Asymptotically Minimal Interval Contractors Based on the Centered Form — Application to the Stability Analysis of Linear Time-Delayed Differential Equations

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Abstract

This paper proposes a new interval-based contractor for nonlinear equations which is minimal when dealing with narrow boxes. The method is based on the centered form classically used by interval algorithms combined with a Gauss Jordan band diagonalization preconditioning. As an illustration in stability analysis, we propose to compute the set of all parameters of a characteristic function of linear time-delayed equations which have at least one zero in the imaginary axis. Our approach is able compute a guaranteed and accurate enclosure of the solution set faster than existing approaches.

Keywords: interval analysis, contractor, centered form, stability

1 Introduction

Interval analysis is an efficient tool used for solving rigorously complex nonlinear problems involving bounded uncertainties [7, 20, 34]. Many interval algorithms are based on the notion of *interval contractor* [8] (or *contractor* for short) which is an operator which shrinks an axis-aligned box $[\mathbf{x}]$ of $\mathbb{R}^n$ without removing any point of the solution set $\mathbb{X}$. The set $\mathbb{X}$ is assumed to be defined by equations involving the components $x_1, \dots, x_n$ of a vector $\mathbf{x} \in \mathbb{R}^n$.

Combined with a paver [38] which bisects boxes, the contractor builds an outer approximation of the set $\mathbb{X}$. The resulting methodology can be applied in several domains of engineering such as identification [32], localization [21, 14], SLAM [29, 37], vision [11], reachability [13], control [3, 41], calibration [26], etc.

Centered form is one of the most fundamental brick in interval analysis. It is traditionally used to enclose the range of a function over narrow intervals [28, 30, 16]. The quadratic approximation property, guarantees an asymptotically small

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overestimation for sufficiently narrow boxes. Now, the centered form is only for the forward interval evaluation of a function. The backward propagation is not treated by the classical centered form. Now, this backward step is mandatory is we want to implement a propagation process. This is why we need to build an interval contractor which contains not only a forward interval evaluation, but also the backward propagation. In this paper, we propose to use the centered form to build efficient contractors [17] that are optimal when the intervals are narrow. To my knowledge, no other contractor with this asymptotic property exists in the literature.

To achieve this goal, we first get a guaranteed first order enclosure of each equation composing our problem using an interval linearization technique. Then, we combine these constraints preserving the first order approximation using interval linear techniques. More particularly, we propose to use a preconditioning method based on a Gauss-Jordan band diagonalization. We show that our approach is guaranteed to enclose all solutions of the problem and may outperform state of the art techniques on an example taken from the literature.

The main contribution of this paper is that the contractor we propose is asymptotically minimal, *i.e.*, it is minimal when the boxes are small. To the best of my knowledge, such a contractor does not exist in the literature even if some use a linear approximation (see the X-Taylor iteration [1] tested on global minimization problems, [6] which is similar to X-Taylor but for solving inequalities, the interval Newton [28] used for solving square nonlinear systems, or the affine arithmetic [12] which has been used for non-square systems but which is not asymptotically minimal).

Section 2 recalls some useful mathematical notions related to the sensitivity of the solution set of a linear system. Section 3 introduces wrappers to approximate accurately a function over a box. Section 4 defines what is an asymptotically minimal contractor and Section 5 gives an algorithm to generate it. The relevance and the efficiency of our approach are shown in Section 6 on the stability analysis of a linear differential equation with delays. Section 7 concludes the paper.

2 Preliminaries

This section recalls some basic definitions and theorems related to the sensitivity of the solution set of a linear system with respect to small perturbations. They will be used later in the paper to define the asymptotic minimality of our approximation for the solution set.

2.1 Proximity

Denote by $L(\mathbf{a}, \mathbf{b})$ the distance between $\mathbf{a}$ and $\mathbf{b}$ of $\mathbb{R}^n$ induced by the L-norm [5]. As illustrated by Figure 1, the *proximity* of $\mathbb{A}$ to $\mathbb{B}$, where $\mathbb{A}$ and $\mathbb{B}$ are closed subsets of $\mathbb{R}^n$, is defined by

$$h(\mathbb{A},\mathbb{B}) = \sup_{\mathbf{a}\in\mathbb{A}} L(\mathbf{a},\mathbb{B}) \tag{1}$$

where

$$L(\mathbf{a},\mathbb{B}) = \inf_{\mathbf{b}\in\mathbb{B}} L(\mathbf{a},\mathbf{b}). \tag{2}$$

The norm L that will be used later in the algorithm will be the L_∞ norm, even if, in the pictures, for a better visibility, we use the Euclidean L_2 norm.

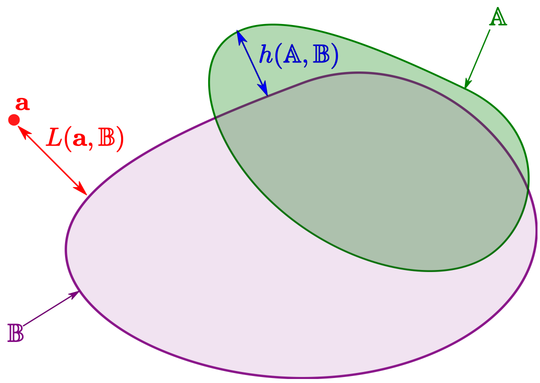

Figure 1: Proximity $h(\mathbb{A},\mathbb{B})$ of $\mathbb{A}$ to $\mathbb{B}$. If we inflate $\mathbb{B}$ by a coefficient of $h(\mathbb{A},\mathbb{B})$, then $\mathbb{B}$ will enclose $\mathbb{A}$

A nested sequence of closed subsets $\mathbb{B}(k) \subset \mathbb{R}^n$, $k \in \mathbb{N}$ is converging to $\mathbf{x}$ if

$$\lim_{k\to\infty} h(\mathbb{B}(k), \{\mathbf{x}\}) = 0. \tag{3}$$

2.2 Linear systems

Consider a system of linear equations of the form $\mathbf{A}\cdot\mathbf{x} = \mathbf{b}$ with more variables than unknows. Denote by $\mathbb{X}$ the solution set. This set can can be a point (if $\mathbf{A}$ is square), a line, a plane, or any affine space. Consider $\bar{\mathbf{x}} \in \mathbb{X}$. If we change just a little the entries for $\mathbf{A}$ and $\mathbf{b}$, the solution set $\mathbb{X}$ will move also. The point $\bar{\mathbf{x}}$ will then probably be outside $\mathbb{X}$, but still close to the new $\mathbb{X}$. The corresponding distance is $L(\bar{\mathbf{x}},\mathbb{X})$. The following proposition allows us to quantify the value for $L(\bar{\mathbf{x}},\mathbb{X})$ or equivalently to provide a sensitivity for the system $\mathbf{A}\cdot\mathbf{x} = \mathbf{b}$.

Proposition. *Consider a point* $\mathbf{x}$ *which satisfies the linear system* $\mathbf{A}\cdot\mathbf{x} = \mathbf{b}$*, where* $\mathbf{A}$ *has independent rows, i.e., which is full rank. Consider a small variation* $d\mathbf{A}$ *of* $\mathbf{A}$*. The quantity*

$$d\mathbf{x} = -\mathbf{A}^\dagger \cdot (d\mathbf{A}\cdot\mathbf{x} + d\mathbf{A}\cdot d\mathbf{x}), \tag{4}$$

where

$$\mathbf{A}^\dagger = \mathbf{A}^T(\mathbf{A}\cdot\mathbf{A}^T)^{-1} \tag{5}$$

is the generalized inverse of $\mathbf{A}$*, satisfies*

$$(\mathbf{A} + d\mathbf{A}) \cdot (\mathbf{x} + d\mathbf{x}) = \mathbf{b}. \tag{6}$$

This proposition tells us that if we move $\mathbf{A}$ a little, then, the solution set for the linear equation moves a little also, at order 1.

