L}\left(S_h(\Omega)\right)} = \left\|E_\phi^T G_\phi^{-1} E_\phi\right\|_2 =: r_h$ holds where $\|\cdot\|_2$ is the matrix 2-norm / the spectral matrix norm (see [5]). Since the non-singularity of the matrix can be verified by computational procedure (see, e.g., [11]), the existence of $[I-A]_h^{-1}$ is usually assumed to be valid([5]).

1.2 Motivation and preliminary results

In this subsection, we describe the previous results mainly obtained in [4], which is the motivation of this study.

Suppose that P_h defined by (4) has the following convergence property

$$\lim_{h\to 0} \|P_h u - u\|_{H_0^1(\Omega)} = 0, \quad \forall u \in H_0^1(\Omega) \tag{5}$$

and that there exists a positive constant $\tilde{C}(h)$ such that $\tilde{C}(h) \to 0$ as $h \to 0$ and satisfying

$$\|\nabla(u - P_h u)\|_{L^2(\Omega)^d} \le \tilde{C}(h)\,\|\triangle u\|_{L^2(\Omega)}\,, \quad \forall u \in H\big(\triangle; L^2(\Omega)\big). \tag{6}$$

The conditions (5) and (6) are satisfied for usual finite element subspaces (see, e.g., [1, 2, 8] etc.). Also, note that the following estimates hold for arbitrary $u \in H_0^1(\Omega)$:

$$\begin{aligned} \|(I-P_h)Au\|_{H_0^1(\Omega)} &\le \tilde{C}(h)\,\|\triangle Au\|_{L^2(\Omega)} \\ &\le \tilde{C}(h)\,\|f'(u_k)\|_{\mathcal{L}\left(H_0^1(\Omega),L^2(\Omega)\right)} \|u\|_{H_0^1(\Omega)}\,. \end{aligned} \tag{7}$$

We now suppose that the linearized operator $f'(u_k)$ is represented as $f'(u_k)u = -b \cdot \nabla u - cu$ for some functions such that $b \in W^{1,\infty}(\Omega)^d$ and $c \in L^\infty(\Omega)$. And we set the following non-negative constants:

$$\begin{aligned} C_1 &:= \|b\|_{L^\infty(\Omega)^d} + C_p \|c\|_{L^\infty(\Omega)}, \\ C_2 &:= \|b\|_{L^\infty(\Omega)^d} + \tilde{C}(h) \|c\|_{L^\infty(\Omega)}, \\ K(h) &:= \tilde{C}(h) \left(C_p \|\nabla \cdot b\|_{L^\infty(\Omega)} + C_1 \right) \end{aligned}$$

where C_p is the Poincaré constant satisfying

$$\|u\|_{L^2(\Omega)} \le C_p \|\nabla u\|_{L^2(\Omega)^d} \quad \forall u \in H_0^1(\Omega).$$

Then we already obtain the following existential condition and estimates of the linearized inverse operator $(I - A)^{-1}$:

Theorem 1 ([7, Theorem 2]). *If $\kappa_h := \tilde{C}(h)\big(r_h K(h) C_1 + C_2\big) < 1$, then $I - A$ is invertible and the following estimate holds:*

$$\left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)} \le \frac{1}{1-\kappa_h} \left\| \begin{pmatrix} r_h\big(1 - C_2\tilde{C}(h)\big) & r_h K(h) \\ r_h C_1 \tilde{C}(h) & 1 \end{pmatrix} \right\|_2 .$$

Moreover, by using the above theorem, if $\{r_h\}_{h>0}$ is a convergent sequence, then we have

$$\begin{aligned} \left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)} &\le \lim_{h\to 0} \frac{1}{1-\kappa_h} \left\| \begin{pmatrix} r_h\big(1 - C_2\tilde{C}(h)\big) & r_h K(h) \\ r_h C_1 \tilde{C}(h) & 1 \end{pmatrix} \right\|_2 \\ &= \left\| \begin{pmatrix} \lim\limits_{h\to 0} r_h & 0 \\ 0 & 1 \end{pmatrix} \right\|_2 \\ &= \max\left\{ \lim_{h\to 0} r_h, 1 \right\}. \end{aligned} \tag{8}$$

In our previous paper [4], by using (8), we presented the following relation:

$$1 \le \left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)} \le \lim_{h\to 0} \left\|[I-A]_h^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}, \tag{9}$$

provided that the limit in (9) actually exists. However, the question remains whether the second inequality of (9) becomes equality. In this paper, we prove that such equality holds true as well as clarify the condition for the existence of $[I - A]_h^{-1}$.

