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The Inventory Control Problem for a Supply Chain With a Mixed Type of Demand Uncertainty

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Abstract

This paper is concerned with a dynamic inventory control system described by a network model where the nodes are warehouses and the arcs represent production and distribution activities. We assume that an uncertain demand may take any value in an assigned interval and we allow that the system is disturbed by noise inputs. These assumptions yield a model with a mix of interval and stochastic demand uncertainties. We use the method of model predictive control to derive the control strategy. To deal with interval uncertainty we use the interval analysis tools and act according to the interval analysis theory. The developed results are illustrated using a numerical example.

Keywords: inventory control, supply chain, network model, model predictive control, interval-stochastic uncertainty, interval analysis, multiobjective optimization, quadratic programming

1 Introduction

Nowadays, most supply chains are multi-echelon and have a complex network structure. Such a network consists of suppliers, manufacturing plants, warehouses, customers, and distribution channels that are organized efficiently to get raw materials, convert them to finished products, and distribute the products to customers. The structure of any multi-echelon supply chain depends on the configuration and location of various echelons with respect to each other. It can be described by a directed network in which the nodes represent warehouses and the arcs are controllable and uncontrollable flows in the network. The controllable flows can be controlled by a system manager. They redistribute resources between the network nodes, possibly process them, and plan deliveries from outside. The uncontrollable flows represent a demand in the network nodes that can be made both by other nodes and from the outside. Supply chain managers always seek to find best decisions to provide products or services for customers in the right quantities, at the right places, and at right times.

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This paper deals with the inventory control problem in a multi-echelon supply chain network. In the classical inventory control theory uncertainty of demand is regarded as stochastic uncertainty. However, in many real cases, there are not enough historical data to estimate parameters of distributions of random variables that affect the system. This fact gives rise to the need to use other approaches to describing uncertainty. An interesting approach based on unknown-but-bounded inputs is proposed in [2, 3]. The studies are devoted to the inventory control problem under an uncertain demand. Unlike the classical stochastic approach, they model demand uncertainty in an unknown-but-bounded way assuming that an uncertain demand may take any values in an assigned set, and nothing else is known about demand behaviour. This makes sense because in practice the upper and lower bounds for an uncertain demand can be inferred from the decision maker's experience or available historical data much more easily and with much more confidence than empirical probability distribution. At the same time, the efficiency of the control strategy strongly depends on the width of the interval of uncertain demand and this interval should be as narrow as possible. To reduce the width of the interval we can use a mixed form of model uncertainty. This is reasonable, for example, when we have partial information about demands. Indeed, for some products we do not have historical demand data, while for others we do. In addition, we can have quite stable orders within given limits from some consumers and random orders from others. In such cases, an uncertain aggregate demand can be decomposed in two sub-vectors, one of which is unknown-but-bounded (interval), and the other is stochastic. These assumptions lead to a mixed interval-stochastic uncertainty which is used further in this study.

We use the model predictive control [4, 14] (MPC) to derive the optimal control strategy. The MPC approach is widely applied in the practice of control and allows for solving complex control problems for systems with various types of uncertainty. For example, the papers [15, 18] consider supply chain networks under conditions of stochastic uncertainty, and the MPC approach allows the authors to develop a control strategy in order to achieve the system robustness, performance and high levels of service. The paper [1] studies stochastic hybrid systems and shows the effectiveness of suggested techniques for a problem of supply chain management. The paper [8] addresses the problem of the model predictive control for discrete systems with random dependent parameters and its possible application to investment portfolio optimization. The papers [6, 12, 17] examine the MPC problem for systems with a polytopic uncertainty description on state-space matrices under diverse input-output constraints. The problem is solved using the minmax approach to the MPC based on linear matrix inequalities. The paper [7] discusses the case of uncertain linear dynamic systems with interval assigned parameters and multiplicative noises in system matrices. By using the minmax MPC based on linear matrix inequalities, the optimal robust control strategy providing the system with stability in the mean-square sense is obtained. But the lack of constraints does not allow the use of the obtained results for inventory management directly, where, as a rule, there are various capacity constraints. The paper [5] is concerned with the inventory control problem under hard constraints in storage levels and controls. A linear objective criterion is used to find the optimal control strategy that minimizes the worst-case storage cost under interval demand uncertainty, but the cost of control actions is not taken into account here. The problem is converted into a linear programming problem with constraints to be solved online that gives the optimal control strategy without a shortage and backlogged demand. However, additional stochastic uncertainty is not assumed here.

This paper considers an inventory control system with mixed additive uncertainty in the presence of constraints in the states of the system and control actions. An uncertain demand is estimated by an interval without any distribution information, and the system is assumed to be disturbed by white noise. To deal with the interval uncertainty we use the interval analysis tools and act according to the interval analysis theory [13]. The influence of stochastic uncertainty leads to the minimization of the conditional expected value of the MPC objective under soft constraints in the states of the system. We transform the system control problem with mixed model uncertainty to a deterministic quadratic programming problem for which there are efficient solution methods and commercial software packages (we used the `quadprog` function provided by the software Optimization Toolbox in the MATLAB environment). Solving this problem online, we get a feedback inventory control strategy with a minimum expected level of storage, but a high level of service.

The paper is organized as follows. Section 2 introduces the problem to be solved. Section 3 presents the main results concerning the optimal control under interval-stochastic demand uncertainty. A numerical example showing the results obtained is presented in Section 4 and conclusions are given in Section 5.