Proof. We have

$$\begin{array}{cl} & (\mathbf{A} + d\mathbf{A}) \cdot (\mathbf{x} + d\mathbf{x}) = \mathbf{b} \\ \Leftrightarrow & \mathbf{A} \cdot \mathbf{x} + \mathbf{A} \cdot d\mathbf{x} + d\mathbf{A} \cdot \mathbf{x} + d\mathbf{A} \cdot d\mathbf{x} = \mathbf{b} \end{array} \tag{7}$$

Thus

$$\mathbf{A} \cdot d\mathbf{x} + d\mathbf{A} \cdot \mathbf{x} + d\mathbf{A} \cdot d\mathbf{x} = \mathbf{0} \tag{8}$$

i.e.

$$\mathbf{A} \cdot d\mathbf{x} = -d\mathbf{A} \cdot \mathbf{x} - d\mathbf{A} \cdot d\mathbf{x} \tag{9}$$

Since $\mathbf{A}$ has independent lines, the solution which minimizes $\|d\mathbf{x}\|$ is

$$d\mathbf{x} = \mathbf{A}^{\dagger} \cdot (-d\mathbf{A} \cdot \mathbf{x} - d\mathbf{A} \cdot d\mathbf{x}). \tag{10}$$

□

Corollary. *Consider the hyperplane*

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{A} \cdot \mathbf{x} = \mathbf{0}\}, \tag{11}$$

where $\mathbf{A}$ *has independent lines. Consider a small variation* $d\mathbf{A}$ *of* $\mathbf{A}$ *with* $\|d\mathbf{A}\| = O(\varepsilon)$ *where* ε *is small. Take a point* $d\mathbf{x} \in \mathcal{P}$ *with* $\|d\mathbf{x}\| = O(\varepsilon)$*. The distance from* $d\mathbf{x}$ *to* $\tilde{\mathcal{P}} = \{\mathbf{x} \in \mathbb{R}^n | (\mathbf{A} + d\mathbf{A}) \cdot \mathbf{x} = \mathbf{0}\}$ *is* $o(\varepsilon)$*, i.e.,* $O(\varepsilon^2)$*.*

Proof. Denote by $\hat{\mathbf{p}}$ the projection of a point $\mathbf{p} \in \mathcal{P}$ on $\tilde{\mathcal{P}}$. From Proposition 2.2, we have

$$\|\hat{\mathbf{p}} - \mathbf{p}\| = O(\varepsilon). \tag{12}$$

If we take $\mathbf{p} = d\mathbf{x}$. We get

$$\|d\hat{\mathbf{x}} - d\mathbf{x}\| = o(\varepsilon) = O(\varepsilon^2) \tag{13}$$

as illustrated by Figure 2. □

3 Wrappers

The approximation of sets using boxes computed using interval analysis generates a strong wrapping effect. It has been shown by several authors that it was possible to get a linear approximation with a better accuracy using other types of sets such as zonotopes [9, 10], constrained zonotopes [39, 35], ellipsoids [33], or doubleton [19]. Before defining the notion of wrapper to quantify the order of approximation we can get, we first recall what is a contractor.

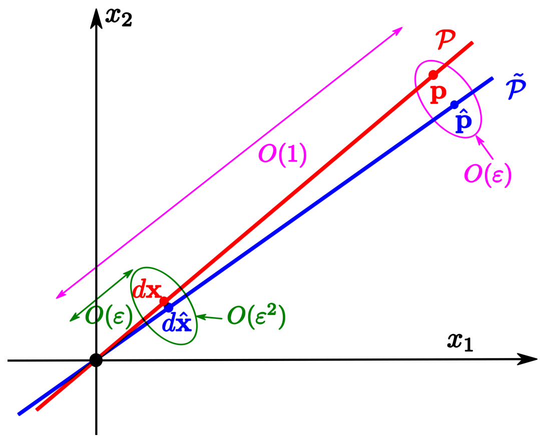

Figure 2: If we move the plane $\mathcal{P}$ of an order ε, a point $\mathbf{p}$ of the plane $\mathcal{P}$ will be at a distance to the new plane $\tilde{\mathcal{P}}$ of an order ε. If we do the same operation with a vector $d\mathbf{x}$ with a norm of order ε, then the distance of $d\mathbf{x}$ to $\tilde{\mathcal{P}}$ is an order ε^2.

Definition. *Denote by* $\mathbb{IR}^n$ *the set of boxes of* $\mathbb{R}^n$*. A* contractor *associated to the closed set* $\mathbb{X} \subset \mathbb{R}^n$ *is a function* $\mathcal{C} : \mathbb{IR}^n \mapsto \mathbb{IR}^n$ *such that*

$$\begin{array}{ll} \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contraction)} \\ [\mathbf{x}] \cap \mathbb{X} \subset \mathcal{C}([\mathbf{x}]) & \text{(consistency)} \end{array}$$

The contractor $\mathcal{C}$ *for* $\mathbb{X}$ *is* minimal *if* $\mathcal{C}([\mathbf{x}]) = [\![[\mathbf{x}] \cap \mathbb{X}]\!]$ *where* $[\![\mathbb{A}]\!]$ *denotes the smallest box enclosing the set* $\mathbb{A}$*.*

The following definition of a *wrapper* extends the concept of contractor and will be needed for convergence analysis.

Definition. *A* wrapper *associated to the closed set* $\mathbb{X} \subset \mathbb{R}^n$ *is a function* $\mathcal{W}$: $\mathbb{IR}^n \mapsto \mathcal{P}(\mathbb{R}^n)$ *such that*

$$\begin{array}{ll} \mathcal{W}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contraction)} \\ [\mathbf{x}] \cap \mathbb{X} \subset \mathcal{W}([\mathbf{x}]) & \text{(consistency)} \\ \mathbf{x} \notin \mathbb{X} \Rightarrow \exists \varepsilon, \forall [\mathbf{x}] \subset B(\mathbf{x}, \varepsilon), \mathcal{W}([\mathbf{x}]) = \emptyset & \text{(accuracy)} \end{array}$$

where $\mathcal{P}(\mathbb{R}^n)$ *is the set of all subsets of* $\mathbb{R}^n$ *and* $B(\mathbf{x}, \varepsilon)$ *is the box with center* $\mathbf{x}$ *and radius* ε*.*

An illustration of a wrapper is given by Figure 3. The set $\mathbb{X}$ is a curve which could be given by an equation. For the box $[\mathbf{a}]$, the set $\mathcal{W}([\mathbf{a}])$ encloses the part of $\mathbb{X}$ which is inside $[\mathbf{a}]$. The accuracy property is illustrated by the box $[\mathbf{b}]$, which satisfies $\mathcal{W}([\mathbf{b}]) = \emptyset$. The box $[\mathbf{b}]$ is inside the box $B(\mathbf{b}, \varepsilon)$ with $\mathbf{b} \notin \mathbb{X}$. This translates the fact if a box $[\mathbf{b}]$ is outside $\mathbb{X}$ and sufficiently small then the wrapper will be able conclude that it is indeed outside $\mathbb{X}$.

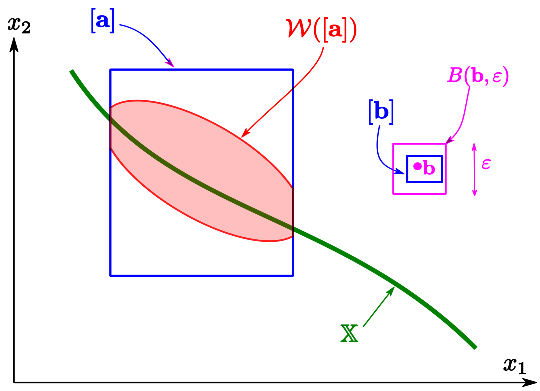

Figure 3: Illustration of a wrapper. $\mathcal{W}([\mathbf{b}])$ is empty which means that $[\mathbf{b}] \cap \mathbb{X} = \emptyset$.

The wrapper $\mathcal{W}$ for $\mathbb{X}$ has an order i at point $\mathbf{x}$ if for all nested sequences of boxes $[\mathbf{x}](k)$ converging to $\mathbf{x}$, we have

$$\lim_{k \to \infty} \frac{h(\mathcal{W}([\mathbf{x}](k)), \mathbb{X})}{(w([\mathbf{x}](k)))^i} = 0 \tag{14}$$

where $w([\mathbf{x}])$ is the width of $[\mathbf{x}]$. In this paper, only the order one will be considered. Denote by $\text{Wrap}(\mathbb{X}, \mathbf{x})$ the set of all wrappers for $\mathbb{X}$ which have an order 1 at point $\mathbf{x}$.

The notion of order is illustrated by Figure 4. The larger is k, the narrower is $[\mathbf{x}](k)$ and more accurate is the approximation.

