The Dombi t-norm-based Group Consensus Measure and Its Applications

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Abstract

In this study, we elaborate on the idea of parametric group consensus measures. Here, we employ the additive generators of Dombi t-norms to construct fuzzy entropies, which are then utilized to generate group consensus measures. Since these additive generators are single-parameter functions, the resulting group consensus measures are also single-parameter mappings. We demonstrate that, given a set of inputs from decision-makers, the group consensus measure behaves as a bounded and non-decreasing function of its parameter. With this in mind, we address the question of which parameter value yields a group consensus measure that most accurately reflects the average perceived consensus level of the group. Next, we present a necessary and sufficient condition for the existence of an optimal value of the parameter in question, and provide algorithmic procedures to find it.

Keywords: fuzzy entropy, group decision-making, group consensus, Dombi operators

1 Introduction

The analysis of decisions made by a group of experts belongs to the field of group decision making (GDM). In GDM situations, the participating experts evaluate one or more possible alternatives either quantitatively [20, 19] or qualitatively [30, 6, 5, 4, 16] to make a final decision. This is often based on some aggregate score of the individual evaluations, but the information content of these scores (as that of measures of central tendency) is limited. Therefore, it is advisable to complement the aggregate score with some measure that informs us about the spread (as measures of spread do in descriptive statistics) of the evaluations. In the field of GDM, this can be performed by employing group consensus measures. These measures are intended to quantify the level of agreement among the decision-makers. In the existing literature, there are several proposals for the favourable properties

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of such measures (see, e.g., [8, 30, 2, 3]) and for the measures themselves (see, e.g., [19, 21, 22]).

In this study, we consider the works [19] and [13] as starting points, and we aim to generate consensus measures using the so-called Dombi t-norms. With this approach, we can achieve an enhanced level of flexibility w.r.t. the consensus measure to be generated. This flexibility is manifested in the property of the proposed parametric consensus measure that it can assume any value between its lower and upper bound. Furthermore, the value of this consensus measure also depends on its parameter value. With this in mind, we present algorithmic procedures to find such a consensus measure whose value in the actual GDM situation is equal to the average level of perceived consensus among the decision-makers. That is, we seek to find a consensus measure which best models the actual decision situation.

The above can be interpreted as we aim to investigate how the value of the consensus measure depends on its parameter value, and how the consensus measure can be optimized such that it best models the perceptions of the participants. Taking this into consideration, we practically join to the comparative studies investigating how the choice of aggregation operators and similarity functions affect the level of consensus (see e.g., [12, 10]) and to those giving recommendations to the participating experts how to change their evaluations to achieve a higher level of consensus (see, e.g., [25, 26, 24]).

In our study, we intend to investigate the behavior of the proposed parametric measure w.r.t. the value of its α parameter, and instead of the highest possible level of consensus, the one modelling best the perceived consensus level is aimed to be found. To the best of our knowledge, the current body of literature does not offer methodologies for the calibration or optimization of parametric group consensus measures aimed at capturing the perceived aggregate consensus among decision-makers. Consequently, the approach of adjusting a consensus measure via its parameter to align with the perceived average consensus level within a decision-making group represents a novel and distinctive contribution.

The rest of the paper is structured as follows: In the Preliminaries section, we introduce the basic concepts and definitions our work is built upon. In Section 3, we show how to generate fuzzy entropies using the additive generators of Dombi t-norms, from which we construct the Dombi t-norm-based group consensus measure in Section 4. In this section, we establish a theoretical result that serves as a cornerstone for the optimization of the Dombi t-norm-based group consensus measure through its parameter, with the objective of aligning the measure with the perceived average consensus among decision-makers. Specifically, Theorem 2 provides the foundational basis for a procedure that determines the optimal parameter value of the consensus measure in this context. In Section 5, we outline some practical applications of our theoretical results; i.e., we propose procedures to find from the family of consensus measures the one which best models the perceptions of the decision-makers in the actual GDM situation. We end our study with some concluding remarks and outline future directions of research in Section 6.

2 Preliminaries

In our study, the common notation $\mathbb{R}$ will be used for the real line. Here, we will operate on the extended real line $[-\infty, \infty]$ with the conventions

$$\frac{1}{0} = \infty \quad \text{and} \quad \frac{1}{\infty} = 0.$$

From now on, we will assume that in a group of n entities ($n \in \mathbb{N}$, $n \geq 2$), every group member expresses its opinion on the unit interval $[0, 1]$. Here, $x_i \in [0, 1]$ is the individual input (evaluation) of the ith group member (decision-maker) concerning a given decision alternative, where $i = 1, 2, \ldots, n$. And the higher the value of x_i, the better the ith decision-maker's opinion of the alternative in question. We will refer to the n-dimensional vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0, 1]^n$ as the input vector of a group.

Consensus measures are functions that measure how much the individual inputs of the group members agree with one another. In the seminal paper of Beliakov et al. (see [8]), a consensus measure is defined as a function $C\colon [0, 1]^n \to [0, 1]$ with the following two properties: (1) for any $a \in [0, 1]$, $C(a, a, \ldots, a) = 1$ (called the unanimity of C); (2) for the special case of two inputs, it holds that $C(0, 1) = C(1, 0) = 0$ (referred to as the minimum consensus for $n = 2$). The authors in [8] set the five reasonable additional requirements that a consensus measure should satisfy: symmetry, maximum dissension, reciprocity, replication invariance and monotonicity with respect to the majority. Using their approach, in [19], the authors employed the following definition for a consensus measure.

Definition 1 (cf. [8], [19]). *A function $C\colon [0, 1]^n \to [0, 1]$ is said to be a group consensus measure if it satisfies the following requirements:*

(C1) (Unanimity) For any $a \in [0, 1]$, $C(a, a, \ldots, a) = 1$.

(C2) (Minimum consensus for $n = 2$) For the special case of two inputs, it holds that $C(0, 1) = C(1, 0) = 0$.

(C3) (Symmetry) For any permutation $\rho\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ and input vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0, 1]^n$, it holds that $C(x_1, x_2, \ldots, x_n) = C\left(x_{\rho(1)}, x_{\rho(2)}, \ldots, x_{\rho(n)}\right)$.

(C4) (Maximum dissension) For $n = 2k$, if k of the inputs are equal to zero and k of the inputs are equal to 1, then $C(0, 0, \ldots, 1, 1) = 0$ for all permutations of the input vector.

(C5) (Reciprocity) For any input vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0, 1]^n$, it holds that $C(x_1, x_2, \ldots, x_n) = C\left(N_s(x_1), N_s(x_2), \ldots, N_s(x_n)\right)$, where $N_s(a) = 1 - a$ is the standard fuzzy negation and $a \in [0, 1]$.

(C6) (Replication invariance) For any input vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0, 1]^n$, replicating the inputs does not alter the degree of consensus, i.e., $C(\mathbf{x}) = C(\mathbf{x}, \mathbf{x}) = \ldots = C(\mathbf{x}, \mathbf{x} \ldots, \mathbf{x})$.

(C7) (Monotonicity with respect to the majority) For $n = 2k$ *let half of the inputs be equal and be denoted by* $\mathbf{a} = (a, a, \ldots, a)$*, where* $\mathbf{a} \in [0,1]^k$*. Furthermore let* $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ *and* $\mathbf{y} = (y_1, y_2, \ldots, y_k)$ *be two input vectors, where* $\mathbf{x}, \mathbf{y} \in [0,1]^k$*. If* $|a - x_j| \leq |a - y_j|$ *for all* $j = 1, 2, \ldots, k$*, then* $C(\mathbf{a}, x_1, x_2, \ldots, x_k) \geq C(\mathbf{a}, y_1, y_2, \ldots, y_k)$ *holds for any permutation of the inputs.*

Remark 1. We should add that the (C2) requirement is a special case of the (C4) requirement. That is, with the choice $n = 2$, criterion (C4) is the same as criterion (C2).