2 Problem statement

We consider a dynamic inventory control system with a network structure (supply chain). The evolution of the network is described by the equation:

$$x(k+1) = x(k) + Bu(k) + Cd(k) + Cw(k), \quad k = 0, 1, 2, \dots \tag{1}$$

Here $x(k) \in \mathbb{R}^n$ is the system state whose components represent storage levels in the network nodes at the time k, the initial state $x(0)$ is assumed to be fixed and given; $u(k) \in \mathbb{R}^m$ is the control representing the controllable flows between the network nodes at the time k; $d(k)$, $w(k) \in \mathbb{R}^l$ are the uncertain demand vectors describing the uncontrollable flows in the network nodes at the time k; the matrices $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{n\times l}$ describe the network structure. As the unit of time k we can take, for example, a day, a week, a month, or a longer period.

Interval uncertainty in the system is represented by the vector $d(k)$. We know that $d(k)$ takes its values within a given interval but the rest is unknown:

$$d(k) \in \boldsymbol{D}, \quad k = 0, 1, 2, \dots, \tag{2}$$

where $\boldsymbol{D} \in \mathbb{IR}^l$, $\boldsymbol{D} = \left[\underline{D}, \overline{D}\right] \geq 0$.

In the paper we follow the notation of the informal international standard [11]. Intervals and interval objects (vectors, matrices) are denoted in bold, $\underline{x}, \overline{x}$ are the lower and upper bounds of the interval $\boldsymbol{x}$, $\mathbb{IR}^n = \{\boldsymbol{x} = [\underline{x}, \overline{x}], \underline{x} \leq \overline{x}, \underline{x}, \overline{x} \in \mathbb{R}^n\}$ is the set of all n-dimensional intervals in the classical interval arithmetic $\mathbb{IR}$, $\mathbb{KR}^n = \{\boldsymbol{x} = [\underline{x}, \overline{x}], \underline{x}, \overline{x} \in \mathbb{R}^n\}$ is the set of all n-dimensional intervals in the Kaucher complete interval arithmetic $\mathbb{KR}$ [10, 16].

The uncertain vector $w(k)$ describes white noise with a zero mean and the covariance matrix $\mathsf{E}\{w(k)w^\top(k)\} = W$. This forms stochastic uncertainty in the system.

Additionally, we assume that both expected storage levels and controls must be non-negative and bounded:

$$\mathsf{E}\big\{x(k+1) \bigm| x(k)\big\} \in \boldsymbol{X}, \quad k = 0, 1, 2, \ldots, \tag{3}$$

$$u(k) \in \boldsymbol{U}, \quad k = 0, 1, 2, \ldots, \tag{4}$$

where $\mathsf{E}\{\cdot|\cdot\}$ denotes the conditional expectation; $\boldsymbol{X} \in \mathbb{IR}^n$, $\boldsymbol{X} = \big[0, \overline{X}\big]$; $\boldsymbol{U} \in \mathbb{IR}^q$, $\boldsymbol{U} = \big[0, \overline{U}\big]$. The bounds of the constraints given in (3), (4) define the system's capacities, such as storage limit and order quantity limit. In (3), the lower bound equal to zero means that a shortage of stock is undesirable, but possible. The shortage reduces the service level that is defined as the proportion of demand satisfied. The ideal case is 100% service level. In order to maintain a high service level under the uncertain demand a safety stock is formed. The level of the safety stock for real-life complex, lean, and agile networks can be efficiently determined by the method of the dynamic simulation.

We define the MPC performance index (cost function) as follows:

$$\begin{aligned} J(k+p|k) = \mathsf{E}\Bigg\{\sum_{i=1}^{p}\Big(&\big(x(k+i|k) - x_0\big)^\top Q\big(x(k+i|k) - x_0\big) \\ &- Q_1\big(x(k+i|k) - x_0\big) + u(k+i-1|k)^\top R u(k+i-1|k)\Big) \Bigm| x(k)\Bigg\}, \end{aligned} \tag{5}$$

where $x(k+i|k)$ is the state at the time $k+i$ which is predicted at the time k, $x(k)$ or $x(k|k)$ denotes the state measured at the time k; x_0 is the target level that defines a desired storage level; $u(k+i|k)$ is the predictive control at the time $k+i$ which is computed at the time k; p is the prediction horizon; $Q \in \mathbb{R}^{n\times n}$, $Q_1 \in \mathbb{R}^{1\times n}$ and $R \in \mathbb{R}^{m\times m}$ are the weighting matrices such that Q, R are symmetric positive definite matrices and $Q_1 \geq 0$.

The control goal generally is to keep the state of the system close to the target using little control efforts. But taking into account the fact that we deal with a storage level it is necessary to specify the goal so that the state of the system is close but preferably not below the target level x_0. In cost function (5) the first term $\big(x(k+i|k) - x_0\big)^\top Q\big(x(k+i|k) - x_0\big)$ penalizes the state deviation from the target level, the second linear term $Q_1\big(x(k+i|k) - x_0\big)$ penalizes the state negative deviation from the target level, and the last term $u(k+i-1|k)^\top R u(k+i-1|k)$ penalizes the control efforts.

The problem to be solved is to compute a sequence of the predictive controls $u(k|k), u(k+1|k), \ldots, u(k+p-1|k)$ which minimizes cost index (5):

$$\min_{u(k|k),u(k+1|k),\ldots,u(k+p-1|k)} J(k+p|k),$$

subject to system dynamics (1) and constraints (2), (3), (4).

We reduce the above problem to an interval quadratic programming problem where the uncertain inputs are represented by intervals. Since the input data are interval, the objective value is also interval. We calculate the lower and upper bounds of the objective values of the interval quadratic programming problem analytically using the interval analysis and formulate a two-objective optimization problem. We then transform the problem into a conventional quadratic programming problem with a single objective by using the multi-objective optimization technique [9].

As is standard in the MPC, at the time k we calculate the sequence of predictive controls $u(k|k), u(k+1|k), \ldots, u(k+p-1|k)$, but use only the first of them and obtain the feedback control $u(k) = u(k|k)$ as a function of the state $x(k)$. Then the state $x(k+1)$ is measured, the control horizon is moved by one, and the optimization is repeated at the next time $k+1$. The result is the feedback inventory control strategy $\Phi = \{u(k) = u\left(x(k), k\right), \, k \geq 0\}$.