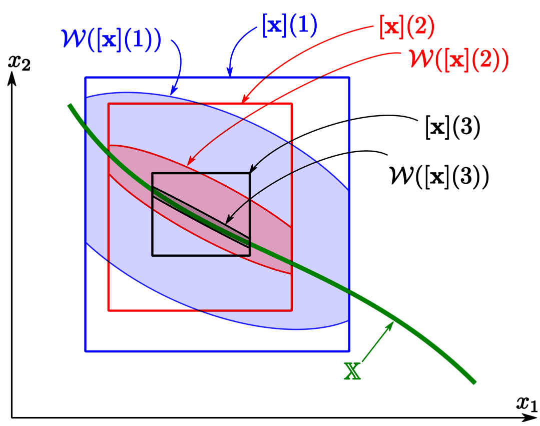

Figure 4: Wrapper of order 1. This wrapper generates a set which fits the shape of the set $\mathbb{X} \cap [\mathbf{x}]$.

Definition. *We define the intersection* $\mathcal{W}$ *of two wrappers* $\mathcal{W}_1$ *and* $\mathcal{W}_2$ *as*

$$\mathcal{W}([\mathbf{x}]) = (\mathcal{W}_1 \cap \mathcal{W}_2)([\mathbf{x}]) = \mathcal{W}_1([\mathbf{x}]) \cap \mathcal{W}_2([\mathbf{x}]). \tag{15}$$

It is trivial to check that if $\mathcal{W}_1$ *is a wrapper for* $\mathbb{X}_1$ *and* $\mathcal{W}_2$ *is a wrapper for* $\mathbb{X}_2$ *then* $\mathcal{W} = \mathcal{W}_1 \cap \mathcal{W}_2$ *is a wrapper for* $\mathbb{X}_1 \cap \mathbb{X}_2$*. Unfortunately, the order of the approximation is not always preserved. The following proposition gives some conditions which allows us to preserve the order 1.*

Proposition. *Given* m *sets* $\mathbb{X}_i = \{\mathbf{x} \in \mathbb{R}^n | f_i(\mathbf{x}) = 0\}$*, where* $f_i : \mathbb{R}^n \mapsto \mathbb{R}$*. Consider* $\mathbb{Z} = \bigcap_i \mathbb{X}_i$ *and a point* $\mathbf{z} \in \mathbb{Z}$. Assume that all $\frac{df_i}{d\mathbf{x}}(\mathbf{z})$ are independent. *If* $\mathcal{W} = \bigcap_i \mathcal{W}_i$*, we have*

$$\forall i, \mathcal{W}_i \in \mathit{Wrap}(\mathbb{X}_i, \mathbf{z}) \Rightarrow \textstyle\bigcap_i \mathcal{W}_i \in \mathit{Wrap}(\mathbb{Z}, \mathbf{z}) \tag{16}$$

Figure 5 illustrates that the intersection of two wrappers of order 1 at $\mathbf{z}$ is generally a wrapper of order 1 at $\mathbf{z}$. In the figure, the set $\mathbb{Z} = \mathbb{X}_1 \cap \mathbb{X}_2$ is the singleton $\{\mathbf{z}\}$. The box $[\mathbf{x}]$ should be interpreted as a narrow box containing $\mathbf{z}$.

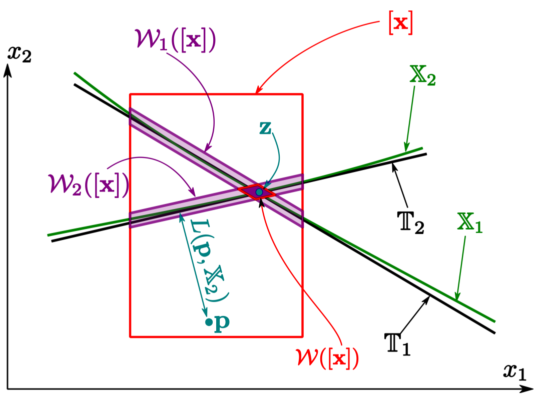

Figure 5: The intersection of two wrappers $\mathcal{W}_1$ and $\mathcal{W}_2$ of order 1 (here red) is a wrapper $\mathcal{W}$ of order 1 for the intersection of the two corresponding sets $\mathbb{X}_1$ and $\mathbb{X}_2$.

Proof. Since $\mathbb{Z} = \bigcap_i \mathbb{X}_i$, $\mathcal{W} = \bigcap_i \mathcal{W}_i$ is a wrapper for $\mathbb{Z}$. We also need to prove that the order of $\mathcal{W}$ is 1 at $\mathbf{z}$. For this, consider a sequence $[\mathbf{x}](k)$ converging to $\mathbf{z}$. When k is large $\varepsilon = w([\mathbf{x}](k))$ is small. For short, let us omit the dependency with respect to k. For all $\mathbf{p} \in [\mathbf{x}]$, we have $\|\mathbf{p} - \mathbf{z}\| = O(\varepsilon)$. If $\mathbb{T}_i$ is the tangent space of $\mathbb{X}_i$ at point $\mathbf{z}$ then

$$L(\mathbf{p}, \mathbb{X}_i) = L(\mathbf{p}, \mathbb{T}_i) + o(\varepsilon). \tag{17}$$

If all $\mathbb{T}_i$ are transverse, we have

$$L(\mathbf{p}, \mathbb{Z}) = L(\mathbf{p}, \textstyle\bigcap_i \mathbb{X}_i) = L(\mathbf{p}, \bigcap_i \mathbb{T}_i) + o(\varepsilon). \tag{18}$$

Take now, $\mathbf{p} \in \mathcal{W}([\mathbf{x}])$. Since $\forall i, L(\mathbf{p}, \mathbb{T}_i) = o(\varepsilon)$ and since the $\mathbb{T}_i$ are transverse, we get that $L(\mathbf{p}, \bigcap_i \mathbb{T}_i) = o(\varepsilon)$. Therefore, from (18), $L(\mathbf{p}, \mathbb{Z}) = o(\varepsilon)$. Since this is true for all $\mathbf{p} \in \mathcal{W}([\mathbf{x}])$, we have

$$h(\mathcal{W}([\mathbf{x}]), \mathbb{Z}) = \sup_{\mathbf{p} \in \mathcal{W}([\mathbf{x}])} L(\mathbf{p}, \mathbb{Z}) = o(\varepsilon) = o(w([\mathbf{x}])). \tag{19}$$

Taking into account the dependency of $[\mathbf{x}]$ in k, we get:

$$\lim_{k \to \infty} \frac{h(\mathcal{W}([\mathbf{x}](k)), \mathbb{Z})}{w([\mathbf{x}](k))} = 0, \tag{20}$$

which proves that $\mathcal{W}$ has an order 1 at point $\mathbf{z}$. □

4 Asymptotically minimal contractor

Consider the special case where wrappers, as defined by Definition 3, generate sets $\mathcal{W}([\mathbf{x}])$ that are boxes of $\mathbb{R}^n$. The order cannot be equal to 1 (it can only be equal to 0), except if $n = 1$. Now, we can use the wrappers of order 1 (which return a set which is not a box, a zonotope, for instance), as an intermediate result, to get contractors with a good accuracy. For this, we will have to compute the smallest possible box which encloses this non-box intermediate approximation.

This section formally defines such accurate contractors which are called *asymptotically minimal*.

Definition. *A contractor for* $\mathbb{X}$ *is* asymptotically minimal *at point* $\mathbf{z} \in \mathbb{X} \subset \mathbb{R}^n$ *if for any nested sequence* $[\mathbf{x}](k)$ *converging to* $\mathbf{z}$*, we have*

$$\lim_{k \to \infty} \frac{h(\mathcal{C}([\mathbf{x}](k)), [\![[\mathbf{x}](k) \cap \mathbb{X}]\!])}{w([\mathbf{x}](k))} = 0. \tag{21}$$

Note that since $\mathcal{C}$ is a contractor the quantity $\mathcal{C}([\mathbf{x}](k))$ is a box.

Proposition. *If* $\mathcal{W} \in$ *Wrap*$(\mathbb{X}, \mathbf{z})$*, then, the contractor defined by*

$$\mathcal{C}([\mathbf{x}]) = [\![\mathcal{W}([\mathbf{x}])]\!] \tag{22}$$

is an asymptotically minimal contractor for $\mathbb{X}$ *at* $\mathbf{z}$*.*

An illustration of the proposition is given by Figure 6. The gray part corresponds to the pessimism of the contractor which tends to disappear when $[\mathbf{x}]$ becomes narrow.