Fuzziness measures are used to characterize how much the membership function of a fuzzy set differs from the characteristic function. Suppose that X is a nonempty set and $\mu \colon X \to [0,1]$ is the membership function of a fuzzy set on the universe X. In the pioneer work of De Luca and Termini (see [11]), the authors proposed the following function to characterize the fuzziness of the fuzzy set given by the membership function μ:

$$d(\mu) = \frac{1}{n} \sum_{i=1}^{n} S\left(\mu(x_i)\right),$$

where x_i is an element of a finite set $X = \{x_1, x_2, \ldots, x_n\}$ and

$$S(x) = -\frac{1}{\ln(2)} \left(x \ln(x) + (1 - x) \ln(1 - x)\right), \quad x \in [0,1] \tag{1}$$

(with the convention $0 \cdot \ln(0) = 0$). As the function S in Eq. (1) is formally similar to the Shannon entropy, it is often referred to as a fuzzy entropy. We will utilize the following, more general concept of fuzzy entropy, which was proposed in [19].

Definition 2 (cf. [19])**.** *We say that a function* $F \colon [0,1] \to [0,1]$ *is a fuzzy entropy if* F *satisfies the following requirements:*

(a) F *is continuous on* $[0,1]$*.*

(b) $F(0) = 0$ *and* $F(1) = 0$*.*

(c) F *is strictly increasing on* $\left(0, \frac{1}{2}\right)$ *and* F *is strictly decreasing on* $\left(\frac{1}{2}, 1\right)$*.*

(d) $F(x)$ *has a unique maximum at* $x = \frac{1}{2}$*, and* $F\left(\frac{1}{2}\right) = 1$*.*

(e) $F(x) = F(1 - x)$ *for any* $x \in [0,1]$*.*

It can be easily verified that, with the convention $0 \cdot \ln(0) = 0$, the function S given in Eq. (1) satisfies the criteria for a fuzzy entropy in the sense of Definition 2.

Here, we will make use of the following group consensus measure, which was proposed in [19].

Definition 3 (cf. [19])**.** *Let* $F \colon [0,1] \to [0,1]$ *be a fuzzy entropy in the sense of Definition 2. The mapping* $C_F \colon [0,1]^n \to [0,1]$ *said to be a fuzziness measure-based* *group consensus measure induced by the fuzzy entropy F, if for any input vector* $\mathbf{x} = x_1, x_2, \ldots, x_n \in [0,1]^n$, $C_F(\mathbf{x})$ *is given as*

$$C_F(\mathbf{x}) = 1 - \sum_{i=1}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) F\left(\frac{n-i+1}{n}\right), \tag{2}$$

where $\pi\colon \{0,1,2,\ldots,n+1\} \to \{0,1,2,\ldots,n+1\}$ *is a permutation such that*

$$0 = x_{\pi(0)} \leq x_{\pi(1)} \leq x_{\pi(2)} \leq \ldots \leq x_{\pi(n)} \leq x_{\pi(n+1)} = 1. \tag{3}$$

It was proven in [19] that the function C_F given in Eq. (2) satisfies the requirements (C1)-(C7) for a group consensus measure given in Definition 1. We call the function C_F the fuzziness measure-based consensus measure induced by the fuzzy entropy F. That is, the fuzzy entropy F may be viewed as a generator function of the consensus measure C_F.

3 Fuzzy entropies based on Dombi t-norms

3.1 Fuzzy entropies induced by additive generators of strict t-norms

Let $g\colon [0,1] \to [0,\infty]$ be a continuous and strictly decreasing function with the properties $g(0) = \infty$ and $g(1) = 0$. It is well known (see, e.g., Section 5.1 in [28]) that such a function g is an additive generator of a strict triangular norm (t-norm for short), which is uniquely determined up to a positive constant multiplier of g. The strict t-norm $T\colon [0,1]^2 \to [0,1]$ induced by the additive generator g is

$$T(x,y) = g^{-1}\left(g(x) + g(y)\right). \tag{4}$$

Remark 2. We should add that a continuous and strictly increasing mapping $h\colon [0,1] \to [0,\infty]$ with the properties $h(0) = 0$ and $h(1) = \infty$ is an additive generator of a strict triangular conorm (t-conorm for short), which is uniquely determined up to a positive constant multiplier of h (see, e.g., Section 5.1 in [28]). In continuous valued logic, the strict t-norms (t-conorms, respectively) may be viewed as strictly monotonic conjunction (disjunction, respectively) operators.

Based on Theorem 2 in [17], we see that the function $F_g\colon [0,1] \to [0,1]$, which is given by

$$F_g(x) = 2g^{-1}\left(\frac{1}{2}\left(g(x) + g(1-x)\right)\right), \tag{5}$$

satisfies the requirements for a fuzzy entropy function given in Definition 2 (also see [13] and [15]), where function g is an additive generator of a strict t-norm. Based on Eq. (5) and Eq. (2), the group consensus measure $C_{F_g}\colon [0,1]^n \to [0,1]$ induced by the fuzzy entropy F_g is

$$\begin{aligned} C_{F_g}(\mathbf{x}) &= 1 - \sum_{i=1}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) F_g\left(\frac{n-i+1}{n}\right) \\ &= 1 - 2\sum_{i=1}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) g^{-1}\left(\frac{1}{2}\left(g\left(\frac{n-i+1}{n}\right)\right.\right. \\ &\left.\left. + g\left(\frac{i-1}{n}\right)\right)\right), \end{aligned} \tag{6}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$ is an input vector and π is a permutation on $\{0, 1, 2, \ldots, n+1\}$ satisfying Eq. (3).

Notice that the fuzzy entropy F_g generated by g as in Eq. (5) can be written as

$$F_g(x) = 2M_g(x, 1-x), \tag{7}$$

where $M_g(x, 1-x)$ is the quasi-arithmetic mean (see Definition 27 in [23]) of x and $1-x$ ($x \in [0,1]$) induced by the generator function g. That is, $M_g(x, 1-x)$ is given as

$$M_g(x, 1-x) = g^{-1}\left(\frac{1}{2}\left(g(x) + g(1-x)\right)\right). \tag{8}$$

This means that the quasi-arithmetic mean M_g can be used to generate the fuzzy entropy F_g, while F_g can be used to generate a group consensus measure. For other practical applications of quasi-arithmetic means see, e.g., [18, 29, 27]. The following theorem provides a necessary and sufficient condition for the identity of fuzzy entropies induced by additive generators of strict t-norms using Eq. (5).

Theorem 1. *Let $g_1, g_2 \colon [0,1] \to [0,\infty]$ be additive generators of two strict t-norms, and let $F_{g_1}, F_{g_2} \colon [0,1] \to [0,1]$ be the fuzzy entropies induced by g_1 and g_2, respectively, according to Eq. (5), i.e.,*

$$F_{g_1}(x) = 2g_1^{-1}\left(\frac{1}{2}\left(g_1(x) + g_1(1-x)\right)\right)$$

and

$$F_{g_2}(x) = 2g_2^{-1}\left(\frac{1}{2}\left(g_2(x) + g_2(1-x)\right)\right).$$

Then, for any $x \in [0,1]$,

$$F_{g_1}(x) = F_{g_2}(x) \tag{9}$$

holds if and only if

$$g_1(x) = a\, g_2(x) \tag{10}$$

for some $a > 0$ constant.