3 Main results

The following theorem gives the sequence of predictive controls $\{u(k|k), u(k+1|k), \ldots, u(k+p-1|k)\}$ at the time k.

Theorem. *The vector of predictive controls*

$$\tilde{u}(k) = \left(u(k|k)^\top, \, u(k+1|k)^\top, \ldots, \, u(k+p-1|k)^\top\right)^\top$$

that minimizes performance index (5) *under constraints* (2), (3), (4) *on the trajectories of system* (1) *is defined at each time k as a solution to the quadratic programming problem with the criterion*

$$Y(k+p|k) = \tilde{u}(k)^\top H \tilde{u}(k) + 2G(k)\tilde{u}(k) \tag{6}$$

under the constraints

$$\left(B \; 0_{n\times m} \; 0_{n\times m} \ldots 0_{n\times m}\right)\tilde{u}(k) \in \boldsymbol{X} \ominus \boldsymbol{CD} - x(k), \tag{7}$$

$$\tilde{u}(k) \in \tilde{\boldsymbol{U}}. \tag{8}$$

Here $H, G(k)$ are the block matrices of the type

$$H = \begin{pmatrix} H_{11} & H_{12} & \ldots & H_{1p} \\ H_{21} & H_{22} & \ldots & H_{2p} \\ \vdots & \ddots & \vdots & \\ H_{p1} & H_{p2} & \ldots & H_{pp} \end{pmatrix}, \quad H_{ij} = \begin{cases} (p-j+1)B^\top QB, & i<j, \\ R + (p-j+1)B^\top QB, & i=j, \\ (p-i+1)B^\top QB, & i>j, \end{cases} \tag{9}$$

$$G(k) = \left(\left(x(k) - x_0\right)^\top Q - \frac{1}{2}Q_1\right) BK + \operatorname{mid} \boldsymbol{DF}, \tag{10}$$

where

$$K = \begin{pmatrix} K_1 & K_2 & \dots & K_p \end{pmatrix}, \quad K_i = (p-i+1)I_m,$$

$$F = \begin{pmatrix} F_{11} & F_{12} & \dots & F_{1p} \\ F_{21} & F_{22} & \dots & F_{2p} \\ \vdots & \ddots & \vdots & \\ F_{p1} & F_{p2} & \dots & F_{pp} \end{pmatrix}, \quad F_{ij} = \begin{cases} (p-j+1)C^\top QB, & i \le j, \\ (p-i+1)C^\top QB, & i > j, \end{cases}$$

$0_{n\times m}$ *is the zero matrix of the dimension* $n \times m$*,* I_m *is the unit matrix of the dimension* m*,* $\tilde{\boldsymbol{U}} = \left(\boldsymbol{U}^\top\ \boldsymbol{U}^\top \dots \boldsymbol{U}^\top\right)^\top$*,* $\tilde{\boldsymbol{D}} = \left(\boldsymbol{D}^\top\ \boldsymbol{D}^\top \dots \boldsymbol{D}^\top\right)^\top$*,* $\boldsymbol{CD}$ *is the result of multiplying the real matrix* C *by the interval vector* $\boldsymbol{D}$*,* $\boldsymbol{DF}$ *is the result of multiplying the interval vector* $\tilde{\boldsymbol{D}}^\top$ *by the real matrix* F*,* $\operatorname{mid}\boldsymbol{x}$ *is the midpoint of the interval* $\boldsymbol{x}$*,* $\boldsymbol{x} \ominus \boldsymbol{y} = \left[\underline{x} - \underline{y}, \overline{x} - \overline{y}\right]$ *is the internal subtraction in* $\mathbb{KR}$*.*

Proof. Let us consider performance index (5). By using the fact that the summand

$$\begin{aligned}&(x(k+i|k)-x_0)^\top Q\,(x(k+i|k)-x_0) - Q_1\,(x(k+i|k)-x_0) \\ &+u(k+i-1|k)^\top Ru(k+i-1|k) = x(k+i|k)^\top Qx(k+i|k) - \left(2x_0^\top Q + Q_1\right)x(k+i|k) \\ &\qquad + \left(x_0^\top Q + Q_1\right)x_0 + u(k+i-1|k)^\top Ru(k+i-1|k),\end{aligned}$$

(5) turns into

$$\begin{aligned}J(k+p|k) = \mathsf{E}\Big\{\sum_{i=1}^{p}\Big(&x(k+i|k)^\top Qx(k+i|k) - \left(2x_0^\top Q + Q_1\right)x(k+i|k) \\ &+ u(k+i-1|k)^\top Ru(k+i-1|k)\Big)\ \Big|\ x(k)\Big\} + p\left(x_0^\top Q + Q_1\right)x_0.\end{aligned}$$

To deal with the conditional expectation, we rewrite the index as:

$$\begin{aligned}J(k+p|k) = \mathsf{E}\Big\{&x(k+1|k)^\top Qx(k+1|k) - \left(2x_0^\top Q + Q_1\right)x(k+1|k) \\ &+ u(k|k)^\top Ru(k|k) + \mathsf{E}\Big\{x(k+2|k)^\top Qx(k+2|k) \\ &- \left(2x_0^\top Q + Q_1\right)x(k+2|k) + u(k+1|k)^\top Ru(k+1|k) + \dots \\ &+ \mathsf{E}\big\{x(k+p|k)^\top Qx(k+p|k) - \left(2x_0^\top Q + Q_1\right)x(k+p|k) \\ &+ u(k+p-1|k)^\top Ru(k+p-1|k)\ \big|\ x(k+p-1)\big\} \dots \\ &\Big|\ x(k+1)\Big\}\ \Big|\ x(k)\Big\} + p\left(x_0^\top Q + Q_1\right)x_0.\end{aligned}$$