Proof. The proof is by contradiction. Assume that $\mathcal{C}([\mathbf{x}]) = [\![\mathcal{W}([\mathbf{x}])]\!]$ is not asymptotically minimal in $\mathbf{z}$. From (21), there exists a sequence of nested boxes such converging to $\mathbf{z}$ such that

$$\lim_{k \to \infty} \frac{h([\![\mathcal{W}([\mathbf{x}])(k)]\!], [\![[\mathbf{x}](k) \cap \mathbb{X}]\!])}{w([\mathbf{x}](k))} > 0. \tag{23}$$

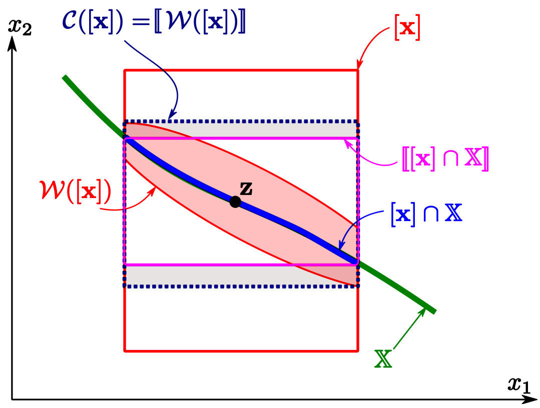

Figure 6: Asymptotic minimal contractor $\mathcal{C}([\mathbf{x}])$. It first computes the set $\mathcal{W}([\mathbf{x}])$ and then encloses in the box $[\![\mathcal{W}([\mathbf{x}])]\!]$.

Since for all $\mathbb{A} \subset \mathbb{R}^n$, and for all box $[\mathbf{b}]$, we have $h([\![\mathbb{A}]\!], [\mathbf{b}]) = h(\mathbb{A}, [\mathbf{b}])$, we have

$$\lim_{k\to\infty} \frac{h(\mathcal{W}([\mathbf{x}])(k), [\![[\mathbf{x}](k) \cap \mathbb{X}]\!])}{w([\mathbf{x}](k))} > 0. \qquad (24)$$

Moreover, since h is monotonic decreasing with respect to its second argument, we get

$$\lim_{k\to\infty} \frac{h(\mathcal{W}([\mathbf{x}])(k), [\mathbf{x}](k) \cap \mathbb{X})}{w([\mathbf{x}](k))} > 0.$$

Since the sequence $[\mathbf{x}](k)$ converges to $\mathbf{z}$, if k is sufficiently large, we have $h(\mathcal{W}([\mathbf{x}])(k), [\mathbf{x}](k) \cap \mathbb{X}) = h(\mathcal{W}([\mathbf{x}])(k), \mathbb{X})$. As a consequence,

$$\lim_{k\to\infty} \frac{h(\mathcal{W}([\mathbf{x}](k)), \mathbb{X})}{w([\mathbf{x}](k))} > 0. \qquad (25)$$

This is inconsistent with the fact that $\mathcal{W}$ has an order 1 in $\mathbf{z}$ (see (14)). □

5 Centered contractor

In this section, we show how to build an asymptotic minimal contractor using the centered form. We will consider functions $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^p$ which are all continuous and differentiable. More precisely, the function $\mathbf{f}$ is described by continuous operator of functions such as $+, -, /, \sin, \exp, \ldots$ As a consequence using interval analysis, we are able to enclose the range of $\mathbf{f}$ and of $\frac{d\mathbf{f}}{d\mathbf{x}}$ over a box $[\mathbf{x}]$. In [28], Moore has proved that if $w([\mathbf{x}]) = O(\varepsilon)$ then using interval computation, we get an enclosure $[\mathbf{f}]([\mathbf{x}])$ for $\mathbf{f}([\mathbf{x}])$ and an enclosure $[\frac{d\mathbf{f}}{d\mathbf{x}}]([\mathbf{x}])$ for $\frac{d\mathbf{f}}{d\mathbf{x}}([\mathbf{x}])$ such that $w([\mathbf{f}]([\mathbf{x}])) = O(\varepsilon)$ and $w(\frac{d\mathbf{f}}{d\mathbf{x}}([\mathbf{x}])) = O(\varepsilon)$.

5.1 Scalar case

Proposition. *Consider the equation $f(\mathbf{x}) = \mathbf{0}$, where $f : \mathbb{R}^n \mapsto \mathbb{R}$ is differentiable. The solution set is*

$$\mathbb{X} \quad = \quad \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) = \mathbf{0}\}. \tag{26}$$

Consider a point $\mathbf{z}$ *such that* $f(\mathbf{z}) = 0$. *Consider a nested sequence* $[\mathbf{x}](k)$ *converging to* $\mathbf{z}$. The function $\mathcal{L} : \mathbb{IR}^n \mapsto \mathcal{P}(\mathbb{R}^n)$ defined as

$$\begin{aligned} \mathcal{L}([\mathbf{x}]) \quad = \{ \quad & \mathbf{x} \in [\mathbf{x}] \mid \exists \mathbf{a} \in [\tfrac{df}{d\mathbf{x}}]([\mathbf{x}]), \\ & f(\mathbf{m}) + \mathbf{a} \cdot (\mathbf{x} - \mathbf{m}) = 0\} \end{aligned} \tag{27}$$

where $\mathbf{m} = \text{center}([\mathbf{x}])$, is a wrapper of order 1, *i.e., it belongs to Wrap*$(\mathbb{X}, \mathbf{z})$. *It will be called the centered wrapper associated with* f.

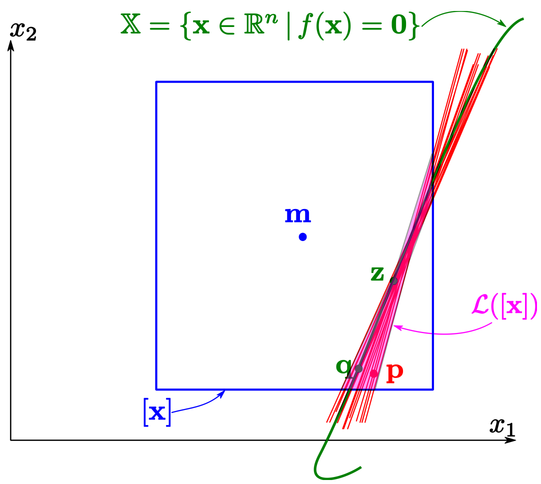

Figure 7: The set $\mathcal{L}([\mathbf{x}])$ (magenta) with a bowtie shape is close to the set $\mathbb{X}$ (here the curve in green). Moreover, $\mathcal{L}([\mathbf{x}])$ encloses $[\mathbf{x}] \cap \mathbb{X}$. The approximation is asymptotically perfect.

Proof. Consider the sequence $[\mathbf{x}](k) \subset \mathbb{R}^n$ converging to $\mathbf{z}$. We assume that $[\mathbf{x}](k)$, or $[\mathbf{x}]$ for short, is narrow, *i.e.*, $w([\mathbf{x}]) = O(\varepsilon)$. If $\mathbf{p} \in \mathcal{L}([\mathbf{x}])$ (see Figure 7) then, for some $\mathbf{a} \in [\mathbf{a}] = [\frac{df}{d\mathbf{x}}]([\mathbf{x}])$, we have

$$f(\mathbf{m}) + \mathbf{a} \cdot (\mathbf{p} - \mathbf{m}) = 0 \tag{28}$$

where $\mathbf{m} = \text{center}([\mathbf{x}])$. From Corollary 2.2, taking $d\mathbf{x} = \mathbf{p} - \mathbf{m} = O(\varepsilon)$ and since $w([\mathbf{a}]) = O(\varepsilon)$, we get that the distance between a point in $\mathcal{L}([\mathbf{x}])$ and the set $\mathbb{X}$ is an $o(\varepsilon)$. We get that

$$h(\mathcal{L}([\mathbf{x}](k)), \mathbb{X}) = o(w([\mathbf{x}](k))) \tag{29}$$

i.e.,

$$\lim_{k\to\infty} \frac{h(\mathcal{L}([\mathbf{x}](k)),\mathbb{X})}{w([\mathbf{x}](k))} = 0. \tag{30}$$

Thus the wrapper $\mathcal{L}$ is of order 1 at $\mathbf{z}$. □

Corollary. *The contractor for $f(\mathbf{x}) = 0$ defined by*

$$\begin{array}{lcl} [x_i] & = & [x_i] \cap \left(m_i - \frac{1}{[a_i]} \left(f(\mathbf{m}) + \sum_{j\neq i} [a_j] \cdot ([x_j] - m_j) \right) \right) \\ [a_j] & = & [\frac{\partial f}{\partial x_j}]([\mathbf{x}]) \end{array} \tag{31}$$

is asymptotically minimal.