2 Main results

In this section, based on the notations and the preliminaries introduced in previous sections, we present the main result on the convergence property for finite element approximations of an inverse elliptic operator. To proceed with the argument, in the following, although it may be duplicated, some new definitions and assumptions are made again. It should also be noted that the intended purpose is achieved under a very common setting of the finite element space and approximation scheme. Let $\mathcal{L}\big(H_0^1(\Omega)\big)$ be a Banach space constituting of a set of bounded linear operators on $H_0^1(\Omega)$ with norm $\|Q\|_{\mathcal{L}\left(H_0^1(\Omega)\right)} := \sup_{0\neq u\in H_0^1(\Omega)} \dfrac{\|Qu\|_{H_0^1(\Omega)}}{\|u\|_{H_0^1(\Omega)}}$ for each $Q \in \mathcal{L}\big(H_0^1(\Omega)\big)$. Therefore, $S_h(\Omega)$ is considered as a finite-dimensional subspace of $H_0^1(\Omega)$ depending on the discretization parameter $h > 0$ with the same inner product and norm as $H_0^1(\Omega)$.

Assumption 1. *Operator $I - A$ is invertible. Namely, there exists $(I - A)^{-1} \in \mathcal{L}\big(H_0^1(\Omega)\big)$.*

Let $P_h \in \mathcal{L}\big(H_0^1(\Omega), S_h(\Omega)\big)$ be an orthogonal projection defined in (4). Then note that $\|P_h\|_{\mathcal{L}\left(H_0^1(\Omega),S_h(\Omega)\right)} \leq 1$ holds. We now assume the following two convergence properties:

Assumption 2. *For an arbitrary $u \in H_0^1(\Omega)$, $P_h u$ converges to u in $H_0^1(\Omega)$ as $h \to 0$.*

Assumption 3. *For each h, there exists a positive constant $C(h)$, which converges to 0 as $h \to 0$, satisfying*

$$\|(I - P_h)Au\|_{H_0^1(\Omega)} \leq C(h)\, \|u\|_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

Assumptions 2 and 3 correspond to (5) and (7), respectively, in the previous section. Therefore, as mentioned in subsection 1.2, these assumptions are quite reasonable conditions for usual finite element subspace $S_h(\Omega) \subset H_0^1(\Omega)$.

Remark 1. From the assumptions 1 and 3, there exists a constant $\delta_A > 0$ such that, for all $h \in (0, \delta_A)$,

$$C(h) < \frac{1}{\|(I - A)^{-1}\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}}. \tag{10}$$

Due to the compactness of operator $P_h A \in \mathcal{L}\big(H_0^1(\Omega), S_h(\Omega)\big)$, we have the following properties.

Lemma 1. *Let δ_A be the same constant in Remark 1. Then, for all $h \in (0, \delta_A)$, there exists a bounded inverse of $I - P_h A$ with estimates*

$$\big\|(I - P_h A)^{-1}\big\|_{\mathcal{L}\left(H_0^1(\Omega)\right)} \leq \frac{\big\|(I - A)^{-1}\big\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}}{1 - C(h)\, \|(I - A)^{-1}\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}}. \tag{11}$$

Proof. For an arbitrary $f \in H_0^1(\Omega)$, we consider the solution $u \in H_0^1(\Omega)$ satisfying:

$$(I - P_h A)u = f. \tag{12}$$

From assumption 1, it is readily seen that (12) is equivalent to the following fixed point equation:

$$u = -(I - A)^{-1}(I - P_h)Au + (I - A)^{-1} f =: T_{h,f}(u). \tag{13}$$

Hence, by using assumption 3, for arbitrary v, $w \in H_0^1(\Omega)$, we have

$$\begin{aligned}\|T_{h,f}(v) - T_{h,f}(w)\|_{H_0^1(\Omega)} &= \left\|(I - A)^{-1}(I - P_h)A(v - w)\right\|_{H_0^1(\Omega)} \\ &\le \left\|(I - A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)} C(h) \|v - w\|_{H_0^1(\Omega)} .\end{aligned}$$