We introduce the notation

$$
\begin{aligned}
J_{k+i} = \mathsf{E}\Big\{ & x(k+i+1|k)^\top Qx(k+i+1|k) - \left(2x_0^\top Q + Q_1\right) x(k+i+1|k) \\
& + u(k+i|k)^\top Ru(k+i|k) + \mathsf{E}\Big\{ x(k+i+2|k)^\top Qx(k+i+2|k) \\
& - \left(2x_0^\top Q + Q_1\right) x(k+i+2|k) + u(k+i+1|k)^\top Ru(k+i+1|k) + \ldots \\
& + \mathsf{E}\big\{ x(k+p|k)^\top Qx(k+p|k) - \left(2x_0^\top Q + Q_1\right) x(k+p|k) \\
& + u(k+p-1|k)^\top Ru(k+p-1|k) \bigm| x(k+p-1) \big\} \ldots \\
& \Big| \; x(k+i+1) \Big\} \; \Big| \; x(k+i) \Big\}.
\end{aligned}
$$

Now it is clear that

$$
\begin{aligned}
J_{k+i} = {} & \mathsf{E}\big\{ x(k+i+1|k)^\top Qx(k+i+1|k) - \left(2x_0^\top Q + Q_1\right) x(k+i+1|k) \\
& + u(k+i|k)^\top Ru(k+i|k) + J_{k+i+1} \bigm| x(k+i) \big\}, \quad i = 0, \ldots, p-1,
\end{aligned} \tag{11}
$$

with $J_{k+p} = 0$ and

$$
J(k+p|k) = J_k + p\left(x_0^\top Q + Q_1\right) x_0. \tag{12}
$$

Using the method of mathematical induction we prove that the following relationship is valid:

$$
\begin{aligned}
J_{k+p-t} = {} & tx(k+p-t|k)^\top Qx(k+p-t|k) \\
& + \Big(2\big(x(k+p-t|k) - x_0\big)^\top Q - Q_1 \Big) \\
& \times \left(\sum_{i=1}^{t} iBu(k+p-i|k) + \sum_{i=1}^{t} iCd(k+p-i) \right) \\
& + \sum_{i=1}^{t} u(k+p-i|k)^\top \Bigg(2\sum_{j=1}^{i-1} jB^\top QBu(k+p-j|k) \\
& + (iB^\top QB + R)u(k+p-i|k) \Bigg) + 2\sum_{i=1}^{t} \Bigg(\sum_{j=1}^{i} jd(k+p-j)^\top \\
& + \sum_{j=i+1}^{t} id(k+p-j)^\top \Bigg) C^\top QBu(k+p-i|k) \\
& + \sum_{i=1}^{t} d(k+p-i)^\top C^\top QC \left(2\sum_{j=1}^{i-1} jd(k+p-j) + id(k+p-i) \right) \\
& + \operatorname{tr}\left\{ \frac{t(t+1)}{2} C^\top QCW \right\} - t(2x_0^\top Q + Q_1)x(k+p-t|k), \quad t = 1, \ldots, p,
\end{aligned} \tag{13}
$$

where $\operatorname{tr}\{\cdot\}$ is the trace of a matrix.

At first, let us consider the case for $p = 1$. From (11) for $i = p - 1$ we have

$$\begin{aligned} J_{k+p-1} = \mathsf{E}\big\{ & x(k+p|k)^\top Q x(k+p|k) - \big(2x_0^\top Q + Q_1\big)\, x(k+p|k) \\ & + u(k+p-1|k)^\top R u(k+p-1|k) \;\big|\; x(k+p-1)\big\}. \end{aligned} \tag{14}$$

Substituting $x(k+p|k)$ by its expression in terms of $x(k+p-1|k)$ from (1) in (14) and taking the conventional mathematical expectation, we get

$$\begin{aligned} J_{k+p-1} = {} & x(k+p-1|k)^\top Q x(k+p-1|k) + \Big(2\big(x(k+p-1|k) - x_0\big)^\top Q - Q_1\Big) \\ & \times \big(Bu(k+p-1|k) + Cd(k+p-1)\big) \\ & + u(k+p-1|k)^\top (B^\top Q B + R) u(k+p-1|k) \\ & + 2d(k+p-1)^\top C^\top Q B u(k+p-1|k) + d(k+p-1)^\top C^\top Q C d(k+p-1) \\ & + \mathrm{tr}\{C^\top Q C W\} - \big(2x_0^\top Q + Q_1\big)\, x(k+p-1|k), \end{aligned}$$

and this coincides with (13) if $t = 1$.
Now let us suppose that relationship (13) is valid for some t, and show that (13) is valid for $t+1$. Indeed, from recursive expression (11) we obtain

$$\begin{aligned} J_{k+p-t-1} = \mathsf{E}\big\{ & x(k+p-t|k)^\top Q x(k+p-t|k) - \big(2x_0^\top Q + Q_1\big)\, x(k+p-t|k) \\ & + u(k+p-t-1|k)^\top R u(k+p-t-1|k) + J_{k+p-t} \;\big|\; x(k+p-t)\big\}. \end{aligned} \tag{15}$$