Remark. Before starting the proof, it is important to recall an important notion on interval propagation. Consider an equation of the form

$$1 + a_1(x_1 - 2) + a_2(x_2 - 3) = 0,$$

with $a_1 \in [a_1], a_2 \in [a_2], x_1 \in [x_1], x_2 \in [x_2]$. The smallest box $[\mathbf{y}] = [y_1] \times [y_2]$ which encloses the set

$$\{(x_1, x_2) \in [\mathbf{x}] \,|\, \exists a_1 \in [a_1], \exists a_2 \in [a_2],\, 1 + a_1(x_1 - 2) + a_2(x_2 - 3) = 0\}$$

where $[\mathbf{x}] = [x_1] \times [x_2]$, is defined by

$$\begin{array}{ll} [y_1] & = [x_1] \cap \left(2 - \frac{1}{[a_1]} \left(1 + [a_2]([x_2] - 3)\right)\right) \\ [y_2] & = [x_2] \cap \left(3 - \frac{1}{[a_2]} \left(1 + [a_1]([x_1] - 2)\right)\right) \end{array}$$

This corresponds to a forward-backward contraction in our special case. As shown in [27], $[\mathbf{y}]$ is indeed the smallest because both x_1 and x_2 occur only once in the equation $1 + a_1(x_1 - 2) + a_2(x_2 - 3) = 0$. It is related to what Moore calls the dependency problem [28]. When we have more than one equation, such as for instance,

$$\begin{array}{lcl} 1 + a_{11}(x_1 - 2) + a_{12}(x_2 - 3) & = & 0 \\ 1 + a_{21}(x_1 - 2) + a_{22}(x_2 - 3) & = & 0 \end{array}$$

the forward-backward contraction will not yield the minimal contraction. This is due to the fact that in the system of two equations, x_1 and x_2 occur twice and not once.

Proof. Define $\mathcal{L}([\mathbf{x}])$ as in (27). From Proposition 4, $\mathcal{L} \in \text{Wrap}(\mathbb{X}, \mathbf{z})$. The contractor $\mathcal{C}([\mathbf{x}]) = [\![\mathcal{L}([\mathbf{x}])]\!]$ is an asymptotically minimal contractor. Now the set $\mathcal{L}([\mathbf{x}])$ can be defined as the set of all $\mathbf{x}$ which satisfy the following constraint

$$\left\{ \begin{array}{c} f(\mathbf{m}) + \mathbf{a} \cdot (\mathbf{x} - \mathbf{m}) = 0 \\ \text{with } \mathbf{a} \in [\frac{df}{d\mathbf{x}}]([\mathbf{x}]) \\ \text{and } \mathbf{m} = \text{center}([\mathbf{x}]) \end{array} \right. \tag{32}$$

Since $\mathbf{x}$ occurs only once in the constraint $f(\mathbf{m}) + \mathbf{a} \cdot (\mathbf{x} - \mathbf{m}) = 0$, an interval forward-backward propagation provides us the minimal contraction [27], *i.e.*, it returns the box $[\![\mathcal{L}([\mathbf{x}])]\!]$. □

5.2 Vector case

Proposition. *Consider the equation* $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, *where* $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^p$ *is differentiable. The solution set is*

$$\mathbb{X} \quad = \quad \{\mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{f}(\mathbf{x}) = \mathbf{0}\}. \tag{33}$$

Consider a point $\mathbf{z}$ *such that* $\mathbf{f}(\mathbf{z}) = \mathbf{0}$ *and a nested sequence* $[\mathbf{x}](k)$ *converging to* $\mathbf{z}$. Assume that all $\frac{df_i}{d\mathbf{x}}(\mathbf{z})$ are independent. *Consider the wrappers* $\mathcal{L}_i : \mathbb{IR}^n \mapsto \mathcal{P}(\mathbb{R}^n)$ *of order 1 for* $f_i(\mathbf{x}) = 0$ *defined by*

$$\mathcal{L}_i([\mathbf{x}]) \quad = \{ \quad \mathbf{x} \in [\mathbf{x}] \,|\, \exists \mathbf{a} \in [\tfrac{df_i}{d\mathbf{x}}]([\mathbf{x}]), f_i(\mathbf{m}) + \mathbf{a} \cdot (\mathbf{x} - \mathbf{m}) = 0\} \tag{34}$$

where $\mathbf{m} =$ *center*$([\mathbf{x}])$. *The operator* $\bigcap_i \mathcal{L}_i$, *belongs to Wrap*$(\mathbb{X}, \mathbf{z})$.

Proof. We have

$$\mathbb{X} = \underbrace{\{\mathbf{x} \in \mathbb{R}^n \,|\, f_1(\mathbf{x}) = 0\}}_{\mathbb{X}_1} \cap \cdots \cap \underbrace{\{\mathbf{x} \in \mathbb{R}^n \,|\, f_p(\mathbf{x}) = 0\}}_{\mathbb{X}_p}.$$

Now, from Proposition 5.1, the $\mathcal{L}_i([\mathbf{x}])$, as defined by 34, belong to Wrap$(\mathbb{X}_i, \mathbf{z})$. From Proposition 3, we get that $\bigcap_i \mathcal{L}_i$ belongs to Wrap$(\mathbb{X}, \mathbf{z})$. □

To compute $\bigcap_i \mathcal{L}_i$, the method proposed for the scalar case is not valid anymore. An interval linear method could be used [31, 1] that are based on an interval version of the simplex algorithms. Now, these methods are not proved to be minimal or asymptotically minimal, which may ruin our objective to get an asymptotically minimal contractor. An other possibility is to use a preconditioning method based on the Gauss-Jordan decomposition, which will be minimal in many cases, such as the test-case that will be treated in Section 6.

5.3 Preconditioning

Consider the equation $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, where $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^p$ is differentiable. Intersecting sets $\mathcal{L}_i([\mathbf{x}])$ as suggested by Proposition 5.2 requires the resolution of interval linear equations. This operation is costly and should be avoided if it has to be repeated a large number of times. Instead of this, we prefer to use a specific preconditioning method.

To understand the principle of the preconditioning, consider the following interval linear system

$$\begin{pmatrix} d_{11} & d_{12} & 0 \\ 0 & d_{22} & d_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{35}$$

where

$$d_{ij} \in [d_{ij}], x_j \in [x_j], b_i \in [b_i] \tag{36}$$

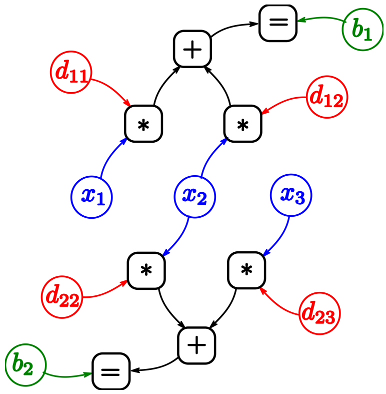

Figure 8: The constraint network has no cycle (it is a tree). Thus the interval propagation is minimal.

The optimal contraction can be obtained by a simple interval propagation. This is due to the fact that the corresponding constraint network has no cycle [27], as illustrated by Figure 8.

Note that no cycle would have been obtained with the following linear system:

$$\begin{pmatrix} d_{11} & d_{12} & 0 & 0 \\ 0 & d_{22} & d_{23} & 0 \\ 0 & 0 & d_{33} & d_{34} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{37}$$

A matrix $\mathbf{D}$ such that the system $\mathbf{D} \cdot \mathbf{x} = \mathbf{b}$ has no cycle can be called a *tree matrix*.

Both systems (35) and (37), for which the matrix $\mathbf{D}$ is a *band* matrix [2], could be obtained from a Gauss Jordan transformation of a linear systems [22]. For instance, if we have a system of the form $\mathbf{Ax} = \mathbf{c}$ where $\mathbf{A}$ is of dimension 3×4 with full rank, there exists a matrix $\mathbf{Q}$ of dimension 3×3 such that

$$\mathbf{Ax} = \mathbf{c} \Leftrightarrow \mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{Q} \cdot \mathbf{c} \tag{38}$$

where $\mathbf{D} = \mathbf{Q} \cdot \mathbf{A}$ has the form given by (37).

Proposition. *Consider a set* $\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{f}(\mathbf{x}) = \mathbf{0}\}$. *Take a narrow box* $[\mathbf{x}]$ *with center* $\mathbf{m}$. *Assume that* $\frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{m})$ *is a tree matrix. An interval propagation on the system*

$$\begin{array}{c} \mathbf{f}(\mathbf{m}) + \mathbf{A} \cdot (\mathbf{x} - \mathbf{m}) = \mathbf{0} \\ \text{with } \mathbf{A} \in [\frac{d\mathbf{f}}{d\mathbf{x}}]([\mathbf{x}]) \\ \text{and } \mathbf{x} \in [\mathbf{x}] \end{array} \tag{39}$$

corresponds to an asymptotically minimal contractor for $\mathbb{X}$.