Proof. Let $M_{g_1}(x, 1-x)$ and $M_{g_2}(x, 1-x)$ be the quasi-arithmetic means of x and $1-x$ induced by the generators g_1 and g_2, respectively, according to Eq. (8). Next, based on [1] (see also p. 226 in [9] and Proposition 7 in [23]), we find that

$$M_{g_1}(x, 1-x) = M_{g_2}(x, 1-x) \tag{11}$$

holds if and only if

$$g_1(x) = a\, g_2(x) + b$$

for some real a, b constants, $a \neq 0$ ($x \in [0,1]$). Taking into account the fact that g_1 and g_2 are both continuous and strictly decreasing mappings from $[0,1]$ to $[0,\infty]$ with the properties $g_1(0) = g_2(0) = \infty$ and $g_1(1) = g_2(1) = 0$, we find that b necessarily has to have the value of zero and $a > 0$. Noting Eq. (7), we see that Eq. (11) is equivalent to the equation

$$F_{g_1}(x) = F_{g_2}(x).$$

Therefore, Eq. (9) holds if and only if Eq. (10) holds for some constant $a > 0$. □

Remark 3. Notice that for any fixed input vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$, the group consensus measure induced by the additive generator g of a strict t-norm according to Eq. (6) depends only on the generator g. Hence, an immediate consequence of Theorem 1 is that if $C_{F_{g_1}}$ and $C_{F_{g_2}}$ are two group consensus measures induced by the additive generators g_1 and g_2 of two strict t-norms, respectively, then for any input vector $\mathbf{x}$,

$$C_{F_{g_1}}(\mathbf{x}) = C_{F_{g_2}}(\mathbf{x})$$

holds if and only if $g_1(x) = a\, g_2(x)$ for some constant $a > 0$ ($x \in [0,1]$). This means that the group consensus measure induced by the fuzzy entropy F_g using Eq. (6) is uniquely determined up to a positive constant multiplier of the generator function g.

3.2 Fuzzy entropies induced by additive generators of Dombi t-norms

Let $\alpha \in \mathbb{R} \setminus \{0\}$. One can see that the function $g_\alpha \colon [0,1] \to [0,\infty]$, which is given by

$$g_\alpha(x) = \left(\frac{1-x}{x}\right)^\alpha, \tag{12}$$

is an additive generator of a strict t-norm (strict t-conorm, respectively) if $\alpha > 0$ ($\alpha < 0$, respectively). The t-norms and t-conorms induced by the additive generator g_α are known as the Dombi operators [13]. The inverse function of g_α is the $g_\alpha^{-1} \colon [0,\infty] \to [0,1]$ function that is given by

$$g_\alpha^{-1}(x) = \frac{1}{1 + x^{\frac{1}{\alpha}}}. \tag{13}$$

Let $\alpha > 0$. Using Eq. (4), the Dombi t-norm $T_\alpha\colon [0,1]^2 \to [0,1]$ induced by g_α is

$$T_\alpha(x,y) = g_\alpha^{-1}\left(g_\alpha(x) + g_\alpha(y)\right) = \frac{1}{1 + \left(\left(\frac{1-x}{x}\right)^\alpha + \left(\frac{1-y}{y}\right)^\alpha\right)^{\frac{1}{\alpha}}}.$$

Noting Eq. (5), the fuzzy entropy $F_\alpha\colon [0,1] \to [0,1]$ induced by g_α is

$$\begin{aligned} F_\alpha(x) &= 2g_\alpha^{-1}\left(\frac{1}{2}\left(g_\alpha(x) + g_\alpha(1-x)\right)\right) \\ &= \frac{2}{1 + \left(\frac{1}{2}\left(\left(\frac{1-x}{x}\right)^\alpha + \left(\frac{x}{1-x}\right)^\alpha\right)\right)^{\frac{1}{\alpha}}}. \end{aligned} \tag{14}$$

Figure 1 shows some example plots of fuzzy entropies induced by additive generators of Dombi t-norms.

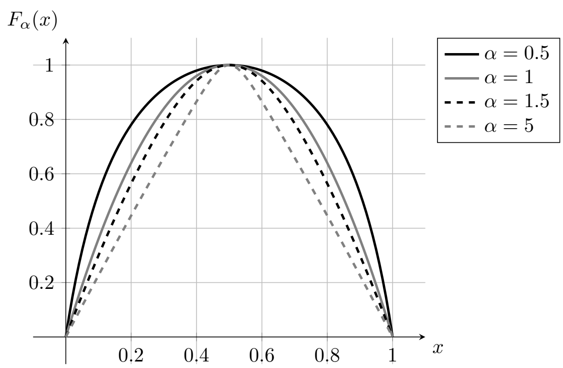

Figure 1: Example plots of the fuzzy entropy $F_\alpha(x)$ for various values of parameter α.

Lemma 1 states some important properties of fuzzy entropies induced by additive generators of Dombi t-norms.

Lemma 1. *Let $\alpha > 0$, let $g_\alpha\colon [0,1] \to [0,\infty]$ be an additive generator of the Dombi t-norm given in Eq. (12), and let $F_\alpha\colon [0,1] \to [0,1]$ be the fuzzy entropy induced by g_α as in Eq. (14). Then, F_α has the following properties:*

(a) $F_\alpha(0) = F_\alpha(1) = 0$;

(b) F_α *is a concave function on* $(0,1)$;

(c) For any $x \in (0,1)$,

$$\lim_{\alpha \to 0} F_\alpha(x) = 1;$$

(d) For any $x \in (0,1)$*,*

$$\lim_{\alpha\to\infty} F_\alpha(x) = \begin{cases} 2x, & \textit{if } x \in \left(0, \frac{1}{2}\right] \\ 2(1-x), & \textit{if } x \in \left(\frac{1}{2}, 1\right). \end{cases}$$

Proof.

Proof of (a). Since for any $\alpha > 0$, F_α is a fuzzy entropy (i.e., F_α is induced by an additive generator of a strict t-norm and so, based on Theorem 2 in [17], F_α satisfies the requirements for a fuzzy entropy), we readily have that $F_\alpha(0) = F_\alpha(1) = 0$.

Proof of (b). Using the second derivative of F_α, it can be verified that for any $\alpha > 0$ and $x \in (0,1)$, $\frac{\mathrm{d}^2 F_\alpha(x)}{\mathrm{d}x^2} < 0$, which means that F_α is a concave function on $(0,1)$.

Proof of (c). Let $x \in (0,1)$ have a fixed value. Then,

$$\lim_{\alpha\to 0} F_\alpha(x) = \frac{2}{1 + \lim_{\alpha\to 0}\left(\left(\frac{1}{2}\left(\left(\frac{1-x}{x}\right)^\alpha + \left(\frac{x}{1-x}\right)^\alpha\right)\right)^{\frac{1}{\alpha}}\right)}$$

Using the L'Hospital rule, for any $x \in (0,1)$, we have:

$$\begin{aligned}
\lim_{\alpha\to 0}\left(\frac{1}{2}\left(\left(\frac{1-x}{x}\right)^\alpha + \left(\frac{x}{1-x}\right)^\alpha\right)\right)^{\frac{1}{\alpha}} &= \mathrm{e}^{\lim_{\alpha\to 0}\frac{\ln\left(\frac{1}{2}\left(\left(\frac{1-x}{x}\right)^\alpha + \left(\frac{x}{1-x}\right)^\alpha\right)\right)}{\alpha}} \\
&= \mathrm{e}^{\lim_{\alpha\to 0}\frac{\frac{\ln\left(\frac{1-x}{x}\right)\left(\frac{1-x}{x}\right)^\alpha + \ln\left(\frac{x}{1-x}\right)\left(\frac{x}{1-x}\right)^\alpha}{\left(\frac{x}{1-x}\right)^\alpha + \left(\frac{1-x}{x}\right)^\alpha}}{1}} \\
&= \mathrm{e}^{\frac{\ln\left(\frac{1-x}{x}\frac{x}{1-x}\right)}{2}} = \mathrm{e}^{\frac{\ln(1)}{2}} = 1,
\end{aligned}$$

from which

$$\lim_{\alpha\to 0} F_\alpha(x) = 1$$

follows.