Now we will substitute $x(k+p-t|k)$ by its expression in terms of $x(k+p-t-1|k)$ from (1) in (15) and J_{k+p-t} by its expression from (13). Expanding the conventional expectation and transforming the expression, we obtain that

$$\begin{aligned} J_{k+p-t-1} = {} & (t+1) x(k+p-t-1|k)^\top Q x(k+p-t-1|k) \\ & + \Big(2\big(x(k+p-t-1|k) - x_0\big)^\top Q - Q_1\Big) \\ & \times \left(\sum_{i=1}^{t+1} iBu(k+p-i|k) + \sum_{i=1}^{t+1} iCd(k+p-i)\right) \\ & + \sum_{i=1}^{t+1} u(k+p-i|k)^\top \left(2\sum_{j=1}^{i-1} jB^\top Q B u(k+p-j|k)\right. \\ & \left. + (iB^\top Q B + R) u(k+p-i|k)\right) + 2\sum_{i=1}^{t+1}\left(\sum_{j=1}^{i} jd(k+p-j)^\top\right. \\ & \left. + \sum_{j=i+1}^{t+1} id(k+p-j)^\top\right) C^\top Q B u(k+p-i|k) \\ & + \sum_{i=1}^{t} d(k+p-i)^\top C^\top Q C \left(2\sum_{j=1}^{i-1} jd(k+p-j) + id(k+p-i)\right) \\ & + \mathrm{tr}\left\{\frac{(t+1)(t+2)}{2} C^\top Q C W\right\} - (t+1)(2x_0^\top Q + Q_1) x(k+p-t-1|k). \end{aligned} \tag{16}$$

Formula (16) coincides with (13) if t is replaced by $t+1$, and hence, according to the mathematical induction rule, formula (13) is valid for all $t = 1, \dots, p$.
Using the fact that (13) gives an expression for J_k with $t = p$, we get from (12):

$$\begin{aligned} J(k+p|k) &= px(k|k)^\top Qx(k|k) + \Big(2\big(x(k|k) - x_0\big)^\top Q - Q_1\Big) \\ &\quad \times \left(\sum_{i=1}^{p} iBu(k+p-i|k) + \sum_{i=1}^{p} iCd(k+p-i)\right) \\ &\quad + \sum_{i=1}^{p} u(k+p-i|k)^\top \left(2\sum_{j=1}^{i-1} jB^\top QBu(k+p-j|k)\right. \\ &\quad \left. + \big(iB^\top QB + R\big)u(k+p-i|k)\right) + 2\sum_{i=1}^{p}\left(\sum_{j=1}^{i} jd(k+p-j)^\top\right. \\ &\quad \left. + \sum_{j=i+1}^{p} id(k+p-j)^\top\right) C^\top QBu(k+p-i|k) \\ &\quad + \sum_{i=1}^{p} d(k+p-i)^\top C^\top QC\left(2\sum_{j=1}^{i-1} jd(k+p-j) + id(k+p-i)\right) \\ &\quad + \operatorname{tr}\left\{\frac{p(p+1)}{2} C^\top QCW\right\} - p\left(2x_0^\top Q + Q_1\right)x(k|k) + p\left(x_0^\top Q + Q_1\right)x_0. \end{aligned}$$

Eliminating all the terms that do not depend on the controls u and do not influence the optimum, we obtain

$$\begin{aligned} \mathcal{J}(k+p|k) &= \Big(2\big(x(k|k) - x_0\big)^\top Q - Q_1\Big) \sum_{i=1}^{p} iBu(k+p-i|k) \\ &\quad + \sum_{i=1}^{p} u(k+p-i|k)^\top \left(2\sum_{j=1}^{i-1} jB^\top QBu(k+p-j|k)\right. \\ &\quad \left. + \big(iB^\top QB + R\big)u(k+p-i|k)\right) + 2\sum_{i=1}^{p}\left(\sum_{j=1}^{i} jd(k+p-j)^\top\right. \\ &\quad \left. + \sum_{j=i+1}^{p} id(k+p-j)^\top\right) C^\top QBu(k+p-i|k). \end{aligned} \tag{17}$$

Expression (17) can be rewritten in a matrix form as:

$$\mathcal{J}(k+p|k) = \tilde{u}(k)^\top H\tilde{u}(k) + 2\mathcal{G}(k)\tilde{u}(k), \tag{18}$$

where $\tilde{u}(k) = \big(u(k|k)^\top,\ u(k+1|k)^\top, \dots,\ u(k+p-1|k)^\top\big)^\top$, H is given by (9),

$$\mathcal{G}(k) = \left(\big(x(k) - x_0\big)^\top Q - \frac{1}{2}Q_1\right) BK + \tilde{d}(k)^\top F,$$

and

$$\tilde{d}(k) = \left(d(k)^\top, d(k+1)^\top, \dots, d(k+p-1)^\top\right)^\top, \ \tilde{d}(k) \in \tilde{\boldsymbol{D}}. \tag{19}$$

Now we will consider constraints (3), (4). It is clear that (4) leads to constraint (8). Using the expression in terms of $x(k)$ from (1) instead of $x(k+1)$ in (3) we get

$$\begin{aligned} \mathsf{E}\{x(k+1) \mid x(k)\} &= \mathsf{E}\{x(k) + Bu(k) + Cd(k) + Cw(k) \mid x(k)\} \\ &= x(k) + Bu(k) + Cd(k) \in \boldsymbol{X}. \end{aligned} \tag{20}$$

Under constraint (2), condition (20) turns into the next inclusion:

$$x(k) + Bu(k) + \boldsymbol{CD} \in \boldsymbol{X},$$

that is valid if and only if

$$x(k) + Bu(k) \in \boldsymbol{X} \ominus \boldsymbol{CD}.$$

Then for the observed state of the system $x(k)$ at each sampling time k the control $u(k)$ must satisfy

$$Bu(k) \in \boldsymbol{X} \ominus \boldsymbol{CD} - x(k),$$

which is consistent with (7).