Proof. The interval matrix $[\mathbf{A}] = [\frac{d\mathbf{f}}{d\mathbf{x}}]([\mathbf{x}])$ is such that $w([\mathbf{A}]) = O(\varepsilon)$, where $\varepsilon = w([\mathbf{x}])$. Now, Proposition 2.2 tells us that if we move $\mathbf{A}$ a little (at order 0), then, the solution set for the linear equation moves a little also, at order 1. Due to the fact that the contractor $\mathcal{C}$ resulting from the interval propagation is minimal for $\mathbf{A} = \frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{m})$, we get that the contractor obtained by an elementary interval propagation is asymptotically minimal. □

Corollary. *Consider a set* $\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{f}(\mathbf{x}) = \mathbf{0}\}$. *Take a narrow box* $[\mathbf{x}]$ *with center* $\mathbf{m}$. *Define* $\mathbf{Q}$ *such that* $\mathbf{Q} \cdot \frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{m})$ *is a tree matrix. An interval propagation on the system*

$$\begin{array}{c} \mathbf{Q} \cdot \mathbf{f}(\mathbf{m}) + \mathbf{Q} \cdot \mathbf{A} \cdot (\mathbf{x} - \mathbf{m}) = \mathbf{0} \\ \text{with } \mathbf{A} \in [\frac{d\mathbf{f}}{d\mathbf{x}}]([\mathbf{x}]) \\ \text{and } \mathbf{x} \in [\mathbf{x}] \end{array} \quad (40)$$

corresponds to an asymptotically minimal contractor for $\mathbb{X}$.

Proof. It suffices to apply Proposition 5.3 where $\mathbf{f}(\mathbf{x})$ should be replaced by $\mathbf{Q} \cdot \mathbf{f}(\mathbf{x})$. □

5.4 Algorithm

Consider the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ and take a box $[\mathbf{x}]$. We assume that we have an analytical expression for $\mathbf{f}$, so that we have an inclusion function for $\mathbf{f}$ and its Jacobian matrix $\frac{d\mathbf{f}}{d\mathbf{x}}$. The following algorithm corresponds to a centered contractor.

Input:	$\mathbf{f}$,$[\mathbf{x}]$
1	$\mathbf{m} = \text{center}([\mathbf{x}])$
2	Compute the Gauss-Jordan matrix $\mathbf{Q}$ for $\frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{m})$
3	Define $\mathbf{g}(\mathbf{x}) = \mathbf{Q} \cdot \mathbf{f}(\mathbf{x})$
4	For $i \in \{1, \ldots, p\}$
5	For $j \in \{1, \ldots, n\}$
6	$[\mathbf{a}] = [\frac{\partial g_i}{\partial \mathbf{x}}]([\mathbf{x}])$
7	$[s] = \sum_{k \neq j} [a_k] \cdot ([x_k] - m_k)$
8	$[x_j] = [x_j] \cap \frac{1}{[a_j]} (-g_i(\mathbf{m}) - [s])$
9	Return $[\mathbf{x}]$

- Step 1 takes the center $\mathbf{m}$ of $[\mathbf{x}]$ in order to form a linear approximation for $\mathbf{f}$ in $[\mathbf{x}]$:

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{m}) + \frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{m}) \cdot (\mathbf{x} - \mathbf{m}). \quad (41)$$

- Step 2 returns an invertible $m \times m$ matrix $\mathbf{Q}$ such that $\mathbf{A} = \mathbf{Q} \cdot \frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{m})$ is a band matrix. The matrix $\mathbf{Q}$ is chosen by a Gauss-Jordan algorithm. The new system to be solved is now

$$\mathbf{Q} \cdot \mathbf{f}(\mathbf{x}) = \mathbf{0}. \quad (42)$$

- Step 3 defines the function $\mathbf{g}(\mathbf{x}) = \mathbf{Q} \cdot \mathbf{f}(\mathbf{x})$. We need to solve $\mathbf{g}(\mathbf{x}) = \mathbf{0}$ in the box $[\mathbf{x}] - \mathbf{m}$. The main difference compared to the previous system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ is that its linear approximation

$$\mathbf{g}(\mathbf{x}) = \mathbf{g}(\mathbf{m}) + \mathbf{A} \cdot (\mathbf{x} - \mathbf{m}) \tag{43}$$

  is such that $\mathbf{A}$ is a band matrix.

- Step 4-9 define the set of constraints

$$\begin{cases} \mathbf{0} = \mathbf{g}(\mathbf{m}) + \mathbf{A} \cdot (\mathbf{x} - \mathbf{m}) \\ \quad \text{with } \mathbf{A} \in [\frac{d\mathbf{g}}{d\mathbf{x}}]([\mathbf{x}]) \\ \quad\quad \text{and } \mathbf{x} \in [\mathbf{x}] \end{cases} \tag{44}$$

  and performs an interval propagation. Due to the fact that the system has no cycle (at first order), from Corollary 5.3, we get that the propagation is asymptotically minimal.

6 Test case

Interval methods have been shown to be very powerful for the stability analysis of linear systems [23]. We have chosen to consider the linear time-delay system [40] given by

$$\ddot{x} + 2\dot{x}(t - p_1) + x(t - p_2) = 0 \tag{45}$$

but other types of linear systems [25] with fractional orders could be considered as well. Its characteristic function is

$$\theta(\mathbf{p}, s) = s^2 + 2se^{-sp_1} + e^{-sp_2}. \tag{46}$$

For a given $\mathbf{p} = (p_1, p_2)$, the location of the roots for $\theta(\mathbf{p}, s)$ provides an information concerning the stability of the system. For instance, if all roots are on the half left of the complex plane, then the system is stable. The stability changes when one root crosses the imaginary line. This is the reason why we are interested in characterizing the set

$$\mathcal{P} = \{\mathbf{p} \,|\, \exists \omega > 0,\, \theta(\mathbf{p}, j\omega) = 0\}. \tag{47}$$

which corresponds to the set of parameters for which the roots are at the stability boundary. Since that for all $\mathbf{p}$ and for all ω, we have $\theta(\mathbf{p}, j\omega) = \theta(\mathbf{p}, -j\omega)$, we classically impose $\omega > 0$. Now

$$\begin{array}{ll} & \theta(p_1, p_2, j\omega) \\ = & -\omega^2 + 2j\omega e^{-j\omega p_1} + e^{-j\omega p_2} \\ = & -\omega^2 + 2j\omega(\cos(\omega p_1) - j\sin(\omega p_1)) \\ & + \cos(\omega p_2) - j\sin(\omega p_2) \\ = & -\omega^2 + 2\omega\sin(\omega p_1) + \cos(\omega p_2) \\ & + j \cdot (2\omega\cos(\omega p_1) - \sin(\omega p_2)) \end{array} \tag{48}$$

We have

$$\begin{array}{ll} & \theta(p_1,p_2,j\omega)=0 \\ \Leftrightarrow & \underbrace{\begin{pmatrix} -\omega^2+2\omega\sin(\omega p_1)+\cos(\omega p_2) \\ 2\omega\cos(\omega p_1)-\sin(\omega p_2) \end{pmatrix}}_{\mathbf{f}(p_1,p_2,\omega)}=0 \end{array} \quad (49)$$

Take $[p_1]=[0,2.5]$, $[p_2]=[1,4]$, $[\omega]=[0,10]$ and let us characterize the set $\mathcal{P}$ using the centered contractor. Using a branch and prune algorithm such as SIVIA (see e.g. [18]) with an accuracy of $\varepsilon=2^{-8}$ with an HC4 algorithm [7, 4] (the state of the art), we get the paving of Figure 9 in 4 sec. The number of boxes of the approximation is 43173. Similar results were obtained were obtained on the same example in [24].

With an accuracy of $\varepsilon=2^{-4}$ with the centered contractor given in Section 5.4, we get the paving of Figure 10 in 1.2 sec. The number of boxes of the approximation is 282 (instead of 43173), for a more accurate approximation.

With an accuracy of $\varepsilon=2^{-8}$ with the centered contractor, we get the thin curve represented on Figure 11. This curve is made with the small boxes generated by the paver, which shows the quality of the approximation. The big blue boxes are those already painted in the green box $[\mathbf{a}]$ of Figure 10.