Proof of (d). Let $x \in \left(0, \frac{1}{2}\right)$ have a fixed value. Then, $\frac{x}{1-x} < 1$ and so

$$\begin{aligned}
\lim_{\alpha\to\infty} F_\alpha(x) &= \frac{2}{1 + \lim_{\alpha\to\infty}\left(\left(\frac{1}{2}\left(\left(\frac{1-x}{x}\right)^\alpha + \left(\frac{x}{1-x}\right)^\alpha\right)\right)^{\frac{1}{\alpha}}\right)} \\
&= \frac{2}{1 + \frac{1-x}{x}} = 2x.
\end{aligned}$$

Now, let $x = \frac{1}{2}$. Then,

$$\lim_{\alpha\to\infty} F_\alpha(x) = \frac{2}{1+1} = 1 = 2x.$$

Hence for any $x \in \left(0, \frac{1}{2}\right]$, $\lim_{\alpha\to\infty} F_\alpha(x) = 2x$.

Now, let $x \in \left(\frac{1}{2}, 1\right)$ have a fixed value. Then, $\frac{1-x}{x} < 1$ and so

$$\lim_{\alpha\to\infty} F_\alpha(x) = \frac{2}{1 + \lim_{\alpha\to\infty}\left(\left(\frac{1}{2}\left(\left(\frac{1-x}{x}\right)^\alpha + \left(\frac{x}{1-x}\right)^\alpha\right)\right)^{\frac{1}{\alpha}}\right)} = \frac{2}{1+\frac{x}{1-x}} = 2(1-x).$$

□

4 Group consensus measures induced by additive generators of Dombi t-norms

Based on Eq. (6) and Eq. (14), the group consensus measure $C_{F_\alpha} : [0,1]^n \to [0,1]$ induced by the fuzzy entropy F_α is

$$\begin{aligned} C_{F_\alpha}(\mathbf{x}) &= 1 - \sum_{i=1}^{n}\left(x_{\pi(i)} - x_{\pi(i-1)}\right) F_\alpha\left(\frac{n-i+1}{n}\right) \\ &= 1 - 2\sum_{i=1}^{n}\frac{x_{\pi(i)} - x_{\pi(i-1)}}{1 + \left(\frac{1}{2}\left(\left(\frac{i-1}{n-i+1}\right)^\alpha + \left(\frac{n-i+1}{i-1}\right)^\alpha\right)\right)^{\frac{1}{\alpha}}}, \end{aligned} \tag{15}$$

where $\alpha > 0$, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$ is an input vector and π is a permutation on $\{0, 1, 2, \ldots, n+1\}$ satisfying Eq. (3). We shall call C_{F_α} the *Dombi t-norm-based group consensus measure.*

The following theorem demonstrates important properties of the group consensus measure C_{F_α}, which enhance its practical application.

Theorem 2. *Let $\alpha > 0$, $n \in \mathbb{N}$, $n \geq 2$, and let $C_{F_\alpha} : [0,1]^n \to [0,1]$ be the Dombi t-norm-based group consensus measure given in Eq. (15). Furthermore, let $c \in [0,1]$ be a constant. Suppose that $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$ is an arbitrarily fixed input vector, and d is the number of distinct values in the set $\{x_1, x_2, \ldots, x_n\}$. Consider the equation*

$$C_{F_\alpha}(\mathbf{x}) = c. \tag{16}$$

Then the following hold:

(a) If $d = 1$ (i.e., the case of unanimity holds), then

(a1) if $c = 1$, then any $\alpha \in (0, \infty)$ is a solution of Eq. (16);

(a2) if $c \neq 1$, then Eq. (16) has no solution.

(b) If $d = 2$, n is even and the input values are evenly distributed, say half of the input values are $a \in [0,1]$ and the other half of the input values are $b \in [0,1]$, then

(b1) if $c = 1 - |b - a|$*, then any* $\alpha \in (0, \infty)$ *is a solution of Eq.* (16)*;*

(b2) if $c \neq 1 - |b - a|$*, then Eq.* (16) *has no solution.*

(c) If

(i) $d = 2$*,* n *is even and the input values are unevenly distributed, or*

(ii) $d = 2$*, and* n *is odd, or*

(iii) $d \geq 3$*,*

then

(c1) Eq. (16) *has a unique solution for* α*, if*

$$m < c < M,$$

where

$$m = 1 + x_{\pi(1)} - x_{\pi(n)}, \tag{17}$$

and for an even n*,* M *is given by*

$$\begin{aligned} M = 1 - 2 \sum_{i=1}^{\frac{n}{2}} \left(x_{\pi(i)} - x_{\pi(i-1)} \right) \frac{i-1}{n} \\ - 2 \sum_{i=\frac{n}{2}+1}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)} \right) \frac{n-i+1}{n}, \end{aligned} \tag{18}$$

for an odd n*,* M *is given by*

$$\begin{aligned} M = 1 - 2 \sum_{i=1}^{\frac{n+1}{2}} \left(x_{\pi(i)} - x_{\pi(i-1)} \right) \frac{i-1}{n} \\ - 2 \sum_{i=\frac{n+3}{2}}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)} \right) \frac{n-i+1}{n}, \end{aligned} \tag{19}$$

and π *is a permutation on* $\{0, 1, 2, \ldots, n+1\}$ *satisfying Eq.* (3)*;*

(c2) Eq. (16) *has no solution, if*

$$0 < c \leq m \quad \text{or} \quad M \leq c < 1,$$

where m *is given by Eq.* (17)*, and for an even (odd, respectively)* n*,* M *is given by Eq.* (18) *(Eq.* (19)*, respectively).*

Proof. With the substitution

$$z_i = \frac{n-i+1}{n} \tag{20}$$

$C_{F_\alpha}(\mathbf{x})$ can be written as

$$C_{F_\alpha}(\mathbf{x}) = 1 - D_{F_\alpha}(\mathbf{x}), \tag{21}$$

where

$$D_{F_\alpha}(\mathbf{x}) = \sum_{i=1}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) \frac{2}{1 + \left(\frac{1}{2}\left(\left(\frac{1-z_i}{z_i}\right)^{\alpha} + \left(\frac{z_i}{1-z_i}\right)^{\alpha}\right)\right)^{\frac{1}{\alpha}}}. \tag{22}$$

Using the derivative of

$$p_i(\alpha) = \left(\frac{1}{2}\left(\left(\frac{1-z_i}{z_i}\right)^{\alpha} + \left(\frac{z_i}{1-z_i}\right)^{\alpha}\right)\right)^{\frac{1}{\alpha}}, \tag{23}$$

with respect to α, we see that

(I) If $z_i \neq \frac{1}{2}$, then

$$\frac{\mathrm{d}p_i(\alpha)}{\mathrm{d}\alpha} > 0,$$

which means that for a $z_i \neq \frac{1}{2}$, $p_i(\alpha)$ is a strictly increasing function of α;

(II) If $z_i = \frac{1}{2}$, then $p_i(\alpha) = 1$.

Proof of (a). Assume that $d = 1$, i.e., the unanimity holds. In this case, every member in $D_{F_\alpha}(\mathbf{x})$ is zero regardless the value of $\alpha \in (0, \infty)$. That is, $C_{F_\alpha}(\mathbf{x}) = 1$ and so if $c = 1$, then Eq. (16) holds for any $\alpha \in (0, \infty)$. Next, since $C_{F_\alpha}(\mathbf{x}) = 1$, if $c \neq 1$, then $C_{F_\alpha}(\mathbf{x}) \neq c$, which means that for any $c \neq 1$, Eq. (16) has no solution.