We come to the problem of minimizing quadratic function (18) with interval data (19) subject to constraints (7), (8). To handle the interval data in (18) we convert the problem of interval quadratic programming into the following two-objective optimization problem:

$$\begin{gathered} \min_{\tilde{u}(k)} \underline{\mathcal{J}}(k+p|k) = \tilde{u}(k)^\top H \tilde{u}(k) + 2\underline{\mathcal{G}}(k)\tilde{u}(k), \\ \min_{\tilde{u}(k)} \overline{\mathcal{J}}(k+p|k) = \tilde{u}(k)^\top H \tilde{u}(k) + 2\overline{\mathcal{G}}(k)\tilde{u}(k), \\ \text{subject to (7), (8),} \end{gathered} \tag{21}$$

where the first objective function is the lower bound of interval quadratic function (18) over the interval $\tilde{\boldsymbol{D}}$, and the second is its upper bound. In (21)

$$\underline{\mathcal{G}}(k) = \left(\left(x(k) - x_0\right)^\top Q - \frac{1}{2}Q_1\right) BK + \underline{DF},$$
$$\overline{\mathcal{G}}(k) = \left(\left(x(k) - x_0\right)^\top Q - \frac{1}{2}Q_1\right) BK + \overline{DF},$$

and $\underline{DF}, \overline{DF}$ are the lower and upper bounds of the possible values of $\tilde{d}(k)^\top F$ over the interval $\tilde{\boldsymbol{D}}$.

According to the multi-objective optimization technique [9], problem (21) can be transformed into a quadratic programming problem with a single objective. Based on the scalarization method (the weighting objectives method), we obtain an equivalent compromise single objective optimisation problem where the objective is chosen as a weighted sum of the original criteria:

$$\min_{\tilde{u}(k)} Y(k+p|k) = \lambda_1 \underline{\mathcal{J}}(k+p|k) + \lambda_2 \overline{\mathcal{J}}(k+p|k)$$
$$\text{subject to (7), (8),}$$

where $\lambda_1, \lambda_2 \geq 0$ are the weighting coefficients that represent the relative importance of each criterion, $\lambda_1 + \lambda_2 = 1$. At various weights, we can express varies preferences to estimate the performance objective. For example, $\lambda_1 = 1$ means the optimistic estimate, $\lambda_2 = 1$ states the pessimistic estimate, $\lambda_1 = \lambda_2 = 0.5$ indicates the neutral estimate. It can be tuned manually until the controller reflects the desired behaviour. From our experience, if the demand values are more or less evenly distributed within its intervals, the equal weights give quite good results. Assuming equal weights in objective functions (21) we obtain the following result:

$$\begin{aligned} Y(k+p|k) &= \left(\underline{\mathcal{J}}(k+p|k) + \overline{\mathcal{J}}(k+p|k)\right)/2 \\ &= \tilde{u}(k)^\top H \tilde{u}(k) + 2\frac{1}{2}\left(\underline{\mathcal{G}}(k) + \overline{\mathcal{G}}(k)\right)\tilde{u}(k) = \tilde{u}(k)^\top H \tilde{u}(k) \\ &+ 2\Big(\big((x(k)-x_0)^\top Q - \frac{1}{2}Q_1\big)BK + \frac{1}{2}\left(\underline{DF} + \overline{DF}\right)\Big)\tilde{u}(k), \end{aligned}$$

that is consistent with (6) and (10). □

At this point, it is worth noting that, due to the interval uncertainty in the system, we can only steer the state to a tube sufficiently close to the target level x_0, and keep the state trajectory on average within the target tube. The target tube is a sequence of the sets that at each time contain all the states whose future trajectory can be kept inside the constraints, for all admissible disturbances [2]. It is clear, the width of this tube depends on the width of the initial uncertainty intervals. Indeed, the problem of keeping the state $x(k)$, on average, in some tube $\boldsymbol{X}(a,b) = [a,b]$ has a solution if and only if, for all $x(k) \in \boldsymbol{X}(a,b)$, there is a control $u(t) \in \boldsymbol{U}$ so that

$$\mathsf{E}\big\{x(k+1) \;\big|\; x(k)\big\} = x(k) + Bu(k) + Cd(k) \in \boldsymbol{X}(a,b)$$

is valid for all $d(k) \in \boldsymbol{D}$. That takes place if and only if

$$x(k) + Bu(k) + \boldsymbol{CD} \in \boldsymbol{X}(a,b),$$

and then

$$x(k) + Bu(k) \in \boldsymbol{X}(a,b) \ominus \boldsymbol{CD}.$$

It makes sense if and only if $\boldsymbol{X}(a,b) \ominus \boldsymbol{CD} \in \mathbb{IR}$, that is $a - \underline{CD} \leq b - \overline{CD}$. We can argue that $\overline{CD} - \underline{CD} \leq b - a$, and $\operatorname{wid} \boldsymbol{CD} \leq \operatorname{wid} \boldsymbol{X}(a,b)$. Therefore, the minimum width of the tube, within which on average the state $x(k)$ can be kept for all possible values of the demand, is given by

$$\operatorname{wid} \boldsymbol{CD} = \overline{CD} - \underline{CD}.$$

Evidently, under an excessive storage level any system must pay a high storage cost. But if the storage level is too low, the system will have a low service level due to the shortage, resulting in lost profits and loss of customer loyalty. To find the trade-off, we need to maintain the minimum level of storage without violating state constraints for all possible realizations of model uncertainty. This is why we suggest setting the target level at zero during the first simulation and waiting for the tube $\boldsymbol{X}(0, \operatorname{wid} \boldsymbol{CD})$ to be received. In this case, the control is obtained by pointing to the order-up-to-level in the sense that

$$x(k) + Bu(k) = -\underline{CD} \tag{22}$$

because of $\boldsymbol{X}(0, \operatorname{wid} \boldsymbol{CD}) \ominus \boldsymbol{CD} = -\underline{CD}$. Thus, the developed feedback control turns out to be a periodic review, order-up-to-level (R, S) strategy, where the review interval R is the unit of time, and the order-up-to-level S is equal to $-\underline{CD}$. If the levels of service in the network nodes are high enough, there is no need to raise the target level x_0. Otherwise, we can gradually increase the target level and form a safety stock until the required levels of service are received.

4 Numerical Problem

Now we will apply the results obtained in Section 3 to an example. Let us consider the fictional production-distribution system represented by Figure 1.