With an accuracy of $\varepsilon=2^{-12}$ with the centered contractor, we get the magenta curve of Figure 12. The big gray boxes are those already painted in the red box $[\mathbf{b}]$ of Figure 11. The fact that, for a small ε, the boxes of the approximation only overlap on their corners illustrates the minimality of the contractor.

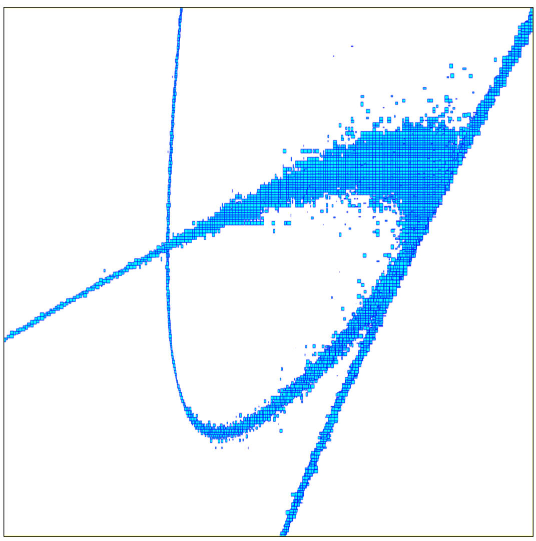

Figure 9: Approximation of the solution set $\mathcal{P}$ with a state of the art contractor (here HC4). The frame box for (p_1,p_2) is $[0,2.5]\times[2,4]$.

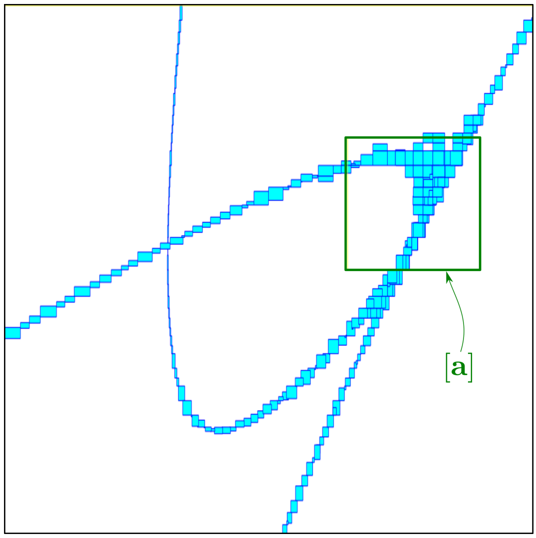

Figure 10: Paving obtained with the centered contractor. The frame box for (p_1, p_2) is $[0, 2.5] \times [2, 4]$.

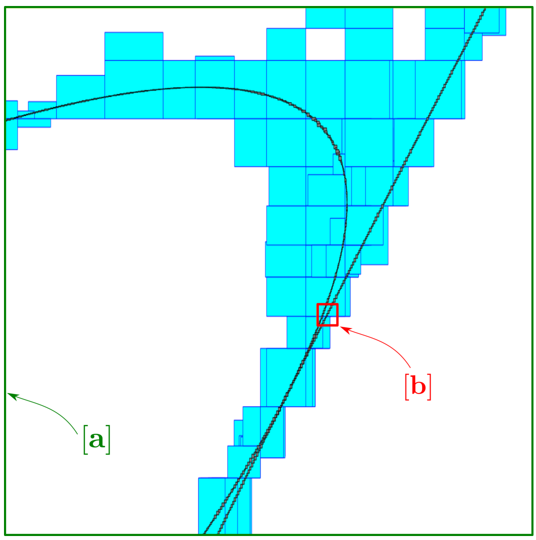

Figure 11: Pavings obtained with the centered contractor in the box $[\mathbf{a}] = [1.3, 1.8] \times [3.0, 3.5]$; Blue: $\varepsilon = 2^{-4}$; Thin: $\varepsilon = 2^{-8}$.

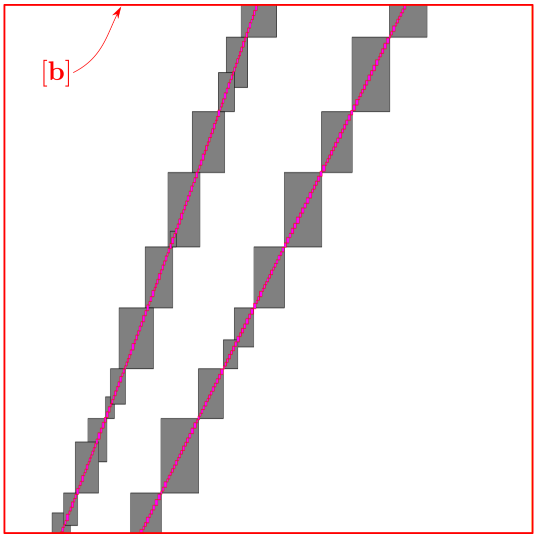

Figure 12: Approximation of the solution set in $[\mathbf{b}] = [1.595, 1.615] \times [3.2, 3.22]$; Gray: $\varepsilon = 2^{-8}$; Magenta: $\varepsilon = 2^{-12}$.

The computing time to get the three Figures 10, 11 and 12 is less than 10 sec. Our results are much more accurate than those obtained in Section 6 of [24].

The code, based on the codac library [36], and an illustrating video are given at `www.ensta-bretagne.fr/jaulin/centered.html`.

7 Conclusion

In this paper, we have proposed a contractor which is asymptotically minimal for the approximation of a curve defined by nonlinear equations. The resulting *centered* contractor is based on the centered form which suppresses the pessimism when the boxes are narrow and when we have a single equation. When we combine several equations, a preconditioning method has been proposed in order to linearize the problem into a system where a tree matrix in involved. The preconditioning has been implemented using a Gauss Jordan band diagonalization method. On an example, we have shown that our centered contractor was able to outperform the state of the art contractor based on a forward-backward propagation.

Other approaches, such as the generalized interval arithmetic [15], the affine arithmetic [12] allows to get first order approximation of the constraints. As for our paper, these arithmetics can obviously model the affine dependencies between quantities with an error that shrinks quadratically with the size of the input intervals. Now, this linear approximation is only valid when we have a single constraint and can thus not be used to build asymptotically minimal contractors without some improvements. Our approach does not require the implementation of a new arithmetic since it only uses the standard interval arithmetic. Moreover, our approach generates a contractor that can be combined with other existing contractors enforcing the efficiency of the resolution.

References

[1] Araya, I., Trombettoni, G., and Neveu, B. A contractor based on convex interval taylor. In Beldiceanu, N., Jussien, N., and Pinson, E., editors, *Proceedings of the 9th International Conference on Integration of AI and OR Techniques in Contraint Programming for Combinatorial Optimzation Problems*, Volume 7298 of *Lecture Notes in Computer Science*, pages 1–16. Springer, 2012. DOI: 10.1007/978-3-642-29828-8_1.

[2] Atkinson, K. *An Introduction to Numerical Analysis (2nd edition)*. John Wiley, 1989. DOI: 10.1002/0471667196.ess1837.

[3] Auer, E., Rauh, A., Hofer, E., and Luther, W. Validated modeling of mechanical systems with SmartMOBILE: Improvement of performance by ValEncIA-IVP. In Hertling, P., Hoffmann, C. M., Luther, W., and Revol, N., editors, *Revised papers of the International Seminar on Reliable Implementation of Real Number Algorithms: Theory and Practice*, Volume 5045 of *Lecture Notes in Computer Science*, pages 1–27. Springer, 2008. DOI: 10.1007/978-3-540-85521-7_1.

[4] Benhamou, F., Goualard, F., Granvilliers, L., and Puget, J. F. Revising hull and box consistency. In Schreye, D. D., editor, *Proceedings of the 1999 International Conference on Logic Programming*, pages 230–244. MIT Press, 1999. DOI: 10.7551/mitpress/4304.003.0024.

[5] Berger, M. *Geometry I and II.* Springer-Verlag, Berlin, 1987. ISBN: 9783540116585 and ISBN: 9783540170150.

[6] Braems, I., Kieffer, M., Walter, É., and Jaulin, L. *Set Computation, Computation of Volumes and Data Safety.* In Krämer, W. and von Gudenberg, J. W., editors, *Scientific Computing, Validated Numerics, Interval Methods*, pages 267–278. Kluwer Academic Publishers, 2001. DOI: 10.1007/978-1-4757-6484-0_22.