If h is sufficiently small that (10) holds, then $T_{h,f}$ is a contraction map. Therefore, $T_{h,f}$ has a unique fixed point $u \in H_0^1(\Omega)$ satisfying (12) by Banach's fixed point theorem. Furthermore, the arbitrariness of f implies that $I - P_h A$ is a bijection on $H_0^1(\Omega)$ for such an h.
Also, by some simple calculation using (13) with assumption 3, we obtain

$$\left\|(I - P_h A)^{-1} f\right\|_{H_0^1(\Omega)} \le \frac{\left\|(I - A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}}{1 - \|(I - A)^{-1}\|_{\mathcal{L}\left(H_0^1(\Omega)\right)} C(h)} \|f\|_{H_0^1(\Omega)} ,$$

which yields the desired estimates (11). □

Note that

$$(I - P_h A)u_h = P_h(I - A)u_h, \quad \forall u_h \in S_h(\Omega)$$

holds. This fact means that $I - P_h A$ is equal to $P_h(I - A)$ on $S_h(\Omega)$, namely, $(I - P_h A)|_{S_h(\Omega)} = P_h(I - A)|_{S_h(\Omega)}$ holds. Therefore, let define $[I - A]_h \in \mathcal{L}(S_h(\Omega))$ by $[I - A]_h := P_h(I - A)|_{S_h(\Omega)}$. The following lemma gives an invertibility condition of $[I - A]_h$, and estimates for the norm of $[I - A]_h^{-1}$.

Lemma 2. *Under the same conditions as in Lemma 1, for all $h \in (0, \delta_A)$, there exists a inverse of $[I - A]_h$ and the following estimate holds*

$$\left\|[I - A]_h^{-1}\right\|_{\mathcal{L}\left(S_h(\Omega)\right)} \le \left\|(I - P_h A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)} . \tag{14}$$

Proof. For an $f_h \in S_h(\Omega)$, if $P_h(I - A)f_h = 0$, then

$$\begin{aligned}f_h &= P_h A f_h \\ &= -(I - P_h)Af_h + Af_h.\end{aligned}$$

Hence we have

$$(I - A)f_h = -(I - P_h)Af_h.$$

Namely,

$$f_h = -(I-A)^{-1}(I-P_h)Af_h.$$

Therefore, by assumption 3, we have

$$\begin{aligned} ||f_h||_{H_0^1(\Omega)} &\le ||(I-A)^{-1}||_{\mathcal{L}\left(H_0^1(\Omega)\right)} ||(I-P_h)Af_h||_{H_0^1(\Omega)} \\ &\le ||(I-A)^{-1}||_{\mathcal{L}\left(H_0^1(\Omega)\right)} C(h) ||f_h||_{H_0^1(\Omega)}, \end{aligned}$$

which yields $f_h = 0$ from (10). Taking notice that the existence and uniqueness of the solution are equivalent for the finite dimensional linear equation on $S_h(\Omega)$, the invertibility of $[I-A]_h$ follows immediately.
Next, observe that

$$\begin{aligned} \left\|(I-P_hA)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)} &= \sup_{0\neq f\in H_0^1(\Omega)} \frac{\|f\|_{H_0^1(\Omega)}}{\|(I-P_hA)f\|_{H_0^1(\Omega)}} \\ &\ge \sup_{0\neq f_h\in S_h(\Omega)} \frac{\|f_h\|_{H_0^1(\Omega)}}{\|(I-P_hA)f_h\|_{H_0^1(\Omega)}} \\ &= \sup_{0\neq f_h\in S_h(\Omega)} \frac{\|f_h\|_{H_0^1(\Omega)}}{\|P_h(I-A)f_h\|_{H_0^1(\Omega)}} \\ &= \left\|[I-A]_h^{-1}\right\|_{\mathcal{L}\left(S_h(\Omega)\right)}, \end{aligned}$$

which completes the proof of (14). □

On the convergence of $(I-P_hA)^{-1}$, we have the following lemma:

Lemma 3. *The following convergence property holds:*

$$\lim_{h\to 0} \left\|(I-P_hA)^{-1} - (I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)} = 0$$

Proof. Let δ_A be the same constant defined above. Therefore, for each $h \in (0, \delta_A)$, $I - P_hA$ is invertible on $H_0^1(\Omega)$ by lemma 1. For an arbitrary $f \in H_0^1(\Omega)$, we set $u := (I-A)^{-1}f \in H_0^1(\Omega)$ and $w(h) := (I-P_hA)^{-1}f \in H_0^1(\Omega)$. Then we have

$$(I-A)u = f \quad \text{and} \quad (I-P_hA)w(h) = f.$$

Hence, we obtain

$$(I-A)\big(u - w(h)\big) = (I-P_hA)w(h) - (I-A)w(h),$$

which is rewritten as

$$u - w(h) = (I-A)^{-1}(I-P_h)Aw(h).$$

From assumption 3, we obtain

$$\begin{aligned}\|u-w(h)\|_{H_0^1(\Omega)} &\le \left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)} C(h)\,\|w(h)\|_{H_0^1(\Omega)} \\ &\le \left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)} C(h)\left(\|u-w(h)\|_{H_0^1(\Omega)}+\|u\|_{H_0^1(\Omega)}\right).\end{aligned}$$