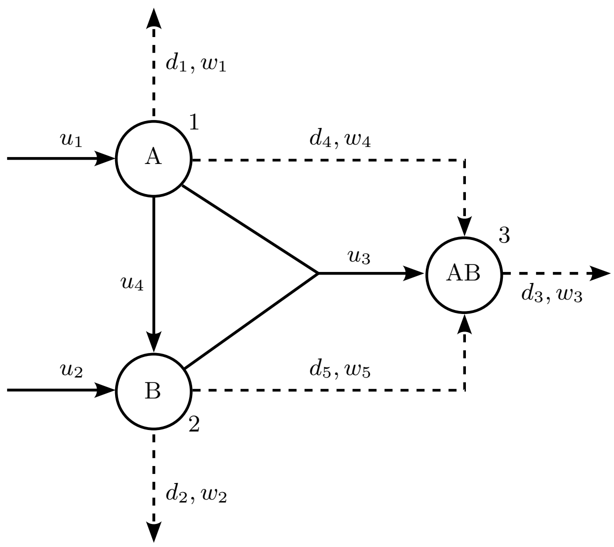

Figure 1: The network structure of a production-distribution system with three nodes and controllable (solid) and uncontrollable (dashed) flows between them

The system has three interdependent production-distribution centres, represented by three nodes. Nodes 1 and 2 make products A and B, these products are

used later for making product AB in node 3. The controllable flows $u_1,\ u_2$ describe the production levels of A in node 1 and B in node 2, respectively, per unit of time, u_3 describes a production line in node 3 which takes some amount of products A and B to produce the same amount AB in node 3. The arc u_4 models an additional flexible capacity present in the system which can be split in any proportion between two production lines A and B. If the arc u_4 works at full force, the flexible capacity is fully used to produce B, while if it works at zero force, the flexible capacity is fully used to produce A. The uncontrollable flows represent the demand in the network nodes that can arise from outside and other nodes. The arcs $d_1,\ d_2,\ d_3$ represent demands for products A, B and AB. And there are the redistribution arcs $d_4,\ d_5$ which represent demands that may unpredictably require A or AB, and B or AB, respectively.
The structural matrices B and C for the system have the form:

$$B = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix}.$$

The constraints in the states and controls are given as follows

$$\boldsymbol{X} = \left([0,130] \quad [0,120] \quad [0,150]\right)^\top,$$

$$\boldsymbol{U} = \left([0,170] \quad [0,50] \quad [0,100] \quad [0,70]\right)^\top.$$

The demand $d(k)$ takes values within the interval vector

$$\boldsymbol{D} = \left([5,25] \quad [20,30] \quad [60,80] \quad [0,20] \quad [0,10]\right)^\top.$$

This example is an adapted version of the example from [2]. The system contains the white noise $w(k)$ with a zero mean and the covariance matrix

$$W = \operatorname{diag}\left(\sigma_1^2, \sigma_2^2, \ldots, \sigma_l^2\right),\ \sigma_i^2 = 0.25 \operatorname{wid} \boldsymbol{D}_i.$$

We assume that the demand cannot be backlogged and that demands during stock-outs are completely lost. The initial storage level is $x(0) = (130\ \ 120\ \ 150)^\top$ and the target storage level is $x_0 = (0\ 0\ 0)^\top$. The weighting matrices are chosen as $Q = I_n, Q_1 = (1\ 1\ 1)^\top, R = I_m$, the prediction horizon is $p = 6$, the problem is solved for 100 time steps. We carried out modelling and simulation in MATLAB. The simulation results are presented in Figures 2, 3, 4, 5.

Figure 2 shows the time behaviour of demands in the network. Normally, they fluctuate inside the given intervals, but there are some peaks lying outside their lower and upper bounds. This is the influence of random disturbances that can cause the demand to leave the predicted interval. We take them into account only in the expected way, and this is reflected in customer service levels. But in our case, decrease in the service levels is insignificant. As the simulation showed, we received high levels of service in the network nodes. They are maintained at the

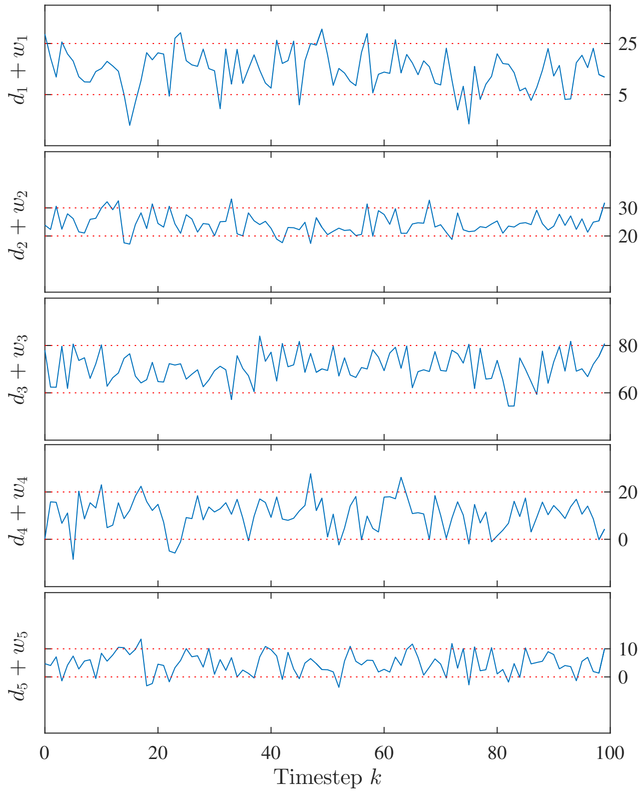

Figure 2: The dynamics of $d_i + w_i$ (solid blue) and the lower and upper bounds of the demand intervals $[\underline{D_i}, \overline{D_i}]$ (dashed red), $i = 1, \ldots, 5$

level of 98.72% in node 1, 99.98% in node 2, and 99.67% in node 3. In this case, there is no need to increase the target level x_0 to form a safety stock.