Proof of (b). Suppose that $d = 2$, n is even and the input values are evenly distributed, say half of the input values are $a \in [0, 1]$ and the other half of the input values are $b \in [0, 1]$. Then, for any $i \in \{1, 2, \dots, n\} \setminus \{\frac{n}{2} + 1\}$, $x_{\pi(i)} - x_{\pi(i-1)} = 0$. If $i = \frac{n}{2} + 1$, then $x_{\pi(i)} - x_{\pi(i-1)} = |b - a|$, $z_i = \frac{1}{2}$ and $p_i(\alpha) = 1$. Therefore, if the conditions of case (b) hold, then $C_{F_\alpha}(\mathbf{x}) = 1 - |b - a|$ regardless the value of $\alpha \in (0, \infty)$. This immediately implies that under the conditions of case (b), the assertions (b1) and (b2) are valid.

Proof of (c). Now, we will show that under the conditions of case (c), $C_{F_\alpha}(\mathbf{x})$ is a strictly increasing function of α with the limits $\lim_{\alpha \to 0} C_{F_\alpha}(\mathbf{x}) = m$ and $\lim_{\alpha \to \infty} C_{F_\alpha}(\mathbf{x}) = M$, where m is given by Eq. (17), and for an even (odd, respectively) n, M is given by Eq. (18) (Eq. (19), respectively).

Assume that $d = 2$, n is even and the input values are unevenly distributed. Noting Eq. (3), we have that $x_{\pi(i)} - x_{\pi(i-1)} \geq 0$, for $i \in \{1, 2, \dots, n\}$. Furthermore, for any $i \in \{1, 2, \dots, n\} \setminus \{\frac{n}{2} + 1\}$, $z_i \neq \frac{1}{2}$ and for $i = \frac{n}{2} + 1$, $z_i = \frac{1}{2}$. Since the two distinct input values are unevenly distributed, there is exactly one $i^* \in \{1, 2, \dots, n\} \setminus \{\frac{n}{2} + 1\}$ for which $x_{\pi(i^*)} - x_{\pi(i^*-1)} > 0$ and $z_{i^*} \neq \frac{1}{2}$, while for any $i \in \{1, 2, \dots, n\} \setminus \{i^*\}$, $x_{\pi(i)} - x_{\pi(i-1)} = 0$. This means that only the i^*th member in $D_{F_\alpha}(\mathbf{x})$ depends on α, while all the other members in $D_{F_\alpha}(\mathbf{x})$ are equal to zero. Based on (I), $p_{i^*}(\alpha)$ is a strictly increasing function of α, and so $D_{F_\alpha}(\mathbf{x})$ is a strictly decreasing function of α. Therefore, based on Eq. (21), $C_{F_\alpha}(\mathbf{x})$ is a strictly increasing function of α.

Now, assume that $d = 2$, and n is odd. In this case, for any $i \in \{1, 2, \dots, n\}$, $z_i \neq \frac{1}{2}$ and there is exactly one $i^* \in \{1, 2, \dots, n\}$ for which $x_{\pi(i^*)} - x_{\pi(i^*-1)} > 0$, while for any $i \in \{1, 2, \dots, n\} \setminus \{i^*\}$, $x_{\pi(i)} - x_{\pi(i-1)} = 0$. Hence, only the i^*th member in $D_{F_\alpha}(\mathbf{x})$ depends on α, while all the other members in $D_{F_\alpha}(\mathbf{x})$ are equal to zero. Taking into account (I), we find that $p_{i^*}(\alpha)$ is a strictly increasing function of α, and so $D_{F_\alpha}(\mathbf{x})$ is a strictly decreasing function of α. This, based on Eq. (21), implies that $C_{F_\alpha}(\mathbf{x})$ is a strictly increasing function of α.

Now, assume that $d \geq 3$. Noting Eq. (3), (I) and (II), for $i \in \{1, 2, \dots, n\}$, $x_{\pi(i)} - x_{\pi(i-1)} \geq 0$ and $p_i(\alpha)$ is a non-decreasing function. Furthermore, since $d \geq 3$, there exist at least two elements, say j and k ($j \neq k$), in the set $\{1, 2, \dots, n\}$ such that $x_{\pi(j)} - x_{\pi(j-1)} > 0$ and $x_{\pi(k)} - x_{\pi(k-1)} > 0$, and at least one of the z_j and z_k values is not equal to $\frac{1}{2}$. Therefore, based on (I), at least one of the $p_j(\alpha)$ and $p_k(\alpha)$ functions is a strictly increasing function of α. Hence, we find that $D_{F_\alpha}(\mathbf{x})$ is a strictly decreasing function of α, which implies that $C_{F_\alpha}(\mathbf{x})$ is a strictly increasing function of α.

Based on the above findings, we conclude that under the conditions of case (c) (i.e., for (i), (ii) and (iii)), $C_{F_\alpha}(\mathbf{x})$ is a strictly increasing function of α. Next, based on properties (a) and (c) in Lemma 1, with z_i given in Eq. (20), and noting that $z_1 = 1$ and $F_\alpha(1) = 0$, and for $i = 2, \dots, n$, $z_i \in (0, 1)$, we have

$$
\begin{aligned}
\lim_{\alpha \to 0} C_{F_\alpha}(\mathbf{x}) &= 1 - \left(x_{\pi(1)} - x_{\pi(0)}\right) F_\alpha(z_1) \\
&\quad - \sum_{i=2}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) \left(\lim_{\alpha \to 0} F_\alpha(z_i)\right) \\
&= 1 - \sum_{i=2}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) \left(\lim_{\alpha \to 0} F_\alpha(z_i)\right) \\
&= 1 - \sum_{i=2}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) = 1 + x_{\pi_{(1)}} - x_{\pi(n)}.
\end{aligned} \tag{24}
$$

Taking into account properties (a) and (d) in Lemma 1, with z_i given in Eq. (20), and noting that $z_1 = 1$, $F_\alpha(1) = 0$, and for $i = 2, \dots, n$, $z_i \in (0, 1)$, we have

$$
\begin{aligned}
\lim_{\alpha \to \infty} C_{F_\alpha}(\mathbf{x}) &= 1 - \left(x_{\pi(1)} - x_{\pi(0)}\right) F_\alpha(z_1) \\
&\quad - \sum_{i=2}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) \left(\lim_{\alpha \to \infty} F_\alpha(z_i)\right) \\
&= 1 - \sum_{i=2}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) \left(\lim_{\alpha \to \infty} F_\alpha(z_i)\right) \\
&= 1 - \sum_{\substack{i \in \{2, \dots, n\} \\ i : z_i \leq \frac{1}{2}}} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) 2 z_i \\
&\quad - \sum_{\substack{i \in \{2, \dots, n\} \\ i : z_i > \frac{1}{2}}} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) 2 \left(1 - z_i\right).
\end{aligned} \tag{25}
$$

Notice that $z_i \leq \frac{1}{2}$ is equivalent to

- $i \geq \frac{n}{2} + 1$, if i is even
- $i \geq \frac{n+3}{2}$, if i is odd,

and $z_i > \frac{1}{2}$ is equivalent to

- $i \leq \frac{n}{2}$, if i is even
- $i \leq \frac{n+1}{2}$, if i is odd.