Hence we have

$$\left(1-C(h)\left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}\right)\|u-w(h)\|_{H_0^1(\Omega)} \le C(h)\left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}\|u\|_{H_0^1(\Omega)}.$$

Taking notice of (10),

$$\|u-w(h)\|_{H_0^1(\Omega)} \le \frac{C(h)\left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}}{1-C(h)\left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}}\|u\|_{H_0^1(\Omega)}.$$

Namely, it holds that

$$\begin{aligned}&\left\|(I-A)^{-1}f-(I-P_hA)^{-1}f\right\|_{H_0^1(\Omega)} \\ &\le \frac{C(h)\left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}}{1-C(h)\left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}}\left\|(I-A)^{-1}f\right\|_{H_0^1(\Omega)} \\ &\le \frac{C(h)\left\|(I-A)^{-1}\right\|^2_{\mathcal{L}\left(H_0^1(\Omega)\right)}}{1-C(h)\left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}}\|f\|_{H_0^1(\Omega)}.\end{aligned}$$

Therefore, we obtain the following convergence property:

$$\left\|(I-P_hA)^{-1}-(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)} \le \frac{C(h)\left\|(I-A)^{-1}\right\|^2_{\mathcal{L}\left(H_0^1(\Omega)\right)}}{1-C(h)\left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}} \to 0$$

as $h \to 0$, which yields the desired conclusion. □

Theorem 2. *The following convergence property holds for each* $f \in H_0^1(\Omega)$.

$$\lim_{h\to 0}\left\|[I-A]_h^{-1}P_hf-(I-A)^{-1}f\right\|_{H_0^1(\Omega)} = 0. \tag{15}$$

Proof. Let δ_A be a positive constant satisfying condition (10) and let h be a fixed parameter in $(0,\delta_A)$. Then, there exists $[I-A]_h^{-1} \in \mathcal{L}\big(S_h(\Omega)\big)$ by lemma 2. For each $f \in H_0^1(\Omega)$, we set $u := (I-A)^{-1}f \in H_0^1(\Omega)$ and $u_h := [I-A]_h^{-1}P_hf \in S_h(\Omega)$. By the definition, we have

$$\begin{aligned}f-P_hf &= (I-A)u-P_h(I-A)u_h \\ &= (I-P_hA)(u-u_h)+(I-A)u-(I-P_hA)u \\ &= (I-P_hA)(u-u_h)-(I-P_h)Au.\end{aligned}$$

Noting that there also exists $(I - P_hA)^{-1} \in \mathcal{L}\big(H_0^1(\Omega)\big)$ by lemma 1, from the assumption 3 and (11), we obtain, by using the above equality,

$$\begin{aligned}
\|u - u_h\|_{H_0^1(\Omega)} &= \left\|(I - P_hA)^{-1}\big(f - P_hf + (I - P_h)Au\big)\right\|_{H_0^1(\Omega)} \\
&\leq \frac{\left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}}{1 - C(h)\left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}}\Big(\|f - P_hf\|_{H_0^1(\Omega)} + C(h)\|u\|_{H_0^1(\Omega)}\Big) \\
&\leq \frac{\left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}}{1 - C(h)\left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}}\Big(\|f - P_hf\|_{H_0^1(\Omega)} \\
&\qquad + C(h)\left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}\|f\|_{H_0^1(\Omega)}\Big). \qquad (16)
\end{aligned}$$

The right-hand side of (16) converges to 0 as $h \to 0$ by the assumptions 2 and 3. Thus, (15) is proved. □

Now we present the norm convergence theorem, which is the main result of this paper.