Figure 3 demonstrates the controls in the network. The average time required to compute the control actions within a time step using the `quadprog` function was about 0.005 seconds. It is worth noting that the arc u_2 works at full force. The flexible capacity is divided between the production lines A and B ($u_4 > 0$). This means that the constraint in u_2 is limiting.

Figure 4 presents the inventory dynamics in the network nodes under the optimal control strategy. In all the nodes, a decreasing trend of the storage levels can be observed. In our case, $\boldsymbol{CD} = ([-45, -5]\,[-40, -20]\,[-80, -30])^\top$ and $\operatorname{wid} \boldsymbol{CD} = (40\;\;20\;\;50)^\top$. Figure 4 shows that starting from some timestep, the state trajectory on average lies within the minimal tube $\boldsymbol{X}(0, \operatorname{wid} \boldsymbol{CD})$.

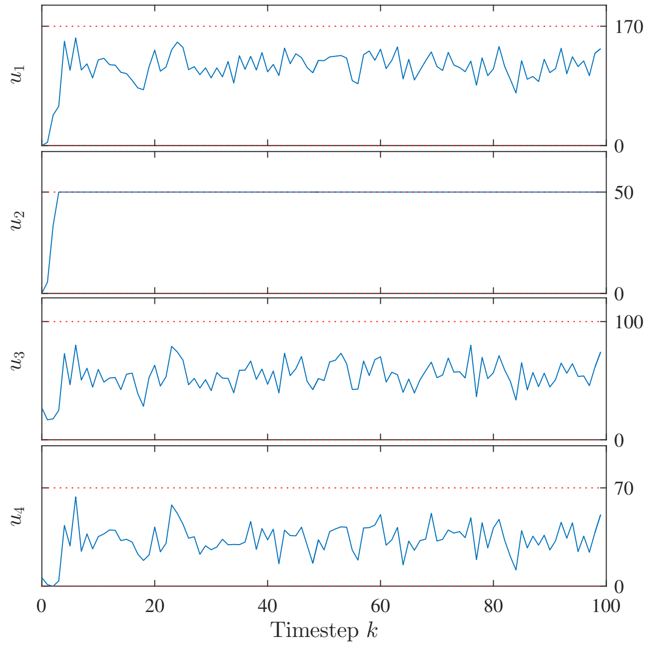

Figure 3: The trajectories of the controls u_1, u_2, u_3, u_4 (solid blue) and its constraints (dashed red)

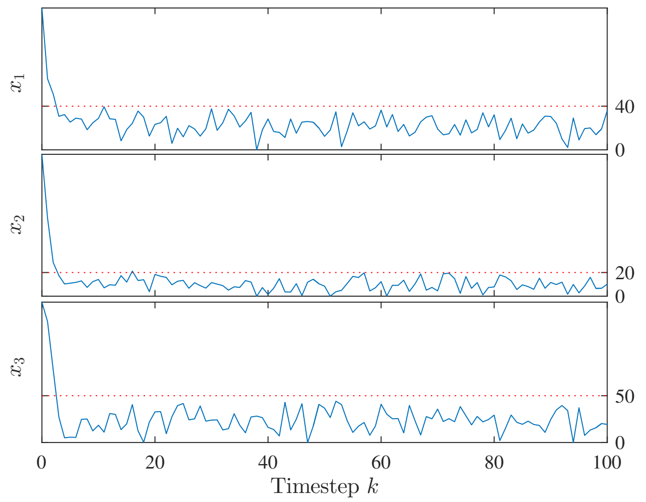

Figure 4: The trajectories of the states x_i (solid blue) and the levels wid $\boldsymbol{CD}_i$ (dashed red), $i = 1, 2, 3$

Figure 5 shows the order-up-to-levels which starting from some point in time are constant and equal to $-\underline{CD} = (45\ \ 40\ \ 80)^{\top}$. This fact is consistent with (22).

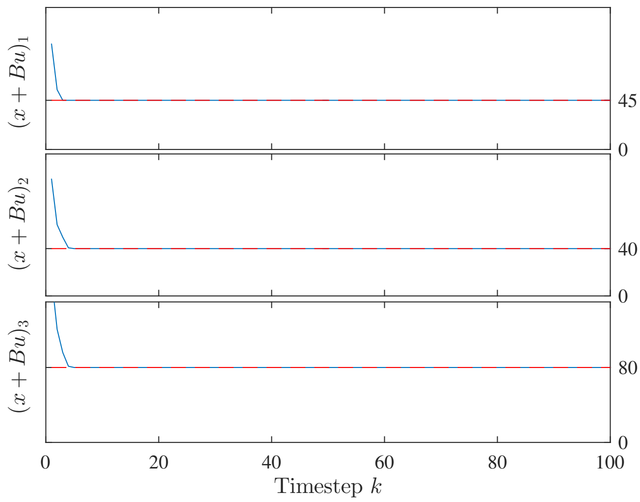

Figure 5: The order-up-to-levels (solid blue) and $-\underline{CD}$ (dashed red)

5 Conclusions and further research

In this study, we considered a supply chain network under interval and stochastic uncertainties. The mixed type of uncertainty is preferred in many cases since it is close to real life. We used the integrated approach to inventory control, with all the network nodes optimized simultaneously. We applied the MPC approach and reduced the problem to a constrained quadratic programming problem which can be solved using efficient techniques. As a result, we developed a feedback inventory control strategy with a high level of service.

However, there are still a number of issues that need to be addressed, such as the case of nonstationary demand, multiplicative noise, storage loss, and the conditions for the existence of controls to fulfill any values of possible demands under interval-stochastic uncertainty. These are the points of possible future research.

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