Therefore, if n is even, then the right hand side of Eq. (25) can be continued as

$$1 - \sum_{\substack{i \in \{2,\ldots,n\} \\ i: z_i \leq \frac{1}{2}}} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) 2z_i - \sum_{\substack{i \in \{2,\ldots,n\} \\ i: z_i > \frac{1}{2}}} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) 2\left(1 - z_i\right) = M,$$

where, using the definition of z_i and the fact that $z_1 = 1$ and $F_\alpha(1) = 0$, for an even n, M can be written as

$$M = 1 - 2\sum_{i=1}^{\frac{n}{2}} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) \frac{i-1}{n} - 2 \sum_{i=\frac{n}{2}+1}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) \frac{n-i+1}{n},$$

and for an odd n, M can be written as

$$M = 1 - 2\sum_{i=1}^{\frac{n+1}{2}} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) \frac{i-1}{n} - 2 \sum_{i=\frac{n+3}{2}}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) \frac{n-i+1}{n}.$$

That is, we found that under the conditions of case (c) (i.e., for (i), (ii) and (iii)), $C_{F_\alpha}(\mathbf{x})$ is a strictly increasing function of α, and

$$\lim_{\alpha \to 0} C_{F_\alpha}(\mathbf{x}) = m \quad \text{and} \quad \lim_{\alpha \to \infty} C_{F_\alpha}(\mathbf{x}) = M,$$

where m is given by Eq. (17), and for an even (odd, respectively) n, M is given by Eq. (18) (Eq. (19), respectively). Hence, if $m < c < M$, then there exist exactly one value of α for which $C_{F_\alpha}(\mathbf{x}) = c$ holds. Thus, under the conditions of case (c), the assertion (c1) is valid.

Based on the above findings, we immediately see that under the conditions of case (c) (i.e., for (i), (ii) and (iii)), if $0 < c \leq m$ or $M \leq c < 1$, then there is no α for which $C_{F_\alpha}(\mathbf{x}) = c$ holds. □

Remark 4. It is worth noting that the case of maximum dissension may be viewed as a special instance of case (b) in Theorem 2. Namely, with $a = 1$ and $b = 0$, or with $a = 0$ and $b = 1$, case (b) in Theorem 2 coincides with the case of maximum dissension. Notice that based on case (b) in Theorem 2, for $a = 1$ and $b = 0$, or for $a = 0$ and $b = 1$, $C_{F_\alpha}(\mathbf{x}) = 1 - |0 - 1| = 0$, which is the requirement (C4) (i.e., maximum dissension) for a consensus measure given in Definition 1.

Remark 5. It should be added that if the conditions of case (b) in Theorem 2 hold, i.e., n is even and half of the input values are $a \in [0, 1]$ and the other half of the input values are $b \in [0, 1]$, then m and M given by Eq. (17) and Eq. (18), respectively, are

$$m = 1 + x_{\pi(1)} - x_{\pi(n)} = 1 - |b - a|$$

and

$$\begin{aligned} M &= 1 - 2\sum_{i=1}^{\frac{n}{2}} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) \frac{i-1}{n} - 2 \sum_{i=\frac{n}{2}+1}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) \frac{n-i+1}{n} \\ &= 1 - 0 - 2 \cdot |b - a| \cdot \frac{1}{2} = 1 - |b - a|. \end{aligned}$$

This means that under the conditions of case (b) in Theorem 2, the lower and upper limits of $C_{F_\alpha}(\mathbf{x})$ with respect to α coincide, which is in line with the finding that in this case, $C_{F_\alpha}(\mathbf{x}) = 1 - |b - a|$ is a constant regardless the value of $\alpha \in (0, \infty)$.

5 Practical applications of the Dombi t-norm-based group consensus measure

5.1 Modeling the average perceived consensus value of a group

Notice that for a given input vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0, 1]^n$, the value of the group consensus measure $C_{F_\alpha}(\mathbf{x})$ given in Eq. (15) depends on the value of parameter α. Knowing all the individual inputs, each decision-maker has a perceived level of consensus concerning the decision-making situation. Suppose that each decision-maker expresses the perceived level of consensus by a value in the unit interval $[0, 1]$, such that the larger the value, the higher the perceived level of consensus. Let $c_i \in [0, 1]$ be the perceived consensus value of the ith decision-maker, $i = 1, 2, \ldots, n$. Then,

$$\bar{c} = \frac{1}{n} \sum_{i=1}^{n} c_i \tag{26}$$

is the average perceived consensus value of the group. Given the value of $\bar{c}$, the question naturally arises if there exists a value of α for which the group consensus measure $C_{F_\alpha}(\mathbf{x})$ equals the average perceived consensus $\bar{c}$, i.e.,

$$C_{F_\alpha}(\mathbf{x}) = \bar{c}. \tag{27}$$

We can answer this question using Theorem 2. Exploiting this theorem, we can formulate the following procedure to find an optimal value of parameter α, denoted by α_{opt}, for which the Dombi t-norm-based group consensus measure given in Eq. (15) best models the average perceived consensus of a group.

Procedure 1: Optimizing the Dombi t-norm-based group consensus measure for one alternative

Inputs: Input vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$ of a group concerning a decision alternative.

Perceived consensus value of the ith decision-maker $c_i \in [0,1]$, where $i = 1, 2, \ldots, n$.

Step 1: Compute the value of average perceived group consensus $\bar{c}$ as in Eq. (26).

Step 2: Identify the number of distinct values d in the input vector, consider the distribution of the inputs and identify which case in Theorem 2 pertains:

- If $d = 1$, then
  - If $\bar{c} = 1$, then any $\alpha \in (0, \infty)$ is a solution of Eq. (27). Consider an arbitrary $\alpha_{\text{opt}} \in (0, \infty)$.
  - If $\bar{c} \neq 1$, then Eq. (27) has no solution. Consider an arbitrary $\alpha_{\text{opt}} \in (0, \infty)$.

  End procedure.
- If $d = 2$ and the input values are evenly distributed, i.e., half of the input values are $a \in [0,1]$ and the other half of the input values are $b \in [0,1]$, then
  - If $c = 1 - |b - a|$, then any $\alpha \in (0, \infty)$ is a solution of Eq. (16). Consider an arbitrary $\alpha_{\text{opt}} \in (0, \infty)$.
  - If $c \neq 1 - |b - a|$, then Eq. (27) has no solution. Consider an arbitrary $\alpha_{\text{opt}} \in (0, \infty)$.

  End procedure.
- Otherwise proceed to Step 3.

Step 3: Compute m as in Eq. (17) and for an even (odd, respectively) n, compute M as in Eq. (18) (Eq. (19), respectively).

Step 4: If $m < \bar{c} < M$, then approximate the unique solution α_{opt} of Eq. (27) using the bisection method. For the algorithm of bisection method and a comparison with other root-finding methods see [7]. If $0 < \bar{c} \leq m$ or $M \leq \bar{c} < 1$, then Eq. (27) has no solution. In this case, for $\bar{c} \leq m$, consider a small value of α_{opt}, e.g., $\alpha_{\text{opt}} = 10^{-6}$, and for $\bar{c} \geq M$, consider a large value of α_{opt}, e.g., $\alpha_{\text{opt}} = 10^{6}$.

Output: α_{opt} for which $C_{F_{\alpha_{\text{opt}}}}(\mathbf{x})$ best models $\bar{c}$.

Example 1. Let us assume that a faculty of a university is considering to unify the statistical softwares applied in teaching throughout the BSc programmes. As there are several professors being active in teaching statistics related subjects, they would like to take a collective decision and also to measure the level of agreement within the group formed by the professors hereinafter referred to as experts. Therefore, they ask 6 experts to participate in the project which includes 3 phases:

1. The experts evaluate the suitability of the R software with a single value $x_i \in [0,1]$, where the greater the value of x_i, the more suitable the software is considered to be applied in all courses by expert i.

2. The evaluations are aggregated and a decision is made about the suitability of R.

3. The experts are finally asked to evaluate the level of consensus regarding their evaluations about R - now already knowing the evaluations of the other experts. The evaluations of consensus, also called the perceived consensus values $c_i \in [0,1]$ have the following interpretation: the greater the value of c_i, the higher the perceived consensus of expert i w.r.t. the decision about the R software.

Table 1 demonstrates 3 different scenarios, in each of which the perceived consensus values have a different distribution.