[7] Cébério, M. and Granvilliers, L. Solving nonlinear systems by constraint inversion and interval arithmetic. In Campbell, J. A. and Roanes-Lozano, E., editors, *Proceedings of the International Conference on Artificial Intelligence and Symbolic Computation*, Volume 1930 of *Lecture Notes in Computer Science*, pages 127–141. Springer, 2000. DOI: 10.1007/3-540-44990-6_10.

[8] Chabert, G. and Jaulin, L. Contractor programming. *Artificial Intelligence*, 173:1079–1100, 2009. DOI: 10.1016/J.ARTINT.2009.03.002.

[9] Combastel, C. Zonotopes and Kalman observers: Gain optimality under distinct uncertainty paradigms and robust convergence. *Automatica*, 55:265–273, 2015. DOI: 10.1016/j.automatica.2015.03.008.

[10] Combastel, C. Functional sets with typed symbols: Mixed zonotopes and polynotopes for hybrid nonlinear reachability and filtering. *Automatica*, 143:110457, 2022. DOI: 10.1016/j.automatica.2022.110457.

[11] Ehambram, A., Voges, R., and Wagner, B. Stereo-visual-lidar sensor fusion using set-membership methods. In *Proceedings of the 17th IEEE International Conference on Automation Science and Engineering*, pages 1132–1139. IEEE, 2021. DOI: 10.1109/CASE49439.2021.9551516.

[12] Figueiredo, L. D. and Stolfi, J. Affine arithmetic: concepts and applications. *Numerical Algorithms*, 37(1):147–158, 2004. DOI: 10.1023/B:NUMA.0000049462.70970.B6.

[13] Goubault, E. and Putot, S. Robust under-approximations and application to reachability of non-linear control systems with disturbances. *IEEE Control System Letters*, 4(4):928–933, 2020. DOI: 10.1109/LCSYS.2020.2997261.

[14] Guyonneau, R., Lagrange, S., and Hardouin, L. A visibility information for multi-robot localization. In *Proceedings of IEEE/RSJ International Conference on Intelligent Robots and Systems*, pages 1426–1431, 2013. DOI: 10.1109/IROS.2013.6696536.

[15] Hansen, E. R. A generalized interval arithmetic. In Nickel, K., editor, *Proceedings of the International Symposium on Interval Mathemantics*, Volume 29, pages 7–18. Springer, 1975. DOI: 10.1007/3-540-07170-9_2.

[16] Hansen, E. R. *Global Optimization using Interval Analysis*. Marcel Dekker, New York, NY, 1992. ISBN: 9780824786960.

[17] Jaulin, L., Kieffer, M., Didrit, O., and Walter, E. *Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control and Robotics*. Springer-Verlag, London, 2001. DOI: 10.1007/978-1-4471-0249-6_2.

[18] Jaulin, L. and Walter, E. Set inversion via interval analysis for nonlinear bounded-error estimation. *Automatica*, 29(4):1053–1064, 1993. DOI: 10.1016/0005-1098(93)90106-4.

[19] Kapela, T., Mrozek, M., Wilczak, D., and Zgliczynski, P. CAPD::DynSys: A flexible C++ toolbox for rigorous numerical analysis of dynamical systems. *Communications in Nonlinear Science and Numerical Simulation*, 101:105578, 2021. DOI: 10.1016/j.cnsns.2020.105578.

[20] Kreinovich, V., Lakeyev, A., Rohn, J., and Kahl, P. Computational complexity and feasibility of data processing and interval computations. *Reliable Computing*, 4(4):405–409, 1998. DOI: 10.1023/A:1024484203503.

[21] Langerwisch, M. and Wagner, B. Guaranteed mobile robot tracking using robust interval constraint propagation. In *Proceedings of the 5th International Conference on Intelligent Robotics and Applications, Part II*, Volume 7507 of *Lecture Notes in Computer Science*, pages 354–365. Springer, 2012. DOI: 10.1007/978-3-642-33515-0_36.

[22] Leon, S. *Linear Algebra with Applications (8th edition)*. Pearson, 2009. ISBN: 9780136009290.

[23] Malan, S. A., Milanese, M., and Taragna, M. Robust analysis and design of control systems using interval arithmetics. *Automatica*, 33(7):1363–1372, 1997. DOI: 10.1016/S0005-1098(97)00028-9.

[24] Malti, R., Rapaić, M., and Turkulov, V. A unified framework for robust stability analysis of linear irrational systems in the parametric space. *Annual Reviews in Control*, 57, 2024. URL: https://hal.archives-ouvertes.fr/hal-03646956.

[25] Mayoufi, A., Malti, R., Chetoui, M., and Aoun, M. System identification of MISO fractional systems: Parameter and differentiation order estimation. *Automatica*, 141:110268, 2022. DOI: 10.1016/j.automatica.2022.110268.

[26] Merlet, J. and Daney, D. Dimensional synthesis of parallel robots with a guaranteed given accuracy over a specific workspace. In *Proceedings of the 2005 IEEE International Conference on Robotics and Automation*, pages 942–947. IEEE, 2005. DOI: 10.1109/ROBOT.2005.1570238.

[27] Montanari, U. and Rossi, F. Constraint relaxation may be perfect. *Artificial Intelligence*, 48(2):143–170, 1991. DOI: 10.1016/0004-3702(91)90059-S.

[28] Moore, R. *Methods and Applications of Interval Analysis*. Society for Industrial and Applied Mathematics, 1979. DOI: 10.1137/1.9781611970906.

[29] Mustafa, M., Stancu, A., Delanoue, N., and Codres, E. Guaranteed SLAM; an interval approach. *Robotics and Autonomous Systems*, 100:160–170, 2018. DOI: 10.1016/J.ROBOT.2017.11.009.

[30] Neumaier, A. *Interval Methods for Systems of Equations*. Cambridge University Press, Cambridge, UK, 1990. DOI: 10.1002/zamm.19920721114.

[31] Neumaier, A. and Shcherbina, O. Safe bounds in linear and mixed-integer linear programming. *Mathematical Programming*, 99(2):283–296, 2004. DOI: 10.1007/s10107-003-0433-3.

[32] Ramdani, N. and Poignet, P. Robust dynamic experimental identification of robots with set membership uncertainty. *IEEE/ASME Transactions on Mechatronics*, 10(2):253–256, 2005. DOI: 10.1109/TMECH.2005.844703.

[33] Rauh, A. and Jaulin, L. A computationally inexpensive algorithm for determining outer and inner enclosures of nonlinear mappings of ellipsoidal domains. *International Journal of Applied Mathematics and Computer Science*, 31(3):399–415, 2021. DOI: 10.34768/AMCS-2021-0027.

[34] Rauh, A., Senkel, L., Auer, E., and Aschemann, H. Interval methods for real-time capable robust control of solid oxide fuel cell systems. *Mathematics in Computer Science*, 8(3-4):525–542, 2014. DOI: 10.1007/S11786-014-0205-X.

[35] Rego, B., Scott, J., Raimondo, D., and Raffo, G. Set-valued state estimation of nonlinear discrete-time systems with nonlinear invariants based on constrained zonotopes. *Automatica*, 129:109638, 2021. DOI: 10.1016/J.AUTOMATICA.2021.109638.

[36] Rohou, S. Codac (Catalog of domains and contractors). Robex, Lab-STICC, ENSTA-Bretagne, 2021. URL: http://codac.io/.

[37] Rohou, S., Jaulin, L., Mihaylova, L., Bars, F. L., and Veres, S. *Reliable Robot Localization*. Wiley, 2019. DOI: 10.1002/9781119680970.

[38] Sainudiin, R. *Machine Interval Experiments: Accounting for the Physical Limits on Empirical and Numerical Resolutions*. LAP Academic Publishers, Köln, Germany, 2010. ISBN: 9783838315997, URL: https://my.lap-publishing.com/catalogue/details/gb/978-3-8383-1599-7/machine-interval-experiments.

[39] Scott, J., Raimondo, D., Marseglia, G., and Braatz, R. Constrained zonotopes: A new tool for set-based estimation and fault detection. *Automatica*, 69:126–136, 2016. DOI: 10.1016/J.AUTOMATICA.2016.02.036.

[40] Turkulov, V., Rapaić, M., and Malti, R. Stability analysis of time-delay systems in the parametric space. *Automatica*, 157, 2023. DOI: 10.1016/J.AUTOMATICA.2023.111220.

[41] Wan, J., Vehi, J., and Luo, N. A numerical approach to design control invariant sets for constrained nonlinear discrete-time systems with guaranteed optimality. *Journal of Global Optimization*, 44:395–407, 2009. DOI: 10.1007/S10898-008-9334-6.