Theorem 3. *The following norm convergence property holds:*

$$\lim_{h\to 0}\left\|[I-A]_h^{-1}\right\|_{\mathcal{L}\left(S_h(\Omega)\right)} = \left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}.$$

Proof. First, note that, for each fixed $f \in H_0^1(\Omega)$, we have by Theorem 2

$$\left\|(I-A)^{-1}f\right\|_{H_0^1(\Omega)} = \lim_{h\to 0}\left\|[I-A]_h^{-1}P_hf\right\|_{S_h(\Omega)}.$$

Therefore, it holds that

$$\begin{aligned}
\left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)} &= \sup_{||f||_{H_0^1(\Omega)}=1}\left\|(I-A)^{-1}f\right\|_{H_0^1(\Omega)} \\
&= \sup_{||f||_{H_0^1(\Omega)}=1}\lim_{h\to 0}\left\|[I-A]_h^{-1}P_hf\right\|_{S_h(\Omega)}. \qquad (17)
\end{aligned}$$

Moreover, for each $h \in (0, \delta_A)$ and $f \in H_0^1(\Omega)$ with $\|f\|_{H_0^1(\Omega)} = 1$, observe that by using Lemma 2

$$\begin{aligned}
\left\|[I-A]_h^{-1}P_hf\right\|_{S_h(\Omega)} &\leq \left\|[I-A]_h^{-1}\right\|_{\mathcal{L}\left(S_h(\Omega)\right)}||P_hf||_{S_h(\Omega)} \\
&\leq \left\|[I-A]_h^{-1}\right\|_{\mathcal{L}\left(S_h(\Omega)\right)} \qquad (18) \\
&\leq ||(I-P_hA)^{-1}||_{\mathcal{L}\left(H_0^1(\Omega)\right)}. \qquad (19)
\end{aligned}$$

On the other hand, by Lemma 3, it holds that the right-hand side of (19) converges to $\left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}$ as $h \to 0$. Combining this fact with (17)-(19) we can show that $\lim_{h\to 0} \left\|[I-A]_h^{-1}\right\|_{\mathcal{L}\left(S_h(\Omega)\right)}$ exists and equals $\left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}$. Indeed, we take the limit inferior and limit superior of (18) and (19),

$$\begin{aligned}\lim_{h\to 0} \left\|[I-A]_h^{-1} P_h f\right\|_{S_h(\Omega)} &\le \liminf_{h\to 0} \left\|[I-A]_h^{-1}\right\|_{\mathcal{L}\left(S_h(\Omega)\right)} \\ &\le \limsup_{h\to 0} \left\|[I-A]_h^{-1}\right\|_{\mathcal{L}\left(S_h(\Omega)\right)} \\ &\le \left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)} \end{aligned} \tag{20}$$

holds. Here, the last inequality follows from Lemma 3. Taking notice that the inequalities, except for the first left-hand sides in (20) is independent of f, we obtain from (17)

$$\begin{aligned}\left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)} &= \sup_{||f||_{H_0^1(\Omega)}=1} \lim_{h\to 0} \left\|[I-A]_h^{-1} P_h f\right\|_{S_h(\Omega)} \\ &\le \liminf_{h\to 0} \left\|[I-A]_h^{-1}\right\|_{\mathcal{L}\left(S_h(\Omega)\right)} \\ &\le \left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)}\end{aligned}$$

Combining the above with (20), we have

$$\liminf_{h\to 0} \left\|[I-A]_h^{-1}\right\|_{\mathcal{L}\left(S_h(\Omega)\right)} = \limsup_{h\to 0} \left\|[I-A]_h^{-1}\right\|_{\mathcal{L}\left(S_h(\Omega)\right)} = \left\|(I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)},$$

which yields the desired conclusion. □

Remark 2. Note that the result of Theorem 3 does not mean
$[I-A]_h^{-1} P_h \to (I-A)^{-1}$ as $h \to 0$ in $\mathcal{L}\left(H_0^1(\Omega)\right)$.
Actually, if $\lim_{h\to 0} \left\|[I-A]_h^{-1} P_h - (I-A)^{-1}\right\|_{\mathcal{L}\left(H_0^1(\Omega)\right)} = 0$ holds, then considering the particular case: $A \equiv 0$, it implies that $\lim_{h\to 0} \|P_h - I\|_{\mathcal{L}\left(H_0^1(\Omega)\right)} = 0$. From the fact that P_h is a finite dimensional operator, this contradicts that the identity operator I is not compact on the infinite dimensional space $\mathcal{L}\left(H_0^1(\Omega)\right)$.

3 Conclusion

We presented the convergence theorem of $[I-A]_h^{-1} P_h$ to $(I-A)^{-1}$ as $h \to 0$ in Theorem 2, and we also established the norm convergence theorem in Theorem 3. Moreover, Lemma 2 is important as a theoretical result for the existence of the Galerkin approximation for $(I-A)^{-1}$. It is also expected that these results can be extended for the more general linear compact operator A, e.g., corresponding to the biharmonic problems, under similar assumptions to 1, 2, and 3.

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