Table 1: Numerical example about one input and 3 different scenarios of perceived consensus values

	i :	1	2	3	4	5	6	m	M	$\bar{c}$
Case	x_i :	0.2	0.9	0.8	0.7	0.3	0.25	0.3	0.45	
1	c_i	0.4	0.5	0.3	0.6	0.8	0.5			0.5167
2	c_i	0.2	0.25	0.5	0.15	0.1	0.2			0.2333
3	c_i	0.4	0.35	0.5	0.45	0.15	0.25			0.3500

In this example, the inputs range from $x_1 = 0.2$ to $x_2 = 0.9$, and m can be obtained using Eq. (17), i.e., $m = 1 + 0.2 - 0.9 = 0.3$. The value of m can also be interpreted as the total length of the $[0,1]$ interval falling outside the range of the inputs. This also means that the lower the range the inputs come from, the higher the lower bound of the consensus level will be, which is quite intuitive as in such a case the agreement is high, indeed.

As in this example n is an even number, M is given by Eq. (18) resulting in $M = 0.45$. Here, all values of the input vector are different and as such, with the notation of Theorem 2, $d = 6$ holds. Consequently, Eq. (27) has a unique solution if

$$m < \bar{c} < M.$$

Now, considering the average perceived consensus levels in each case, we see that the condition of having a unique solution only holds in Case 3. In Case 1, the level of consensus is overestimated by the experts, while in Case 2, it is underestimated, and as such considering the bounded nature of the Dombi t-norm-based consensus measure, there is no $\alpha \in (0, \infty)$ for which Eq. (27) holds. In Case 3, using the bisection method, the optimal value of α is found at $\alpha_{\text{opt}} = 0.48015$. Figure 2 provides a graphical demonstration of the above findings.

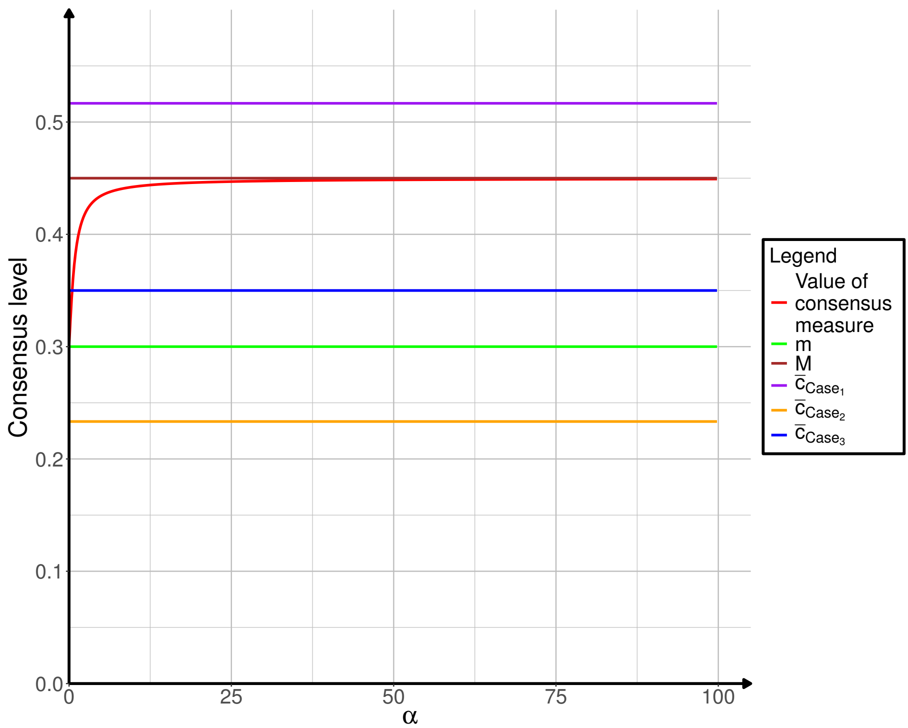

Figure 2: The value of consensus measure as function of parameter α.

5.2 Optimizing C_{F_α} in the case of multiple alternatives

Now, suppose that a group of n entities ($n \in \mathbb{N}$, $n \geq 2$) evaluates r alternatives ($r \in \mathbb{N}$, $r \geq 2$). Let $\mathbf{x}_j \in [0,1]^n$ and $\overline{c}_j \in [0,1]$ denote the input vector and the value of average perceived consensus for the jth alternative, respectively, where $j = 1, 2, \ldots, r$. It is an interesting question what should be the value of α to have the perceived consensus values $\overline{c}_1, \overline{c}_2, \ldots, \overline{c}_r$ best modeled by the Dombi t-norm-based group consensus measures $C_{F_\alpha}(\mathbf{x}_1), C_{F_\alpha}(\mathbf{x}_2), \ldots, C_{F_\alpha}(\mathbf{x}_r)$, respectively. We can answer this question determining $\alpha = \alpha_{\text{opt}}$ such that

$$\sum_{j=1}^{r} \left(C_{F_{\alpha_{\text{opt}}}}(\mathbf{x}_j) - \bar{c}_j \right)^2 = \min_{\alpha \in (0,\infty)} \left(\sum_{j=1}^{r} \left(C_{F_\alpha}(\mathbf{x}_j) - \bar{c}_j \right)^2 \right). \tag{28}$$

Let us consider the bijective mapping $\alpha\colon (0,1) \to (0,\infty)$ that is given by

$$\alpha(\nu) = \frac{1-\nu}{\nu}. \tag{29}$$

Using the inverse of the function in Eq. (29), for any $\alpha \in (0,\infty)$ the corresponding value of ν is

$$\nu(\alpha) = \frac{1}{1+\alpha}.$$

Noting Eq. (29), instead of $C_{F_\alpha}(\mathbf{x})$ given in Eq. (15), we can use the following form of the Dombi t-norm-based group consensus measure:

$$C_{F_\nu}(\mathbf{x}) = 1 - 2\sum_{i=1}^{n} \frac{x_{\pi(i)} - x_{\pi(i-1)}}{1 + \left(\frac{1}{2} \left(\left(\frac{i-1}{n-i+1} \right)^{\frac{1-\nu}{\nu}} + \left(\frac{n-i+1}{i-1} \right)^{\frac{1-\nu}{\nu}} \right) \right)^{\frac{\nu}{1-\nu}}}, \tag{30}$$

where $\nu \in (0,1)$, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$ is an input vector and π is a permutation on $\{0,1,2,\ldots,n+1\}$ satisfying Eq. (3). Using $C_{F_\nu}(\mathbf{x})$ and the bijection given in Eq. (29), the optimization problem given in Eq. (28) is equivalent to

$$\sum_{j=1}^{r} \left(C_{F_{\nu_{\text{opt}}}}(\mathbf{x}_j) - \bar{c}_j \right)^2 = \min_{\nu \in (0,1)} \left(\sum_{j=1}^{r} \left(C_{F_\nu}(\mathbf{x}_j) - \bar{c}_j \right)^2 \right). \tag{31}$$

Once we have the value of $\nu = \nu_{\text{opt}}$ that minimizes the function

$$Q(\nu) = \sum_{j=1}^{r} \left(C_{F_\nu}(\mathbf{x}_j) - \bar{c}_j \right)^2, \quad \nu \in (0,1), \tag{32}$$

we get $\alpha_{\text{opt}} \in (0,\infty)$ as

$$\alpha_{\text{opt}} = \frac{1-\nu_{\text{opt}}}{\nu_{\text{opt}}}.$$

Notice that in the optimization problem given in Eq. (28), the unknown value of α_{opt} is in the interval $(0,\infty)$, while in the case of the optimization problem given in Eq. (31), we need to find the value of ν_{opt} in the bounded interval $(0,1)$. Since a quasi optimal value of ν can be easily found even by a simple brute-force search method in the interval $(0,1)$, it is easier to find a quasi optimal solution for the optimization problem given in Eq. (31) than for the one given in Eq. (28). Based on the above findings, we can formulate the following method to find a quasi optimal value of α for which the Dombi t-norm-based group consensus measure best models the perceived values of group consensus in the case of multiple alternatives.

Procedure 2: Optimizing the Dombi t-norm-based group consensus measure for multiple alternatives

Inputs: Input vectors $\mathbf{x}_j = (x_{1j}, x_{2j}, \ldots, x_{nj}) \in [0,1]^n$ of a group concerning all decision alternatives.

Perceived consensus value of the ith decision-maker regarding the jth decision alternative $c_{ij} \in [0,1]$, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, r$.

Step 1: Compute the value of average perceived group consensus $\overline{c}_j$ for each $j = 1, 2, \ldots, r$ as $\overline{c}_j = \frac{1}{n}\sum_{i=1}^{n} c_{i,j}$.

Step 2: Consider a reasonably large amount of different ν values, where $\nu \in (0,1)$.

Step 3: Calculate the value of $Q(\nu)$ given in Eq. (32) for all the values of ν considered in Step 2 and identify the value of $\nu_{opt} = \nu$ which minimizes $Q(\nu)$.

Step 4: Calculate $\alpha_{\text{opt}} = \frac{1-\nu_{\text{opt}}}{\nu_{\text{opt}}}$.

Output: α_{opt}, a quasi optimal value of α for which $C_{F_{\alpha_{\text{opt}}}}$ best models all $\overline{c}_j$, where $j = 1, 2, \ldots, r$.

Example 2. Let us again assume that a faculty of a university is considering to unify the statistical software applied in teaching throughout the BSc programmes, but now they have to evaluate 3 alternatives as follows:

1. The experts evaluate the suitability of each software alternative with a single value $x_{ij} \in [0,1]$, where the greater the value of x_{ij}, the more suitable software j is considered to be applied in all courses by expert i.

2. The evaluations are aggregated and the candidate software with the highest aggregated evaluation is selected.

3. The experts are finally asked to evaluate the level of consensus regarding their evaluations of each alternative - now already knowing the evaluations of the other experts. The evaluations of consensus, also called the perceived consensus values $c_{ij} \in [0,1]$ have the following interpretation: the greater the value of c_{ij}, the higher the perceived consensus of expert i w.r.t. the decision about software j.

Table 2 contains the x_{ij} evaluations and the c_{ij} perceived consensus values.

We can clearly see from Table 2, that in case of R and SPSS the level of consensus is overestimated by the experts (i.e., no unique solution can be found in terms of α), while a unique solution does exist in case of Excel, as in this case the average perceived consensus level $\overline{c}_{\text{Excel}}$ falls between the corresponding bounds, i.e., $m_{\text{Excel}} < \overline{c}_{\text{Excel}} < M_{\text{Excel}}$ holds.

In case of R and SPSS, a very large value of α might be expected. However, in this case the optimization shall be done by considering each alternative at the

Table 2: Evaluations and perceived consensus levels from 6 decision-makers w.r.t. 3 different software alternatives

	i :	1	2	3	4	5	6	m	M	$\bar{c}_j$
$j = 1$: R	$x_{i,1}$:	0.2	0.9	0.8	0.7	0.3	0.25	0.3	0.45	
	$c_{i,1}$	0.4	0.5	0.3	0.6	0.8	0.5			0.5167
$j = 2$: Excel	$x_{i,2}$	0.7	0.6	0.75	0.8	0.5	0.7	0.7	0.85	
	$c_{i,2}$	0.9	0.75	0.65	0.8	0.7	0.9			0.7833
$j = 3$: SPSS	$x_{i,3}$	0.5	0.8	0.75	0.9	0.4	0.6	0.5	0.6833	
	$c_{i,3}$	0.6	0.9	0.75	0.8	0.85	0.55			0.7417

same time. Based on the above line of thinking, we expect a reasonably large α_{opt} as a result of Procedure 2. And indeed, a quasi optimal value of α is $\alpha_{\text{opt}} = 332.3333$. This means that we can best model the perceived level of consensus for all alternatives with $\alpha_{\text{opt}} = 332.3333$.

In Figure 3, we demonstrate how sensitive the quasi optimal solution is to changes in the average perceived consensus levels. Here, on the horizontal and vertical axes, several possible value pairs of the average perceived consensus levels regarding Excel and SPSS are plotted. For each combination of them, the colours indicate the range into which the quasi optimal solution w.r.t. α falls. The three panels in Figure 3 correspond to the three cases regarding the perceived consensus values of R used earlier in Table 1.

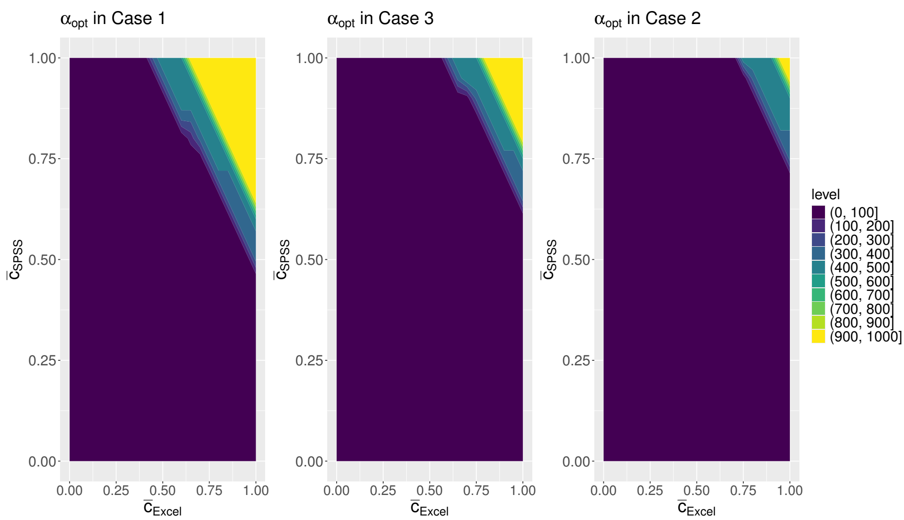

Figure 3: Quasi optimal values of α for several combinations of the averages of perceived consensus

We can see that when the average perceived consensus levels regarding the decision about Excel and SPSS are not too high, the quasi optimal solutions come from the $(0, 100]$ range, and as these averages increase, the quasi optimal solutions increase as well. This can be explained as follows. When the averages of the perceived consensus values are high, a high value of α is needed to decrease the squared deviation of the computed consensus measures from the average perceived consensus levels. We can also observe that as the level of average perceived consensus w.r.t. R decreases ($\bar{c}_{\mathrm{R,\ Case\ 1}} > \bar{c}_{\mathrm{R,\ Case\ 3}} > \bar{c}_{\mathrm{R,\ Case\ 2}}$), the area corresponding to the high-valued quasi optimal solutions also decreases. This is due to the fact that lower averages of the perceived consensus values in the optimizable sum of squared deviations call for a lower quasi optimal value of α.

6 Future research plans

In this study, we came up with the idea of a parametric family of consensus measures. Here, we demonstrated that the flexibility of the group consensus measures proposed in [19] can be further enhanced using the additive generators of the Dombi t-norms to construct fuzzy entropies. This entropies can be used to generate group consensus measures. After examining the properties of the proposed measures, we provided algorithmic procedures to identify the (quasi) optimal value of the α parameter to have a consensus measure modeling best the average perceived consensus level(s) of decision-makers.

In this study, we did not consider consensus as a process, i.e., we considered the level of consensus at one single time, and investigating the so-called consensus reaching process (CRP) fell outside the scope of our study. But it is of course an interesting research question how the choice of the α parameter affects the CRP. This might be investigated in a future study.

As part of our future research, we would also like to study how the additive generator of the generalized Dombi operators (see [14]) can be employed for constructing group consensus measures.

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